Exact travelling wave solutions for the generalized nonlinear Schrödinger (GNLS) equation with a source by Extended tanh–coth, sine–cosine and Exp-Function methods

M. Yaghobi Moghaddam a, A. Asgari b,*, H. Yazdani b

a Department of Mineral Processing, Engineering Faculty, University of Kerman, Kerman, Iran
b Department of Civil Engineering, University of Kerman, P.O. Box 76169-133, Kerman, Iran

Article info

Keywords:
Generalized nonlinear Schrödinger equation
Extended tanh–coth method
Sine–cosine method
Exp-Function method

Abstract

The capability of Extended tanh–coth, sine–cosine and Exp-Function methods as alternative approaches to obtain the analytic solution of different types of applied differential equations in engineering mathematics has been revealed. In this study, the generalized nonlinear Schrödinger (GNLS) equation is solved by three different methods. To obtain the single-soliton solutions for the equation, the Extended tanh–coth and sine–cosine methods are used. Furthermore, for this nonlinear evolution equation the Exp-Function method is applied to derive various travelling wave solution. Results show that while the first two procedures easily provide a concise solution, the Exp-Function method provides a powerful mathematical means for solving nonlinear evolution equations in mathematical physics.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

The nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, chemistry, biology, fluid dynamics, plasma, optical fibers and other areas of engineering. Many efforts have been made to study NPDEs. One of the most exciting advances of nonlinear science and theoretical physics has been a development of methods that look for exact solutions for nonlinear evolution equations. The availability of symbolic computations such as Mathematica or Maple, has popularized direct seeking for exact solutions of nonlinear equations.

Therefore, exact solution methods of nonlinear evolution equations have become more and more important resulting in methods like Variation Iteration Method [1,2], Homotopy Perturbation Method [3–8], Exp-Function method [9–15,36,37], the sine–cosine method [16–18], the homogeneous balance method [19], tanh–sech method [20–24] and Extended tanh–coth method [25–29]. Most of exact solutions have been obtained by these methods, including the solitary wave solutions, shock wave solutions, periodic wave solutions, and the like.

In this paper, we propose Extended tanh–coth, sine–cosine and Exp-Function methods to obtain an exact single-soliton and travelling wave solutions of the generalized nonlinear Schrödinger (GNLS) equation with a source. In order to illustrate the effectiveness and convenience of these methods, we consider the GNLS equation in the form [30,31],

\[ iu_t + au_{xx} + bu|u|^2 + icu_{xxx} + id(u|u|^2)_x = k\phi(\xi - vt), \tag{1.1} \]

where \( \phi = \alpha(x - vt) \) is a real function and \( a, b, c, d, k, \alpha, \nu \) and \( w \) are all real.
The GNLS equation (1.1) plays an important role in many nonlinear sciences. It arises as an asymptotic limit for a slowly varying dispersive wave envelope in a nonlinear medium. For example, its significant application in optical soliton communication plasma physics has been proved.

Furthermore, the GNLS equation enjoys many remarkable properties (e.g., bright and dark soliton solutions, Lax pair, Liouville integrability, inverse scattering transformation, conservation laws, Backlund transformation, etc.).

The rest of this paper is as follows: in Sections 2–4, we provide, in a simple way, the mathematical framework of Extended tanh–coth, sine–cosine and Exp-Function methods, respectively. In Section 5, in order to illustrate the application of these methods, generalized nonlinear Schrödinger equation with a source is investigated, and several exact solutions, including soliton like solutions and trigonometric function solutions, are obtained. This paper is concluded in the last section.

2. Tanh method and Extended tanh method

We now describe the tanh method for a given partial differential equation. This method was defined by Malfliet [20] and Fan and Hon [26]. Wazwaz summarized the main steps of using this method as follows [24]:

I. Wazwaz first considered a general form of nonlinear equation:

\[ N(u, u_t, u_x, u_{xx}, \ldots) = 0, \quad (2.1) \]

II. To find the travelling wave solution of Eq. (2.1), he introduced the wave variable:

\[ \xi = k(x + \lambda t), \quad (2.2) \]

so that

\[ u(x, t) = U(\xi), \quad (2.3) \]

Therefore, Eq. (2.1) constructs ODE of form:

\[ N(U, kU', kU'', k^2U''', \ldots) = 0. \quad (2.4) \]

III. If all terms of the resulting ODE contain derivatives in \( \xi \), then by integrating this equation, and by considering the constant of integration to be zero, we obtain a simplified ODE.

