A note on the local convergence of iterative methods based on Adomian decomposition method and 3-node quadrature rule

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Abstract

Darvishi and Barati [M.T. Darvishi, A. Barati, Super cubic iterative methods to solve systems of nonlinear equations, Appl. Math. Comput., 2006, 10.1016/j.amc.2006.11.022] derived a Super cubic method from the Adomian decomposition method to solve systems of nonlinear equations. The authors showed that the method is third-order convergent using classical Taylor expansion but the numerical experiments conducted by them showed that the method exhibits super cubic convergence. In the present work, using Ostrowski’s technique based on point of attraction, we show that their method is in fact fourth-order convergent. We also prove the local convergence of another fourth-order method from 3-node quadrature rule using point of attraction.

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1. Introduction

The problem of approximating a locally unique solution \( x_\ast \) of the nonlinear equations system \( \vec{F}(\vec{x}) = \vec{0} \), of \( n \) equations with \( n \) variables has many applications in mathematics and engineering. The most commonly used solution methods are iterative – when starting from one or several initial approximations, a sequence is constructed such that it converges to a solution of the equations [1]. The classic Newton method (NM), given by

\[
\vec{x}_{k+1} = \vec{N}(\vec{x}_k) = \vec{x}_k - \vec{F}'(\vec{x}_k)^{-1}\vec{F}(\vec{x}_k)
\]

is the one of most popular methods for this problem. It is well known that Newton’s method converges locally and quadratically [2], that is the number of good digits is roughly doubled at each iteration. Third-order and
higher-order methods like the Halley [3] and Chebyshev [3,4] methods have little practical value because of the evaluation of second Frechet-derivative. However, third- and higher-order multipoint methods can be good substitutes because they require the computation of only the function and its first derivative at different points. Recently, such methods were developed [4–9]. One of these methods is Dravishi’s super cubic Newton-type method [7] from Adomian decomposition method [10] given by

$$\bar{x}_{k+1} = \bar{G}_1(\bar{x}_k) = \bar{N}(\bar{x}_k) - \bar{F}'(\bar{N}(\bar{x}_k))^{-1}\bar{F}(\bar{N}(\bar{x}_k)).$$  \hspace{1cm} (2)

In [7], this method has been shown to converge cubically by the classic Taylor expansion. However, it is in fact the two step composite Newton (TSN) method which has fourth-order convergence [11, p. 317].

Another method from Adomian decomposition method is the third-order Newton-type method [8] given by

$$\bar{x}_{k+1} = \bar{G}_2(\bar{x}_k) = \bar{N}(\bar{x}_k) - \bar{F}'(\bar{x}_k)^{-1}\bar{F}(\bar{N}(\bar{x}_k)).$$  \hspace{1cm} (3)

This method is termed as the third-order Newton method with constant Jacobian (JN) [6] and its convergence based on point of attraction is studied and proved in [11, p. 315].

Darvishi’s fourth-order Newton-type (DN) method [9] derived from 3-node quadrature rule [5] ($\tau_1 = 0, \tau_2 = \frac{1}{6}, \tau_3 = 1, L_1 = \frac{1}{6}, L_2 = \frac{5}{6}, L_3 = \frac{5}{6}$) is given by

$$\bar{x}_{k+1} = \bar{G}_3(\bar{x}_k) = \bar{x}_k - \bar{A}(\bar{x}_k)^{-1}\bar{F}(\bar{x}_k),$$  \hspace{1cm} (4)

where

$$\bar{A}(\bar{x}_k) = \sum_i L_i \bar{F}'\left((1 - \tau_i)\bar{x}_k + \tau_i \bar{G}(\bar{x}_k)\right), \quad \tau_i \in (0, 1]$$

and

$$\sum_i L_i = 1, \quad \sum_i \tau_i L_i = \frac{1}{2}, \quad \sum_i \tau_i^2 L_i = \frac{1}{3}, \quad \sum_i \tau_i^3 L_i = \frac{1}{6}. $$

In the present work, we study and prove the point of attraction theorem introduced by Ostrowski [12] of two fourth-order iterative methods defined by Eqs. (2) and (4). The advantage of point of attraction theorem over Taylor expansion is that the former gives more precision on the estimation of the rate of convergence and sometimes the conditions on the function derivatives are weaker, for example we can take the advantage of the Lipschitz continuity of the derivative. Finally, we test the above methods with a numerical example.

