Vertex and edge PI indices of Cartesian product graphs

M.H. Khalifeh\textsuperscript{a}, H. Yousefi-Azaria\textsuperscript{a}, A.R. Ashrafi\textsuperscript{b,∗}

\textsuperscript{a}School of Mathematics, Statistics and Computer Science, University of Tehran, Tehran, Islamic Republic of Iran
\textsuperscript{b}Department of Mathematics, Faculty of Science, University of Kashan, Kashan 87317-51167, Islamic Republic of Iran

Received 24 March 2007; received in revised form 22 June 2007; accepted 26 August 2007
Available online 24 October 2007

Abstract

The Padmakar–Ivan (PI) index of a graph \(G\) is the sum over all edges \(uv\) of \(G\) of the number of edges which are not equidistant from \(u\) and \(v\). In this paper, the notion of vertex PI index of a graph is introduced. We apply this notion to compute an exact expression for the PI index of Cartesian product of graphs. This extends a result by Klavzar [On the PI index: PI-partitions and Cartesian product graphs, MATCH Commun. Math. Comput. Chem. 57 (2007) 573–586] for bipartite graphs. Some important properties of vertex PI index are also investigated.

© 2007 Elsevier B.V. All rights reserved.

MSC: 05C12; 05A15; 05A20; 05C05

Keywords: Edge PI index; Vertex PI index; Cartesian product graph

1. Introduction

Let \(G\) be a connected graph with vertex and edge sets \(V(G)\) and \(E(G)\), respectively. As usual, the distance between the vertices \(u\) and \(v\) of \(G\) is denoted by \(d(u, v)\) and it is defined as the number of edges in a minimal path connecting the vertices \(u\) and \(v\).

A topological index is a real number related to a graph. It must be a structural invariant, i.e., it preserves by every graph automorphisms. There are several topological indices have been defined and many of them have found applications as means to model chemical, pharmaceutical and other properties of molecules.

The Wiener index \(W\) is the first topological index to be used in chemistry. Usage of topological indices in chemistry began in 1947 when chemist Harold Wiener developed the most widely known topological descriptor, the Wiener index, and used it to determine physical properties of types of alkanes known as paraffins [19]. In a graph theoretical language, the Wiener index is equal to the count of all shortest distances in a graph. We encourage the reader to consult the special issues of MATCH Communication in Mathematics and in Computer Chemistry [10], Discrete Applied Mathematics [11,7,8], for information on results on the Wiener index, the chemical meaning of the index and its history.

Let \(G\) be a graph and \(e = uv\) an edge of \(G\). \(n_{eu}(e|G)\) denotes the number of edges lying closer to the vertex \(u\) than the vertex \(v\), and \(n_{ev}(e|G)\) is the number of edges lying closer to the vertex \(v\) than the vertex \(u\). The Padmakar–Ivan (PI) index of a graph \(G\) is defined as \(\text{PI}(G) = \sum_{e \in E(G)} [n_{eu}(e|G) + n_{ev}(e|G)]\), see for details [1–5,13–15]. In this

∗Corresponding author.
E-mail address: ashrafi@kashanu.ac.ir (A.R. Ashrafi).

0166-218X/S - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2007.08.041
definition, edges equidistant from both ends of the edge \( e = uv \) are not counted. We call this index the edge PI index and denote by \( \text{PI}_e(G) \). We also define the vertex PI index of \( G \), \( \text{PI}_v(G) \), as the sum of \([m_{eu}(e|G) + m_{ev}(e|G)] \) over all edges of \( G \), where \( m_{eu}(e|G) \) is the number of vertices lying closer to the vertex \( u \) than the vertex \( v \) and \( m_{ev}(e|G) \) is the number of vertices lying closer to the vertex \( v \) than the vertex \( u \).

The Cartesian product \( G \times H \) of graphs \( G \) and \( H \) has the vertex set \( V(G \times H) = V(G) \times V(H) \) and \( (a, x)(b, y) \) is an edge of \( G \times H \) if \( a = b \) and \( xy \in E(H) \), or \( ab \in E(G) \) and \( x = y \). If \( G_1, G_2, \ldots, G_n \) are graphs then we denote \( G_1 \times \cdots \times G_n \) by \( \bigotimes_{i=1}^n G_i \). The Wiener index of Cartesian product graphs was studied in \([9,20]\). In \([16]\), Klavzar, Rajapakse and Gutman computed the Szeged index of Cartesian product graphs. Here we continue this progress to compute the PI index of Cartesian product graphs. The main result of this paper is as follows:

**Theorem 3.** Let \( G_1, G_2, \ldots, G_n \) be connected graphs. Then \( \text{PI}_v(\bigotimes_{i=1}^n G_i) = \sum_{i=1}^n (\prod_{j=1, j \neq i}^n |V(G_j)|^2) \text{PI}_v(G_i), \) and,

\[
\text{PI}_e \left( \bigotimes_{i=1}^n G_i \right) = \sum_{i=1}^n \left( \prod_{j=1, j \neq i}^n |V(G_j)|^2 \right) \text{PI}_e(G_i) + \sum_{i=1}^n \text{PI}_e(G_i) \sum_{j=1, j \neq i}^n |V(G_j)||E(G_j)| \prod_{k=1, k \neq i, j}^n |V(G_k)|^2.
\]

**Corollary.** If \( G \) is a connected graph then \( \text{PI}_e(G^n) = |V(G)|^{2(n-1)} \text{PI}_e(G) + n(n-1)(|E(G)|/|V(G)|) \text{PI}_v(G) \) and \( \text{PI}_v(G^n) = n|V(G)|^{2(n-1)} \text{PI}_v(G) \).

In \([17]\), Klavzar presented a formula for calculating the PI index of the product of two bipartite graphs. In \([21]\), Yousefi, Manoochehrian and Ashrafi independently from Klavzar, computed an exact formula for the product of \( n \) bipartite graphs. In what follows, we obtain Klavzar’s result by our main theorem. We first prove the following simple lemma.

**Lemma 1.** Let \( G \) be a graph. Then \( \text{PI}_v(G) \leq |E(G)||V(G)| \) with equality if and only if \( G \) is bipartite.

**Proof.** Clearly, \( \text{PI}_v(G) = \sum_{e \in E(G)} [m_{eu}(e|G) + m_{ev}(e|G)] \leq \sum_{e \in E(G)} |V(G)| = |V(G)||E(G)| \), which completes the first part of our lemma. If \( G \) is bipartite then \( G \) does not have cycles of odd length and so \( m_{eu}(e|G) + m_{ev}(e|G) = |V(G)\). Thus \( \text{PI}_v(G) = |V(G)||E(G)| \). Conversely, suppose that \( \text{PI}_v(G) = |V(G)||E(G)| \) and \( G \) is not bipartite. Then \( G \) has a cycle of odd length. We assume that \( G \) has girth \( k \) and choose \( T \) and \( e = xy \) such that \( T \) is a cycle of length \( k \) and \( e \) is one of the edges of \( T \). Then \( m_{eu} + m_{ev} < |V(G)| \), a contradiction. This implies that \( G \) is bipartite. \( \square \)

The previous lemma shows that for a tree \( T \) with exactly \( n \) vertices, \( \text{PI}_v(T) = n(n - 1) \).

**Corollary (Klavzar [17]).** If \( G \) is a bipartite connected graph then \( \text{PI}_e(G^n) = n|V(G)|^{2(n-1)} \text{PI}_e(G) + n(n-1)|E(G)|/|V(G)| \text{PI}_v(G) \).

**Proof.** The proof is straightforward and follows from our main theorem and Lemma 1. \( \square \)

Throughout this paper, we only consider connected graphs. Our notation is standard and taken mainly from \([6,12,18]\). \( K_n \) denotes a complete graph on \( n \) vertices. Suppose \( G \) is a graph, \( e = xy \), \( f = uv \in E(G) \) and \( w \in V(G) \). Define \( d(w, e) = \min\{d(w, x), d(w, y)\} \). We say that \( e \) is parallel to \( f \) and write \( e \parallel f \), if \( d(x, f) = d(y, f) \). Note that parallelism is reflexive but it is neither symmetric nor transitive. Since bipartite graphs do not have cycles of odd length, parallelism in bipartite graphs is symmetric.

**2. Examples.**

In this section we calculate the vertex and edge PI indices of some well-known graphs. Suppose \( G \) is a graph. Define \( N_G(e) = |E| - (n_{eu}(e|G) + n_{ev}(e|G)) \), where \( e \) is an arbitrary edge of the graph \( G \). Clearly, \( \text{PI}_v(G) = |E|^2 - \sum_{e \in E(G)} N_G(e) \). We use this simple equation freely throughout the paper.
Example 1. Let $C_n$ be a cycle graph with $n$ vertices. Then

\[
\text{PI}_v(C_n) = \begin{cases} 
  n^2, & 2 \mid n, \\
  n(n - 1), & 2 \nmid n.
\end{cases}
\]

On the other hand, by [4, Lemma 2],

\[
\text{PI}_e(C_n) = \begin{cases} 
  n(n - 2), & 2 \mid n, \\
  n(n - 1), & 2 \nmid n.
\end{cases}
\]

Example 2. Consider the ladder graph $L_n$, Fig. 1. Clearly, $L_n = P_n \times P_2$, where $P_n$ is a path with $n$ vertices. So, by our main theorem $\text{PI}_v(L_n) = 6n^2 - 4n$ and by [4, Example 1], $\text{PI}_e(L_n) = 8(n - 1)^2$.