IV. Introducing a new independent variable:

\[ Y = \tanh(\xi) \quad \text{(or} \quad Y = \coth(\xi) \text{)} \quad (2.5) \]

leads to a change in the derivatives:

\[ \frac{d}{d\xi} = (1 - Y^2) \frac{d}{dY}, \]

\[ \frac{d^2}{d\xi^2} = (1 - Y^2) \left( -2Y \frac{d}{dY} + (1 - Y^2) \frac{d^2}{dY^2} \right), \quad (2.6) \]

\[ \frac{d^3}{d\xi^3} = (1 - Y^2) \left( (6Y^2 - 2) \frac{d}{dY} - 6Y(1 - Y^2) \frac{d^2}{dY^2} + (1 - Y^2)^2 \frac{d^3}{dY^3} \right), \]

and the remaining derivatives may be derived similarly.

V. Introduce the ansatz and then solution of \( U(\xi) \) is in the form of:

(a) tanh method:

\[ U(\xi) = \sum_{p=0}^{m} a_p Y^p = a_0 + \cdots + a_m Y^m. \quad (2.7) \]

(b) Extended tanh method:

\[ U(\xi) = \sum_{p=-m}^{m} a_p Y^p = a_m Y^{-m} + \cdots + a_0 + \cdots + a_m Y^m, \quad (2.8) \]

where \( m \) is a positive integer which is unknown to be later determined and \( a_p \) are unknown constants.

VI. To determine the parameter \( m \), we usually balance linear terms of the highest order in the resulting equation with the highest order nonlinear terms. With \( m \), determined as described, by balancing the coefficients of \( Y \)'s with the same power in the resulting equation, a system of algebraic equations involving the \( a_p \) \((p = -m, \ldots, 0, \ldots, m)\) \( k \) and \( \lambda \) is derived.

Having these determined parameters, taking into account that in most cases these parameters are positive, by using (2.8), we find an analytic solution in a closed form.
The travelling wave solutions of many nonlinear ODEs and PDEs from soliton theory (and elsewhere) can be expressed as polynomials of hyperbolic or elliptic functions. For instance, the bell-shaped sech-solutions and kink-shaped tanh-solutions model wave phenomena in fluid dynamics, plasmas, elastic media, electrical circuits, optical fibers, chemical reactions, and bio-genetics.

3. Sine–cosine method

Wazwaz has summarized the main steps of using sine–cosine method, as listed below:

I. Introducing the wave variables \( \xi = x - ct \) into the PDE, the following function is obtained:
\[
\phi(u, u_t, u_x, u_{tt}, u_{xx}, u_{xxx}, \ldots) = 0,
\]
where \( u(x, t) \) is travelling wave solution. This allows the following changes:
\[
\frac{\partial}{\partial t} = -c \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial t^2} = c^2 \frac{d^2}{d\xi^2},
\]
and so for the other derivates. Using Eqs. (3.3) and (3.1), the nonlinear PDE (3.1) is changed to a nonlinear ODE:
\[
N(U, -cU', U', c^2U'', U''', -cU', U''' \ldots) = 0.
\]

II. If all terms of the resulting ODE contain derivatives of \( \xi \), then by integrating this equation and considering the constant of integration zero, a simplified ODE is obtained.

III. By virtue of this solution, the ansatz is introduced as:
\[
U(\xi) = u(x, t) = \lambda \sin^\beta(\mu \xi), \quad |\mu \xi| < \frac{\pi}{2},
\]
or,
\[
U(\xi) = u(x, t) = \lambda \cos^\beta(\mu \xi), \quad |\mu \xi| < \frac{\pi}{2\mu},
\]
where \( \lambda, \mu \) and \( \beta \) are parameters to be determined later. \( \mu \) and \( c \) are the wave number and the wave speed, respectively. By sequence differentiating of \( n \)th power of Eqs. (3.5) and (3.6) with respect to \( \xi \) the following equations are attained:
\[
U(\xi) = \lambda \sin^\beta(\mu \xi),
\]
\[
U^n(\xi) = \lambda^n \sin^\beta(\mu \xi),
\]
\[
(U^n)_{\xi\xi} = n\mu \beta^2 \sin^2(\mu \xi) \cos^{n-1}(\mu \xi),
\]
\[
(U^n)_{\xi\xi\xi} = -n^2 \mu^2 \beta^2 \sin^2(\mu \xi) + n\mu \beta \sin^{n-1}(\mu \xi),
\]
\[
(U^n)_{\xi\xi\xi\xi} = 2\mu \beta \sin^{n-1}(\mu \xi) \cos(\mu \xi),
\]
and
\[
U(\xi) = \lambda \cos^\beta(\mu \xi),
\]
\[
U^n(\xi) = \lambda^n \cos^\beta(\mu \xi),
\]
\[
(U^n)_{\xi\xi} = -n\mu \beta^2 \cos^2(\mu \xi) \sin^{n-1}(\mu \xi),
\]
\[
(U^n)_{\xi\xi\xi} = -n^2 \mu^2 \beta^2 \cos^2(\mu \xi) + n\mu \beta \cos^{n-1}(\mu \xi),
\]
\[
(U^n)_{\xi\xi\xi\xi} = 2\mu \beta \cos^{n-1}(\mu \xi) \sin(\mu \xi),
\]
and so for the other derivates.