2. Preliminaries

We shall use the following notations as in [11]: $R_q(\mathcal{F}, \bar{x}_s), Q_q(\mathcal{F}, \bar{x}_s)$ and $O_R(\mathcal{F}, \bar{x}_s), O_Q(\mathcal{F}, \bar{x}_s)$ are the rate and quotient convergence factors and orders of a general iterative process $\mathcal{F}$ at $\bar{x}_s$. For their definitions, see [11]. We state the following important definition:

Definition 1 [11, p. 299]. Let $\bar{G} : \bar{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then $\bar{x}_s$ is a point of attraction of the iteration

$$\bar{x}_{k+1} = \bar{G}(\bar{x}_k), \quad k = 0, 1, \ldots$$ \hspace{1cm} (5)

if there is an open neighbourhood $\bar{S}$ of $\bar{x}_s$ such that $\bar{S} \subset \bar{D}$ and, for any $\bar{x}_0 \in \bar{S}$, the iterates $\{\bar{x}_k\}$ defined by Eq. (5) all lie in $\bar{D}$ and converge to $\bar{x}_s$.

We also state some lemmas and theorems that will be used in this work.

Lemma 2 (Perturbation Lemma [11, p. 45]). Let $\bar{P}, \bar{Q} \in \bar{L}(\mathbb{R}^n)$ and assume that $\bar{P}$ is invertible, with $\|\bar{P}^{-1}\| \leq \alpha$. If $\|\bar{P} - \bar{Q}\| \leq \phi$ and $\phi \alpha \leq 1$, then $\bar{Q}$ is also invertible, and

$$\|\bar{Q}^{-1}\| \leq \frac{\alpha}{1 - \phi \alpha}.$$
Lemma 3 [4]. Let \( \tilde{F} : \overline{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) has a \( p \)th Frechet-derivative [11, p. 61] and its \( \tilde{F}^{(p)} \) be hemicontinuous at each point of a convex set \( \overline{D}_0 \subset \overline{D} \), then for any \( \overline{u}, \overline{v} \subset \overline{D}_0 \), the following holds:

1. If \( \tilde{F}^{(p-1)} \) is Lipschitz continuous, that is,
   \[
   \| \tilde{F}^{(p-1)}(\overline{v}) - \tilde{F}^{(p-1)}(\overline{u}) \| \leq \gamma_{p-1} \| \overline{v} - \overline{u} \|,
   \]
   then,
   \[
   \| \tilde{F}(\overline{v}) - \tilde{F}(\overline{u}) - \sum_{j=1}^{p-1} \frac{1}{j!} \tilde{F}^{(j)}(\overline{u})(\overline{v} - \overline{u})^j \| \leq \frac{\gamma_{p-1}}{p!} \| \overline{v} - \overline{u} \|^p.
   \]
2. If \( \tilde{F}^{(p)} \) is bounded, that is,
   \[
   \| \tilde{F}^{(p)}(\overline{u}) \| \leq K_p,
   \]
   then,
   \[
   \| \tilde{F}(\overline{v}) - \tilde{F}(\overline{u}) - \sum_{j=1}^{p-1} \frac{1}{j!} \tilde{F}^{(j)}(\overline{u})(\overline{v} - \overline{u})^j \| \leq \frac{K_p}{p!} \| \overline{v} - \overline{u} \|^p.
   \]