Consider the graph $G$ whose vertices are the $N$-tuples $b_1 b_2 \cdots b_N$ with $b_i \in \{0, 1, \ldots, n_i - 1\}$, $n_i \geq 2$, and let two vertices be adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a Hamming graph. It is well-known fact that a graph $G$ is a Hamming graph if and only if it can be written in the form $G = \otimes_{i=1}^{N} K_{n_i}$. In the following example, the vertex and edge PI indices of a Hamming graph is computed.

Lemma 2. Let $G$ be a Hamming graph with above parameter. Then

(a) $\text{PI}_v(G) = \prod_{i=1}^{N} n_i^2 \left( N - \sum_{i=1}^{N} \frac{1}{n_i} \right)$,

(b) $\text{PI}_e(G) = \frac{1}{2} \prod_{i=1}^{N} n_i^2 \left( (N + 1) \sum_{i=1}^{N} n_i + (N + 3) \sum_{i=1}^{n} n_i - \sum_{i,j=1}^{N} \frac{n_i}{n_j} - N^2 - 4N \right)$.

Proof. (a) It is easy to see that $\text{PI}_v(K_m) = m(m - 1)$. Since Hamming graph is a product of complete graphs, by Theorem 3,

\[
\text{PI}_v(G) = \text{PI}_v \left( \prod_{i=1}^{N} K_{n_i} \right) = \sum_{i=1}^{N} \text{PI}_v(K_{n_i}) \prod_{j=1, j \neq i}^{N} n_j^2 = \prod_{i=1}^{N} n_i^2 \sum_{i=1}^{N} \frac{n_i - 1}{n_i} = \prod_{i=1}^{N} n_i^2 \left( N - \sum_{i=1}^{N} \frac{1}{n_i} \right).
\]
This completes part (a). Since \( \text{PI}_e(K_m) = m(m-1)(m-2) \), by Theorem 3 and part (a), we have

\[
\text{PI}_e(G) = \sum_{i=1}^{N} n_i(n_i - 1)(n_i - 2) \prod_{j=1, j \neq i}^{N} n_j^2 + \sum_{i=1}^{N} n_i(n_i - 1) \sum_{j=1, j \neq i}^{N} n_j \left( \frac{n_j}{2} \right) \prod_{k=1, k \neq i,j}^{N} n_k^2
\]

\[
= \prod_{i=1}^{N} n_i^2 \left[ \sum_{i=1}^{N} \frac{(n_i - 1)(n_i - 2)}{n_i} + \sum_{i=1}^{N} \left( \frac{n_i - 1}{n_i} \sum_{j=1, j \neq i}^{N} \frac{n_j - 1}{2} \right) \right]
\]

\[
= \prod_{i=1}^{N} n_i^2 \left[ \sum_{i=1}^{N} \frac{(n_i - 1)(n_i - 2)}{n_i} - \frac{1}{2} \sum_{i=1}^{N} (n_i - 1)^2 + \frac{1}{2} \left( \sum_{i=1}^{N} \frac{n_i - 1}{n_i} \right) \left( \sum_{i=1}^{N} (n_i - 1) \right) \right]
\]

\[
= \frac{1}{2} \prod_{i=1}^{N} n_i^2 \left[ \sum_{i=1}^{N} (n_i - 1)(n_i - 3) + \left( N - \sum_{i=1}^{N} \frac{1}{n_i} \right) \left( \sum_{i=1}^{N} n_i \right) - N \right]
\]

\[
= \frac{1}{2} \prod_{i=1}^{N} n_i^2 \left[ \sum_{i=1}^{N} n_i - 4 + 3 \sum_{i=1}^{N} \frac{1}{n_i} + N \sum_{i=1}^{N} n_i - N^2 - \left( \sum_{i=1}^{N} \frac{1}{n_i} \right) \left( \sum_{i=1}^{N} n_i \right) + N \sum_{i=1}^{N} \frac{1}{n_i} \right]
\]

\[
= \frac{1}{2} \prod_{i=1}^{N} n_i^2 \left[ (N + 1) \sum_{i=1}^{N} n_i + (N + 3) \sum_{i=1}^{N} \frac{1}{n_i} - \sum_{i=1}^{N} \frac{n_i}{n_j} - N^2 - 4N \right].
\]

This completes the lemma. \( \square \)

**Corollary.** Let \( Q_n \) denote the hypercube of dimension \( n \) then \( \text{PI}_e(Q_n) = n^{2n-1} \) and \( \text{PI}_e(Q_n) = n(n-1)2^{2n-1} \).