IV. After substituting Eq. (3.7) or Eq. (3.8) into the reduced Eq. (3.4) obtained above, by equating the two sides of the equation a system of algebraic equations is obtained which can be solved with computerized symbolic calculations. This way, the coefficients of all terms with the same power, \( k \), in \( \sin^k(\mu \xi) \) or \( \cos^k(\mu \xi) \), set to zero to get a system of algebraic equations including the unknowns \( \mu, \beta \) and \( \lambda \). All possible values of the parameters \( \mu, \beta \) and \( \lambda \) have been obtained.
4. Summary of Exp-Function method

The Exp-Function method was first proposed by Wu and He in 2006 [10] and systematically studied in [32]. Furthermore, through the investigation of more than 280 references, He [36] presented an excellent study on the concepts of the recently developed asymptotic methods including Exp-Function. In addition, this method was successfully applied to KdV equation with variable coefficients [33], high-dimensional nonlinear evolution equation [14], Burgers and combine KdV–mKdV (Extended KdV) equations [34], etc.

In this section, for a given PDE, we commence by looking for an Exp-Function type solution of the following form in terms of \( \exp(\xi) \):

\[
U(\xi) = \frac{\sum_{n=-\infty}^{d} a_n \exp(n \xi)}{\sum_{m=-\infty}^{\infty} b_m \exp(m \xi)} = \frac{a_0 \exp(c \xi) + \cdots + a_d \exp(-d \xi)}{a_p \exp(p \xi) + \cdots + a_q \exp(-q \xi)},
\]

where \( c, d, p \) and \( q \) are positive integers which are the unknowns to be later determined and \( a_n \) and \( b_m \) are unknown constants.

Alternatively, we can also assume that the solution can be expressed in this form:

\[
U(\xi) = \sum_{j=1}^{n} a_j \varphi^j,
\]

where \( \varphi \) is the solution of the sub-equation \( \varphi' = r + p\varphi + q\varphi^2 \).

By differentiating of Eq. (4.1) with respect to \( \xi \), introducing the result into Eq. (2.4) and equating the arranged coefficients of the same powered \( \exp(\xi) \) to zero, a system of algebraic equations is constructed. The Exp-Function solution of the Eq. (2.1) can be solved by obtaining \( a_n \) and \( b_m \) from this system.

The following important remarks have been made on the Exp-Function method in [35]:

- The expression of the Exp-Function is more general than the sinh-function and the tanh-function, so we can find solutions that are more general in the Exp-Function method.
- Using Maple or Mathematicamakes the solution procedure simpler.
- Although the Exp-Function method can be employed in both the direct way and the sub-equation way, direct use of this method is suggested, not only for its convenience, but because applying Exp-Function in the sub-equation way may lead to losing some information and solutions.
- The Exp-Function method is more convenient and effective than the Extended Fan sub-equation method.

With regard to the fact that each method of solving differential equations gives the answer in a particular form, depending on the initial and boundary conditions, there would be a better method for a particular differential equation. For example, for a problem with exponential function initial condition, using Exp-Function method is preferred. Once the best method of solution is determined, the results can be classified in distinct collections so that the components of each collection recover each other. A collection which satisfies the initial and boundary conditions, is an exact solution of the equation.

The aim of this paper is to employ, for the first time, Extended tanh, sine–cosine and Exp-Function methods for solving the generalized nonlinear Schrödinger (GNLS) equation with a source not considering the initial and boundary conditions.