Theorem 4 [11, Theorem 10.1.6, p. 303]. Let \( \tilde{G} : \overline{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) have a fixed point \( \bar{x} \in \text{Int}(\overline{D}) \). Suppose that \( \tilde{G} \) is Frechet-differentiable at \( \bar{x} \), and that \( \tilde{G}'(\bar{x}) = 0 \). Then \( \bar{x} \) is a point of attraction of the process (5) and \( R_1(\tilde{F}, \bar{x}) = Q_1(\tilde{F}, \bar{x}) = 0 \). Moreover, if in some ball \( \tilde{S} = \tilde{S}(\bar{x}, r) \subset \overline{D} \) the estimate
\[
\| \tilde{G}(\bar{x}) - \tilde{G}(\bar{x})_0 \| \leq \mu_1 \| \bar{x} - \bar{x}_0 \|^q
\]
holds for some \( q > 1 \), then \( O_8(\tilde{F}, \bar{x}) \geq O_8(\tilde{F}, \bar{x})_0 \geq q \). If, in addition, the estimate
\[
\| \tilde{G}(\bar{x}) - \tilde{G}(\bar{x})_0 \| \geq \mu_2 \| \bar{x} - \bar{x}_0 \|^q
\]
holds for some \( \mu_2 > 0 \), then \( O_8(\tilde{F}, \bar{x}) = O_8(\tilde{F}, \bar{x})_0 = q \).

Theorem 5 (Newton Attraction Theorem [11]). Let \( \tilde{F} : \overline{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) be Frechet-differentiable in an open neighbourhood \( \overline{S}_0 \subset \overline{D} \) of a point \( \bar{x} \in \overline{D} \) for which \( \tilde{F}(\bar{x}) = 0 \), and that \( \tilde{F}' \) is continuous at \( \bar{x} \) and \( \tilde{F}'(\bar{x}) \) is non-singular. Then \( \bar{x} \) is a point of attraction of the iteration \( \tilde{F} \) defined by Eq. (1). In addition, if \( \tilde{F}' \) is Lipschitz continuous, then
\[
O_8(\tilde{F}, \bar{x}) \geq O_8(\tilde{F}, \bar{x})_0 \geq 2.
\]

Theorem 6 [11, Theorem 10.2.2, p. 315]. Let \( \tilde{F} : \overline{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) be Frechet-differentiable in an open ball \( \tilde{S} = \tilde{S}(\bar{x}, \delta) \subset \overline{D} \). Assume that \( \tilde{F}' \) is Lipschitz continuous and \( \tilde{F}'(\bar{x}) = 0 \) and that \( \tilde{F}'(\bar{x}) \) is non-singular. Then \( \bar{x} \) is a point of attraction of the iteration \( \tilde{F} \) defined by Eq. (3), and
\[
O_8(\tilde{F}, \bar{x}) \geq O_8(\tilde{F}, \bar{x})_0 \geq 3.
\]
We prove the fourth-order convergence of the TSN method in the section that follows.

3. Attraction theorem for TSN method

Theorem 7. Let \( \tilde{F} : \overline{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) be Frechet-differentiable in an open ball \( \tilde{S} = \tilde{S}(\bar{x}, \delta) \subset \overline{D} \). Assume that \( \tilde{F}' \) is Lipschitz continuous and \( \tilde{F}'(\bar{x}) = 0 \) and that \( \tilde{F}'(\bar{x}) \) is non-singular. Then \( \bar{x} \) is a point of attraction of the iteration \( \tilde{F} \) defined by Eq. (2), and
\[
O_8(\tilde{F}, \bar{x}) \geq O_8(\tilde{F}, \bar{x})_0 \geq 4.
\]
Proof. Since \( \tilde{F}'(\tilde{x}) \) is non-singular
\[
\| \tilde{F}'(\tilde{x}) \| \leq \alpha. \tag{10}
\]
It follows that from Theorem 5 the Newton function defined by Eq. (1) is well defined and satisfies an estimate of the form
\[
\| \tilde{N}(\tilde{x}) - \tilde{x} \| \leq \lambda_i \| \tilde{x} - \tilde{x}_0 \|^2, \quad \text{for all } \tilde{x} \in \tilde{S}_1, \tag{11}
\]
on some ball \( \tilde{S}_1 = \overline{S(\tilde{x}_0, \delta_1)} \subset \tilde{S} \). Suppose \( \delta_2 \) satisfies both
\[
\delta_2^\alpha \leq \frac{\delta_1}{\lambda_1} \quad \text{and} \quad \alpha \gamma_1 \lambda_1 \delta_2^\alpha < \frac{1}{2}.
\]
Let \( \tilde{R}_1 = \tilde{F}'(\tilde{x})^{-1}(\tilde{F}'(\tilde{x}) - \tilde{F}'(\tilde{N}(\tilde{x}))) \).