3. Main results

In this section, we prove the main result of this paper. In the following lemma, some well-known properties of Cartesian product graphs are introduced. We encourage the reader to consult the book of Imrich and Klavzar [12], for more details.

**Lemma 3.** Let \( G \) and \( H \) be graphs. Then we have:

(a) \( |V(G \times H)| = |V(G)| \cdot |V(H)| \) and \( |E(G \times H)| = |E(G)| \cdot |V(H)| + |V(G)| \cdot |E(H)| \).

(b) \( G \times H \) is connected if and only if \( G \) and \( H \) are connected.

(c) If \((a, x)\) and \((b, y)\) are vertices of \( G \times H \) then \( d_{G \times H}((a, x), (b, y)) = d_G(a, b) + d_H(x, y) \).

(d) The Cartesian product is commutative and associative.

In what follows, we introduce the main properties of vertex PI index. We begin with an equality between two indices.

**Lemma 4.** Suppose \( G \) is a connected graph with exactly \( m \) edges and \( n \) vertices, \( m \geq 3 \). Then \( 2|E(G)| \leq \text{PI}_e(G) \leq |E(G)|(|E(G)| - 1) \), \( 2|E(G)| \leq \text{PI}_e(G) \leq |V(G)||E(G)| \). Moreover, \( 3|E(G)| - |E(G)|^2 \leq \text{PI}_e(G) - \text{PL}_v(G) \leq |E(G)|(|V(G)| - 2) \) and \( \text{PL}_v(G) \geq \sum_{i=1}^{n} \text{deg}(v_i) - 2|E(G)| \).

**Proof.** Suppose \( u\bar{v} = e \in E(G) \). If \( \text{deg}(u), \text{deg}(\bar{v}) > 1 \) then \( N(e) \leq |E(G)| - 2 \). If \( u \) or \( v \) is an end vertex then \( N(e) = 1 \leq |E(G)| - 2 \). This implies that \( \text{PI}_e(G) = |E(G)|^2 - \sum_{e \in E(G)} N(e) \geq |E(G)|^2 - |E(G)|(|E(G)| - 2) = 2|E(G)| \).
On the other hand, \( \Pi_v(G) = \sum_{e \in E(G)} [m_{eu}(e|G) + m_{ev}(e|G)] \geq 2|E(G)| \). We also note that

\[
\Pi_e(G) = \sum_{e \in E(G)} [n_{eu}(e|G) + n_{ev}(e|G)] \geq \sum_{uv \in E(G)} (\deg(u) + \deg(v) - 2)
\]

\[
= \sum_{i=1}^n \deg(v_i)^2 - 2|E(G)|. \quad \Box
\]

Let \( G \) be a graph and \( e = uv \in E(G) \). Define \( M(e) = | \{ w \in V(G) : d(u, w) = d(v, w) \} | \). Then \( M(e) = |V(G)| - m_{eu}(e|G) - m_{ev}(e|G) \) and so \( \Pi_v(G) = \sum_{e \in E(G)} [m_{eu}(e|G) + m_{ev}(e|G)] = |V(G)||E(G)| - \sum_{e \in E(G)} M(e) \). We use this relation freely throughout this section.

**Theorem 1.** Suppose \( G \) and \( H \) are arbitrary graphs. Then

\[
\Pi_e(G \times H) = |V(G)||E(G)|\Pi_v(H) + |V(H)||E(H)|\Pi_e(G) + \Pi_e(G) |V(H)|^2 + \Pi_e(H)|V(G)|^2.
\]

**Proof.** Suppose \( V(G) = \{u_1, \ldots, u_s\} \), \( V(H) = \{v_1, \ldots, v_t\} \) and define

\[
A_k = \{(u_k, v_m)(u_k, v_n) : v_m v_n \in E(H)\},
\]

\[
B_k = \{(u_i, v_k)(u_j, v_k) : u_i u_j \in E(G)\},
\]

\[
C_k = \{(u_k, v_i) : v_i \in V(H)\},
\]

\[
D_k = \{(u_i, v_k) : u_i \in V(G)\},
\]

\[
A = \bigcup_{i=1}^s A_k, \quad B = \bigcup_{i=1}^t B_k.
\]

Then it is clear that \( \bigcup_{i=1}^s A_k \bigcup \bigcup_{i=1}^t B_k = E(G \times H) \) and \( \bigcup_{i=1}^s C_k = \bigcup_{i=1}^t D_k = V(G \times H) \). On the other hand, by [12, Corollary 1.35] \( d_{G \times H}((u_i, v_m), (u_j, v_n)) = d_G(u_i, u_j) + d_H(v_m, v_n) \) and so \( d_{G \times H}((u_i, v_m), (u_i, v_n)) = d_H(v_m, v_n) \).