5. Exact solutions of GNLS equation with a source

5.1. Using Extended tanh method

To study the exact travelling wave solutions of the GNLS, Eq. (1.1), we consider a plane wave transformation in this form:

\[
u(x, t) = \psi(\xi) e^{i(\chi x - vt)},
\]

where \( \psi(\xi) \) is a real function. For convenience, let \( \chi = \beta \xi + x_0 \) where \( \beta \) and \( x_0 \) are real constants and \( \xi = \alpha(x - vt) \). Then, by replacing Eq. (5.1) and its appropriate derivatives in Eq. (1.1) and separating the real and imaginary parts of the result, we obtain the following ordinary differential equations:

\[
\begin{align*}
  c \psi'' + (\xi \psi + 2a\beta \psi^2 - 3c \chi^2 \beta^2) \psi' + 3a \psi^2 \psi' &= 0, \\
  (a \psi^2 - 3c \psi^3 \beta) \psi' + (a \beta \psi \psi' + w - ax^2 \beta^2 + c \chi^3 \beta^3) \psi + (b - d \alpha \beta) \psi^3 - k &= 0.
\end{align*}
\]

Integrating Eq. (5.2) once, with respect to \( \xi \), yields:

\[
(4)x^2 (\psi'(\xi)) + p \psi(\xi) + dq^3 \psi(\xi) - C = 0,
\]

where \( p = -v + 2a\beta x - 3cx^2 \beta^2 \) and \( C \) is an integration constant. Since the same function \( \psi(\xi) \) satisfies two Eqs. (5.3) and (5.4), we obtain the following constraint condition:
Substituting Eqs. (5.8)–(5.10) and (5.11) into Eq. (5.6), we obtain the following exact solutions for Eq. (1.1):
\[
\frac{(az^2 - 3\psi^3\beta)}{cx^2} = \frac{(x\beta + w - ax^2\beta^2 + cx^2\beta^3)}{p} = \frac{(b - dx\beta)}{d} = \frac{k}{C}.
\] (5.5)

Our main purpose is to solve Eq. (5.4). Considering the homogeneous balance between \(\psi''(\zeta)\) and \(\psi^3(\zeta)\) in Eq. (5.4) yields \(m = 1\). We suppose that the solution of Eq. (5.4) can be expressed by
\[
\psi(\zeta) = \sum_{p=-1}^{1} a_p Y^p = a_0 Y^{-1} + a_0 + a_1 Y, \tag{5.6}
\]
where \(Y = \tanh(\zeta)\) or \(Y = \coth(\zeta)\).

Substituting Eq. (5.6) into Eq. (5.4) and then setting the coefficients of all independent terms in \(Y\), the following algebraic relations are obtained:
\[
\begin{align*}
&3b_1^2 + 2cx^2 b_1 = 0, \\
&3db_1a_0 = 0, \\
&3da_1b_1^2 + 3db_1a_0^2 - 2cx^2 b_1 + 2ax\beta b_1 - 3cx^2\beta^2 b_1 - vb_1 = 0, \\
&- C + 2ax\beta a_0 - va_0 - 3cx^2\beta^2 a_0 + 6da_1b_1a_0 + da_0^2 = 0. \\
&- 2cx^2 a_1 + 2ax\beta a_1 + 3da_1a_0^2 - 3cx^2\beta^2 a_1 + 3da_1b_1 - va_1 = 0, \\
&3da_0^2 = 0, \\
&2cx^2 a_1 + da_0^2 = 0.
\end{align*}
\] (5.7)

The above equations are cumbersome to solve. Using a modern computer algebra system, say Maple, gives

Case 1:
\[
\begin{align*}
a_1 &= 0, & a_0 &= 0, & b_1 &= b_1, & \alpha &= \alpha, & \beta &= \beta, & a &= a, & c &= c, \\
d &= -\frac{2cx^2}{b_1}, & v &= \frac{-2cx^2 + 2ax\beta - 3cx^2\beta^2}{C}, & C &= 0.
\end{align*}
\] (5.8)

Case 2:
\[
\begin{align*}
a_1 &= a_1, & a_0 &= 0, & b_1 &= 0, & \alpha &= \alpha, & \beta &= \beta, & a &= a, & c &= c, \\
d &= -\frac{2cx^2}{a_1^2}, & v &= \frac{-2cx^2 + 2ax\beta - 3cx^2\beta^2}{C}, & C &= 0.
\end{align*}
\] (5.9)

Case 3:
\[
\begin{align*}
a_1 &= -b_1, & a_0 &= 0, & b_1 &= b_1, & \alpha &= \alpha, & \beta &= \beta, & a &= a, & c &= c, \\
d &= -\frac{2cx^2}{b_1^2}, & v &= \frac{4cx^2 + 2ax\beta - 3cx^2\beta^2}{C}, & C &= 0.
\end{align*}
\] (5.10)

Case 4:
\[
\begin{align*}
a_1 &= b_1, & a_0 &= 0, & b_1 &= b_1, & \alpha &= \alpha, & \beta &= \beta, & a &= a, & c &= c, \\
d &= -\frac{2cx^2}{b_1^2}, & v &= \frac{-8cx^2 + 2ax\beta - 3cx^2\beta^2}{C}, & C &= 0.
\end{align*}
\] (5.11)