By non-singularity of \( \tilde{F}'(\tilde{x}) \) (inequality (10)) and Lipschitz continuity of \( \tilde{F}' \) (inequality (6) with \( p = 2 \)) and inequality (11), we have
\[
\| \tilde{R}_1 \| \leq \| \tilde{F}'(\tilde{x})^{-1}(\tilde{F}'(\tilde{x}) - \tilde{F}'(\tilde{N}(\tilde{x}))) \|
\leq \alpha \gamma_1 \| \tilde{x} - \tilde{N}(\tilde{x}) \| \leq \alpha \gamma_1 \lambda_1 \delta_2^\alpha < \frac{1}{2}.
\]
By Perturbation Lemma, \( \tilde{F}'(\tilde{N}(\tilde{x}))^{-1} \) exists and is given by
\[
\| \tilde{F}'(\tilde{N}(\tilde{x}))^{-1} \| \leq \frac{\| \tilde{F}'(\tilde{N}(\tilde{x}))^{-1} \|}{1 - \| \tilde{R}_1 \|} \leq 2 \alpha. \tag{12}
\]
Consequently, the two step composite Newton function \( \tilde{G}_1(\tilde{x}) \) is well defined in \( \tilde{S}_2 = \overline{S(\tilde{x}_0, \delta_2)} \subset \tilde{S}_1 \).

Now, noting that \( \tilde{F}(\tilde{x}) = \tilde{0} \) and by Schwartz inequality
\[
\| \tilde{G}_1(\tilde{x}) - \tilde{x} \| \leq \| \tilde{F}'(\tilde{N}(\tilde{x}))^{-1}(\tilde{F}'(\tilde{N}(\tilde{x})) - \tilde{F}'(\tilde{N}(\tilde{x}))) \|
\leq \| \tilde{N}(\tilde{x}) - \tilde{N}(\tilde{x}) \| \leq \alpha \gamma_1 \lambda_1 \delta_2^\alpha < \frac{1}{2}.
\]
for \( \tilde{x} \in \tilde{S}_2 \). Using inequality (7) with \( p = 2 \), we have
\[
\| \tilde{F}(\tilde{x}) - \tilde{F}(\tilde{N}(\tilde{x})) - \tilde{F}'(\tilde{N}(\tilde{x}))(\tilde{x} - \tilde{N}(\tilde{x})) \| \leq \frac{3}{2} \| \tilde{x} - \tilde{N}(\tilde{x}) \|^2. \tag{14}
\]
By substituting inequalities (11), (12) and (14) into inequality (13), we finally obtain
\[
\| \tilde{G}_1(\tilde{x}) - \tilde{x} \| \leq \alpha \gamma_1 \lambda_1 ^2 \| \tilde{x} - \tilde{x} \|^4,
\]
which is fourth-order.

In the next section, we prove the attraction theorem for DN’s method.

4. Attraction theorem for DN method

Theorem 8. Assuming that \( \tilde{F}^{(2)} \) is bounded and \( \tilde{F}^{(3)} \) is both bounded and Lipschitz continuous on \( \tilde{S} \). Under the conditions of Theorem 6, \( \tilde{x}_0 \) is a point of attraction of the iteration \( \tilde{F} \) defined by Eq. (4), and
\[
O_k(\tilde{F}, \tilde{x}_0) \geq O_0(\tilde{F}, \tilde{x}_0) \geq 4. \tag{15}
\]
Proof. From Theorem 6, it follows that the mapping \( \tilde{G}_2(\tilde{x}) \) defined by Eq. (3) is well defined and satisfies an estimate of the form
\[
\| \tilde{G}_2(\tilde{x}) - \tilde{x} \| \leq \lambda_2 \| \tilde{x} - \tilde{x}_0 \| ^3 \quad \text{for all } \tilde{x} \in \tilde{S}_1, \tag{16}
\]
on some ball \( \tilde{S}_3 = \overline{S(\tilde{x}_0, \delta_3)} \subset \tilde{S} \). Suppose \( \delta_4 \) satisfies both
\[
\delta_4 ^\beta \leq \frac{\delta_3}{\lambda_2} \quad \text{and} \quad \alpha \gamma_1 \delta_4 ^\beta (1 + \lambda_2 \delta_4 ^\beta) < 1.
\]
Let \( \tilde{R}_2 = \tilde{F}'(\bar{x}_s)^{-1} \left( \tilde{F}'(\bar{x}_s) - \tilde{A}(\bar{x}) \right) \).