**Claim 1.** Suppose \( (u_m, v_i)(u_m, v_j), (u_m, v'_i)(u_m, v'_j) \in A_m \). Then we have \( (u_m, v_i)(u_m, v_j) \parallel (u_m, v'_i)(u_m, v'_j) \) if and only if \( v_i v_j \parallel v'_i v'_j \).

To prove, it is enough to look at the following equalities:

\[
d_H(v_j, v'_j) = \min\{d_H(v_j, v'_j), d_H(v_j, v'_j)\}
\]

\[
= \min\{d_{G \times H}((u_m, v_j), (u_m, v'_j)), d_{G \times H}((u_m, v_j), (u_m, v'_j))\}
\]

\[
= d_{G \times H}((u_m, v_j), (u_m, v'_j)(u_m, v'_j)),
\]

\[
d_H(v_i, v'_i) = \min\{d_H(v_i, v'_i), d_H(v_i, v'_i)\}
\]

\[
= \min\{d_{G \times H}((u_m, v_i), (u_m, v'_i)), d_{G \times H}((u_m, v_i), (u_m, v'_i))\}
\]

\[
= d_{G \times H}((u_m, v_i), (u_m, v'_i)(u_m, v'_i)).
\]
Claim 2. Suppose \((u_m, v_i)(u_m, v_j) \in A_m, (u_k, v_i')(u_k, v_j') \in A_k\). Then \((u_m, v_i)(u_m, v_j) \parallel (u_k, v_i')(u_k, v_j')\) if and only if \(v_i v_j \parallel v_i' v_j'\).

Consider the following equalities:

\[
d_H(v_j, v_i' v_j') + d_G(u_m, u_k) = \min\{d_H(v_j, v_i'), d_H(v_j, v_j')\} + d_G(u_m, u_k) \\
= \min\{d_{G \times H}(u_m, v_j), (u_k, v_i')\}, d_{G \times H}(u_m, v_j), (u_k, v_j')\} \\
= d_{G \times H}(u_m, v_j), (u_k, v_j').
\]

\[
d_H(v_i, v_i' v_j') + d_G(u_m, u_k) = \min\{d_H(v_i, v_i'), d_H(v_i, v_j')\} + d_G(u_m, u_k) \\
= \min\{d_{G \times H}(u_m, v_i), (u_k, v_i')\}, d_{G \times H}(u_m, v_i), (u_k, v_j')\} \\
= d_{G \times H}(u_m, v_i), (u_k, v_j').
\]

Then, \(d_H(v_j, v_i' v_j') = d_H(v_i, v_i' v_j')\) if and only if \((u_m, v_i)(u_m, v_j) \parallel (u_k, v_i')(u_k, v_j')\).

Claim 3. Suppose \((u_m, v_i)(u_m, v_j) \in A_m\). Then \([\{e \in A \mid (u_m, v_i)(u_m, v_j) \parallel e\}] = s \cdot N_H(v_i v_j)\).

By Claim 2, we have \([\{e \in A \mid (u_m, v_i)(u_m, v_j) \parallel e\}] = [\{e \in E(G) \mid v_i v_j \parallel e\}] = N_H(v_i v_j)\). The proof is now follows from the fact that the sets \(A_k\)'s, \(1 \leq k \leq s\), are disjoint.

Claim 4. Suppose \((u_m, v_i)(u_m, v_j) \in A_m\) and \((u_p, v_k)(u_q, v_k) \in B_k\). Then \((u_m, v_i)(u_m, v_j) \parallel (u_p, v_k)(u_q, v_k)\) if and only if \(d_H(v_i, v_k) = d_H(v_j, v_k)\).

Consider the following equalities:

\[
d_G(u_m, u_p u_q) + d_H(v_j, v_k) = \min\{d_G(u_m, u_p), d_G(u_m, u_q)\} + d_H(v_j, v_k) \\
= \min\{d_{G \times H}(u_m, v_j), (u_p, v_k)\}, d_{G \times H}(u_m, v_j), (u_q, v_k)\} \\
= d_{G \times H}(u_m, v_j), (u_p, v_k)(u_q, v_k).
\]

\[
d_G(u_m, u_p u_q) + d_H(v_i, v_k) = \min\{d_G(u_m, u_p), d_G(u_m, u_q)\} + d_H(v_i, v_k) \\
= \min\{d_{G \times H}(u_m, v_i), (u_p, v_k)\}, d_{G \times H}(u_m, v_i), (u_q, v_k)\} \\
= d_{G \times H}(u_m, v_i), (u_p, v_k)(u_q, v_k).
\]

Then, \(d_H(v_j, v_k) = d_H(v_i, v_k)\) if and only if \((u_m, v_i)(u_m, v_j) \parallel (u_p, v_k)(u_q, v_k)\).