Using Eqs. (10), (6) and (16), we have

\[
\| \tilde{R}_2 \| \leq \frac{\gamma_1}{2} \| \bar{x}_s - \bar{x} \| + \frac{\gamma_1}{2} \| \tilde{G}_2(\bar{x}) - \bar{x}_s \|
\]

\[
\leq \frac{\gamma_1 \delta_4}{2} \left( 1 + \lambda_2 \delta_4^2 \right) \leq \frac{1}{2}.
\]

By Perturbation Lemma, \( \tilde{A}(\bar{x})^{-1} \) exists and is given by

\[
\| \tilde{A}(\bar{x})^{-1} \| \leq \frac{\| \tilde{F}'(\bar{x}_s)^{-1} \|}{1 - \| \tilde{R}_2 \|} \leq 2x.
\]

Consequently the mapping \( \tilde{G}_3(\bar{x}) \) defined by Eq. (4) is well defined on \( \bar{S}_4 = \bar{S}(\bar{x}_s, \delta_4) \subset \bar{S}_3 \). Then, for \( \bar{x} \in \bar{S}_4 \),

\[
\| \tilde{G}_3(\bar{x}) - \bar{x}_s \| \leq \| \tilde{A}(\bar{x})^{-1} \| \| \tilde{W}_1 \|,
\]

where

\[
\tilde{W}_1 = \tilde{A}(\bar{x})(\bar{x} - \bar{x}_s) - \tilde{F}(\bar{x}).
\]

By Mean Value Theorem for integrals, Eq. (19) becomes

\[
\tilde{W}_1 = \tilde{W}_2 + \tilde{W}_3(\bar{x}_s - \bar{x})^3 - \tilde{W}_4(\bar{x}_s - \bar{x})^2(\tilde{G}_2(\bar{x}) - \bar{x}_s) - \tilde{W}_5(\bar{x}_s - \bar{x})(\tilde{G}_2(\bar{x}) - \bar{x}_s),
\]

where

\[
\tilde{W}_2 = \tilde{F}'(\bar{x}_s) - \tilde{F}'(\bar{x}) - \tilde{F}'(\bar{x}_s - \bar{x}) - \frac{1}{2} \tilde{F}''(\bar{x})(\bar{x}_s - \bar{x})^2 - \frac{1}{6} \tilde{F}'''(\bar{x})(\bar{x}_s - \bar{x})^3,
\]

\[
\tilde{W}_3 = \int_0^1 \int_0^1 \sum_{i=1}^3 \tau_i^2 L_i \left( \tilde{F}'(\bar{x} + \tau_i t (\tilde{G}_2(\bar{x}) - \bar{x})) - \tilde{F}'(\bar{x}) \right) t \, ds \, dt
\]

\[
\tilde{W}_4 = \int_0^1 \int_0^1 \sum_{i=1}^3 \tau_i^2 L_i \tilde{F}''(\bar{x} + \tau_i t (\tilde{G}_2(\bar{x}) - \bar{x})) t \, ds \, dt,
\]

and

\[
\tilde{W}_5 = \int_0^1 \sum_{i=1}^3 \tau_i L_i \tilde{F}'''(\bar{x} + \tau_i t (\tilde{G}_2(\bar{x}) - \bar{x})) \, dt.
\]

Using Eq. (7) with \( p = 4 \), we can bound \( \tilde{W}_2 \) by

\[
\| \tilde{W}_2 \| \leq \frac{\gamma_3}{24} \| \bar{x}_s - \bar{x} \|^4.
\]