Claim 5. Suppose \((u_m, v_i)(u_m, v_j) \in A_m\). Then \([\{(u_p, v_r)(u_q, v_r) \in B \mid (u_m, v_i)(u_m, v_j) \parallel (u_p, v_r)(u_q, v_r)\}] = |E(G)| M_H(v_i v_j)\).

Choose an element \(u_p u_q \in E(G)\). Since the sets \(B_i\) are disjoint, by Claim 3,

\[
M_H(v_i, v_j) = |\{v_k \in V(H) \mid d(v_i, v_k) = d(v_j, v_k)\}| \\
= |\{(u_p, v_k)(u_q, v_k) \mid (u_m, v_i)(u_m, v_j) \parallel (u_p, v_k)(u_q, v_k); 1 \leq k \leq t\}|.
\]

We now vary \(u_p u_q\) on \(E(G)\). We have

\[
|E(G)| M_H(v_i v_j) = |E(G)| [\{u_p u_q \in E(G) \mid \{(u_p, v_k)(u_q, v_k) \mid (u_m, v_i)(u_m, v_j) \parallel (u_p, v_k)(u_q, v_k); 1 \leq k \leq t\}] \\
= |\{(u_p, v_k)(u_q, v_k) \mid (u_m, v_i)(u_m, v_j) \parallel (u_p, v_k)(u_q, v_k); 1 \leq k \leq t; u_p u_q \in E(G)\}| \\
= |\{e \in B \mid (u_m, v_i)(u_m, v_j) \parallel e\}|.
\]
Claim 6. Suppose \((u_m, v_i)(u_m, v_j) \in E(G \times H)\). Then \(N_{G \times H}(u_m, v_i)(u_m, v_j) = |E(G)|M_H(v_i v_j) + |V(G)|N_H(v_i v_j)\). Moreover, \(\sum_{e \in A} N_{G \times H}(e) = |E(G)||E(H)| |V(G)||V(H)| - |E(G)||V(G)||P_{v(H)} + |V(G)|^2 \sum_{e \in E(H)} N_{H}(e)\).

Since \(A \cap B = \emptyset\), \(N_{G \times H}(u_m, v_i)(u_m, v_j) = |\{ e \in A \mid (u_m, v_i)(u_m, v_j) \}| + |\{ e \in B \mid (u_m, v_i)(u_m, v_j) \}| = |E(G)|M_H(v_i v_j) + |V(G)|N_H(v_i v_j)\). Using this relation, we have

\[
\sum_{e \in A} N_{G \times H}(e) = \sum_{m=1}^{s} \sum_{v_i v_j \in E(H)} (|E(G)|M_H(v_i v_j) + |V(G)|N_H(v_i v_j))
\]

\[
= \sum_{m=1}^{s} \left[ |E(G)| \sum_{v_i v_j \in E(H)} M_H(v_i v_j) + s \sum_{v_i v_j \in E(H)} N_H(v_i v_j) \right]
\]

\[
= |E(G)||V(G)| \sum_{v_i v_j \in E(H)} M_H(v_i v_j) + s^2 \sum_{v_i v_j \in E(H)} N_H(v_i v_j)
\]

\[
= |E(G)||E(H)||V(G)||V(H)| - |E(G)||V(G)||P_{v(H)} + |V(G)|^2 \sum_{e \in E(H)} N_{H}(e).
\]

Since \(G \times H \cong H \times G\), using a similar argument as above, one can see that \(\sum_{e \in A} N_{H \times G}(v_m, u_i)(v_m, u_j) = \sum_{e \in B} N_{G \times H}(u_m, v_i)(u_m, v_j)\). Therefore for computing \(\sum_{e \in B} N_{G \times H}(e)\) it is enough to interchange \(G\) and \(H\) in the second part of Claim 6. We now compute \(P_{v(G \times H)}\).

\[
P_{v(G \times H)} = |E(G \times H)|^2 - \sum_{e \in E(G \times H)} N_{G \times H}(e)
\]

\[
= |E(G \times H)|^2 - \sum_{e \in A} N_{G \times H}(e) - \sum_{e \in B} N_{G \times H}(e)
\]

\[
= |V(G)|^2|E(G)|^2 + |V(H)|^2|E(H)|^2 + |E(G)||V(G)||P_{v(H)}
\]

\[
+ |V(G)|^2 \sum_{e \in E(H)} N_{H}(e) + |V(H)|^2 \sum_{e \in E(G)} N_{G}(e) + |E(H)||V(H)||P_{v(G)}
\]

\[
= |V(H)|^2 \left( |E(G)|^2 - \sum_{e \in E(G)} N_{G}(e) \right) + |E(G)||V(G)||P_{v(H)}
\]

\[
+ |V(G)|^2 \left( |E(H)|^2 - \sum_{e \in E(H)} N_{H}(e) \right) + |E(H)||V(H)||P_{v(G)}
\]

\[
= P_{v(G)}|V(H)|^2 + P_{v(H)}|V(G)|^2 + |V(G)||E(G)||P_{v(H)} + |V(H)||E(H)||P_{v(G)}.
\]

This completes the proof. □

Theorem 2. Suppose \(G\) and \(H\) are arbitrary graphs. Then \(P_{v(G \times H)} = P_{v(G)}|V(H)|^2 + P_{v(H)}|V(G)|^2\).
Proof. Consider the notation of Theorem 1. Suppose \((u_m, v_i)(u_m, v_j) \in A_m\). We first prove that \(d_{G \times H}((u_m, v_i), (u_r, v_k)) = d_{G \times H}((u_m, v_j), (u_r, v_k))\) if and only if \(d_H(v_i, v_k) = d_H(v_j, v_k)\). To see this, we have
\[
\begin{align*}
  d_{G \times H}((u_m, v_i), (u_r, v_k)) &= d_G(u_m, u_r) + d_H(v_i, v_k) \\
  &= d_G(u_m, u_r) + d_H(v_j, v_k) \\
  &= d_{G \times H}((u_m, v_j), (u_r, v_k)).
\end{align*}
\]

Since \(v_i v_j \in E(H)\) and \(C_k = \{u_k\} \times V(H)\), \([(u_k, v_r) \in C_k | d_{G \times H}((u_m, v_i), (u_k, v_r)) = d_{G \times H}((u_m, v_j), (u_k, v_r)) = \|[v \in V(H)| d_H(v, v_i) = d_H(v, v_j)] = M_H(v_i v_j)\) Suppose \(C = \bigcup_{k=1}^{n} C_k\) and \(D = \bigcup_{l=1}^{n} D_l\). Since \(C_k\)'s and also \(D_j\)'s are disjoint, \(\{v \in C | d_{G \times H}((u_m, v_i), v) = d_{G \times H}((u_m, v_j), v)\} = |V(G)| M_H(v_i v_j)\). Thus,
\[
\sum_{e \in A} M_{G \times H}(e) = \sum_{m=1}^{s} \sum_{e \in A_m} M_{G \times H}((u_m, v_i)(u_m, v_j)) = \sum_{m=1}^{s} |V(G)| \sum_{v_i v_j \in E(H)} M_H(v_i, v_j) = |V(G)|^2 \sum_{e \in E(H)} M_H(e).
\]

Again since \(G \times H \cong H \times G\), \(\sum_{e \in B} M_{G \times H}(e) = |V(H)|^2 \sum_{e \in E(G)} M_G(e)\). Therefore,
\[
\begin{align*}
  \Pi_v(G \times H) &= |E(G \times H)||V(G \times H)| - \sum_{e \in E(G \times H)} M_{G \times H}(e) \\
  &= |E(G \times H)||V(G \times H)| - \sum_{e \in A} M_{G \times H}(e) - \sum_{e \in B} M_{G \times H}(e) \\
  &= |V(H)|^2 |V(G)||E(G)| + |V(G)|^2 |V(H)||E(H)| \\
  &\quad - |V(H)|^2 \sum_{e \in G} M_G(e) - |V(G)|^2 \sum_{e \in H} M_H(e) \\
  &= \Pi_v(G)|V(H)|^2 + \Pi_v(H)|V(G)|^2.
\end{align*}
\]

Suppose \(G_1, G_2, \ldots, G_n\) be \(n\) graphs and \(G = \bigotimes_{i=1}^{n} G_i\). Then by Lemma 3, \(|V(G)| = \prod_{i=1}^{n} |V(G_i)|\) and \(|E(G)| = \sum_{i=1}^{n} |E(G_j)|\prod_{i=1, j \neq i}^{n} |V(G_j)|\). We now are ready to state our main result. We have:

**Proof of the Theorem 3.** In Theorem 1, we proved the case of \(n = 2\) for \(\Pi_v\). We continue our argument by induction. Suppose the result is valid for \(n\) graphs. Then we have
\[
\begin{align*}
  \Pi_v \left( \bigotimes_{i=1}^{n+1} G_i \right) &= \Pi_v \left( G_{n+1} \times \bigotimes_{i=1}^{n} G_i \right) \\
  &= |V(G_{n+1})|^2 \Pi_v \left( \bigotimes_{i=1}^{n} G_i \right) + \left|V \left( \bigotimes_{i=1}^{n} G_i \right) \right|^2 \Pi_v(G_{n+1}) \\
  &= |V(G_{n+1})|^2 \sum_{i=1}^{n} \Pi_v(G_i) \prod_{j=1, j \neq i}^{n} |V(G_j)|^2 + \Pi_v(G_{n+1}) \prod_{i=1}^{n} |V(G_i)|^2 \\
  &= \sum_{i=1}^{n+1} \Pi_v(G_i) \prod_{j=1, j \neq i}^{n+1} |V(G_j)|^2,
\end{align*}
\]
as desired. To prove the second part of the theorem, we apply again an inductive argument. In Theorem 1, we proved the case of \(n = 2\). Suppose the result is valid for \(n\) graphs. Then by our assumption, the first part of this theorem and
Theorem 1, we have

\[
\begin{align*}
\Pi_e^{(n+1)}(\bigotimes_{i=1}^n G_i) &= \Pi_e\left(G_{n+1} \times \bigotimes_{i=1}^n G_i\right) \\
&= |V(G_{n+1})|^2 \Pi_e\left(\bigotimes_{i=1}^n G_i\right) + \left|V\left(\bigotimes_{i=1}^n G_i\right)\right|^2 \Pi_e(G_{n+1}) \\
&\quad + |V(G_{n+1})| \cdot |E(G_{n+1})| \cdot \Pi_e\left(\bigotimes_{i=1}^n G_i\right) \\
&\quad + \left|V\left(\bigotimes_{i=1}^n G_i\right)\right| \cdot |E\left(\bigotimes_{i=1}^n G_i\right)| \cdot \Pi_e(G_{n+1}) \\
&= |V(G_{n+1})|^2 \left(\sum_{i=1}^n \Pi_e(G_i) \prod_{j=1, j \neq i}^n |V(G_j)|^2ight) \\
&\quad + \sum_{i=1}^n \Pi_e(G_i) \sum_{j=1,j \neq i}^n |E(G_j)| \cdot |V(G_j)| \prod_{k=1,k \neq i,j}^{n+1} |V(G_k)|^2 \\
&\quad + \Pi_e(G_{n+1}) \prod_{i=1}^n |V(G_i)|^2 + \left|V\left(\bigotimes_{i=1}^n G_i\right)\right| \cdot |E\left(\bigotimes_{i=1}^n G_i\right)| \cdot \Pi_e(G_{n+1}) \\
&= \prod_{i=1}^n |V(G_i)| \cdot \Pi_e(G_{n+1}) \cdot \sum_{j=1}^n |E(G_j)| \prod_{i=1,i \neq j}^n |V(G_i)|^2 \\
&\quad + \sum_{i=1}^n \Pi_e(G_i) \sum_{j=1,j \neq i}^{n+1} |E(G_j)| \cdot |V(G_j)| \prod_{k=1,k \neq i,j}^{n+1} |V(G_k)|^2 \\
&\quad + \Pi_e(G_{n+1}) \prod_{j=1,j \neq i}^n |V(G_j)|^2 + |E(G_{n+1})| \cdot \Pi_e\left(\bigotimes_{i=1}^n G_i\right) \\
&= \sum_{i=1}^n \Pi_e(G_i) \sum_{j=1,j \neq i}^n |E(G_j)| \cdot |V(G_j)| \prod_{k=1,k \neq i,j}^{n+1} |V(G_k)|^2 \\
&\quad + |E(G_{n+1})| \cdot \Pi_e(G_{n+1}) \prod_{j=1,j \neq i}^{n+1} |V(G_j)|^2 \\
&= \sum_{i=1}^{n+1} \Pi_e(G_i) \sum_{j=1,j \neq i}^{n+1} |E(G_j)| \cdot |V(G_j)| \prod_{k=1,k \neq i,j}^{n+1} |V(G_k)|^2.
\end{align*}
\]

This completes the proof. \qed
Acknowledgment

We are greatly indebted to the referees, whose valuable criticisms and suggestions led us to correct the paper. The research of the third author was in part supported by the Center of Excellence of Algebraic Methods and Applications of Isfahan University of Technology.

References