Using Eq. (6) with \( p = 4 \), the bound on \( \tilde{W}_3 \) is given by

\[
\| \tilde{W}_3 \| \leq \int_0^1 \int_0^1 \sum_{i=1}^3 \tau_i^2 L_i \left( \| \tilde{F}'(\bar{x} + \tau_i t (\tilde{G}_2(\bar{x}) - \bar{x})) - \tilde{F}'(\bar{x}) \| \right) t \, ds \, dt
\]

\[
\leq \gamma_3 \left( \sum_{i=1}^3 \tau_i^2 L_i \right) \left( \int_0^1 \int_0^1 \tau_i^2 \, ds \, dt \right) \| \tilde{G}_2(\bar{x}) - \bar{x}_s \|
\]

\[
\leq \frac{\gamma_3}{36} \left( \| \tilde{G}_2(x) - \bar{x}_s \| + \| \bar{x}_s - \bar{x} \| \right).
\]

Similarly, we have

\[
\| \tilde{W}_4 \| \leq \frac{1}{6} K_3 \quad \text{and} \quad \| \tilde{W}_5 \| \leq \frac{1}{2} K_2.
\]
By Triangle and Schwartz inequality, using inequalities (20)–(22), the bound on $\tilde{W}_1$ is given by
\[
\|\tilde{W}_1\| \leq \frac{5\gamma_3}{72} \|\tilde{x} - \bar{x}\|^4 + \frac{\gamma_4}{36} \|G_2(\bar{x}) - \tilde{x}\| \|\tilde{x} - \bar{x}\|^3 + \frac{K_3}{6} \|G_2(\bar{x}) - \tilde{x}\| \|\tilde{x} - \bar{x}\|^2 + \frac{K_2}{2} \|\tilde{G}_2(\bar{x}) - \tilde{x}\| \|\tilde{x} - \bar{x}\|.
\]
This bound further simplifies, using inequality (16), to
\[
\|\tilde{W}_1\| \leq \left(\frac{5\gamma_3}{72} + \frac{\lambda_2K_2}{2}\right) \|\tilde{x} - \bar{x}\|^4 + \frac{\gamma_4}{36} \|\tilde{x} - \bar{x}\|^3 + \frac{\lambda_2K_3}{6} \|\tilde{x} - \bar{x}\|^2 + \frac{\lambda_2\delta_4K_3}{6} \|\tilde{x} - \bar{x}\| \|\tilde{x} - \bar{x}\|^2.
\]
By substituting inequalities (17) and (23) into inequality (18), we finally obtain
\[
\|\tilde{G}_3(\bar{x}) - \tilde{x}\| \leq \left(\frac{5\gamma_3}{36} + \alpha\lambda_2K_2 + \frac{\alpha\lambda_2\delta_4K_3}{18} + \frac{\alpha\lambda_2\delta_4K_3}{3}\right) \|\tilde{x} - \bar{x}\|^4,
\]
which is fourth-order.

Having analyzed and proved the convergence of two fourth-order methods, namely the TSN and DN methods, we next compare it with second order NM and third-order JN methods via a numerical example.

5. Numerical experiment

We consider the Complex Cubic [2,6] which is a system of 2 nonlinear equations:
\[
\begin{align*}
{x_1}^3 - 3x_1x_2 - 1 &= 0, \\
3x_1^2x_2 - x_3 &= 0.
\end{align*}
\]
We solve this system by the NM, JN, TSN and DN methods with initial approximation $\bar{x}_0 = (1.5, 0.5)^T$ to find the one of the solution of the system, $\tilde{x} = (1, 0)^T$. The stopping criterion is when $L_2$ residual $\|\tilde{x}_{k+1} - \tilde{x}_k\|_2$ has dropped below $10^{-13}$. Fig. 1 shows the drop of the logarithm of the residual of the four methods considered as we iterate. We can see that the residual falls more steeply for the four methods and in less iterations. This shows the fourth-order convergence of the methods. The DN method is the fastest method for this problem.

6. Conclusion

In this paper, we have proved the attraction theorem for two fourth-order methods. Our numerical experiment confirms the fourth convergence of the methods. It also shows that the two methods are better than the classic Newton method.
References