Substituting Eqs. (5.8)–(5.10) and (5.11) into Eq. (5.6), we obtain the following exact solutions for Eq. (1.1):
\[
\begin{align*}
\psi_1(\zeta) &= b_1 \coth(\zeta), & \zeta &= \alpha(x - (-2cx^2 + 2ax\beta - 3cx^2\beta^2)t), \tag{5.12} \\
\psi_2(\zeta) &= a_1 \tanh(\zeta), & \zeta &= \alpha(x - (-2cx^2 + 2ax\beta - 3cx^2\beta^2)t), \tag{5.13} \\
\psi_3(\zeta) &= -b_1(\tanh(\zeta) - \coth(\zeta)), \quad \zeta = \alpha(x - (4cx^2 + 2ax\beta - 3cx^2\beta^2)t), \tag{5.14} \\
\psi_4(\zeta) &= b_1(\tanh(\zeta) + \coth(\zeta)), \quad \zeta = \alpha(x - (-8cx^2 + 2ax\beta - 3cx^2\beta^2)t). \tag{5.15}
\end{align*}
\]
or,
\[
\begin{align*}
u_1(\zeta) &= b_1 \coth(\zeta) \times e^{(i\beta - \psi + \rho_0)}), & \zeta &= \alpha(x - (-2cx^2 + 2ax\beta - 3cx^2\beta^2)t), \tag{5.16} \\
u_2(\zeta) &= a_1 \tanh(\zeta) \times e^{(i\beta - \psi + \rho_0)}, & \zeta &= \alpha(x - (-2cx^2 + 2ax\beta - 3cx^2\beta^2)t), \tag{5.17} \\
u_3(\zeta) &= -b_1(\tanh(\zeta) - \coth(\zeta)) \times e^{(i\beta - \psi + \rho_0)}, & \zeta &= \alpha(x - (4cx^2 + 2ax\beta - 3cx^2\beta^2)t), \tag{5.18} \\
u_4(\zeta) &= b_1(\tanh(\zeta) + \coth(\zeta)) \times e^{(i\beta - \psi + \rho_0)}, & \zeta &= \alpha(x - (-8cx^2 + 2ax\beta - 3cx^2\beta^2)t). \tag{5.19}
\end{align*}
\]
Defining \( z \) and \( \beta \) as two imaginary numbers, the obtained solitary solution can be converted into a periodic solution. Therefore, we define:
\[
\begin{align*}
\alpha &= iA, \\
\beta &= iB,
\end{align*}
\]
where \( i = \sqrt{-1} \) and then apply the following transformations:
\[
\begin{align*}
\sinh(i\xi) &= i\sin(\xi), & \cosh(i\xi) &= \cos(\xi), & \tanh(i\xi) &= i\tan(\xi), \\
\coth(i\xi) &= -i\cot(\xi), & \text{sech}(i\xi) &= \sec(\xi), & \csc h(i\xi) &= -i\csc(\xi).
\end{align*}
\]
In Eqs. (5.12)–(5.18) and (5.19); the results are:
\[
\begin{align*}
\psi_1(x, t) &= -ib_1 \cot(Ax - (2cA^3 - 2aA^2B - 3cA^3B^2)t), \\
\psi_2(x, t) &= i\alpha_1 \tan(Ax - (2cA^3 - 2aA^2B - 3cA^3B^2)t), \\
\psi_3(x, t) &= -ib_1 (\tan(Ax + (4cA^3 + 2aA^2B + 3cA^3B^2)t) + \cot(Ax + (4cA^3 + 2aA^2B + 3cA^3B^2)t)), \\
\psi_4(x, t) &= ib_1 (\tan(Ax + (-8cA^3 + 2aA^2B + 3cA^3B^2)t) - \cot(Ax + (-8cA^3 + 2aA^2B + 3cA^3B^2)t)),
\end{align*}
\]
or,
\[
\begin{align*}
u_1(x, t) &= -ib_1 \cot(Ax - (2cA^3 - 2aA^2B - 3cA^3B^2)t) \times e^{i(\beta A(-x - (2cA^3 - 2aA^2B - 3cA^3B^2)t) - \omega x)}, \\
u_2(x, t) &= i\alpha_1 \tan(Ax - (2cA^3 - 2aA^2B - 3cA^3B^2)t) \times e^{i(\beta A(-x - (2cA^3 - 2aA^2B - 3cA^3B^2)t) - \omega x)}, \\
u_3(x, t) &= -ib_1 (\tan(Ax + (4cA^3 + 2aA^2B + 3cA^3B^2)t) + \cot(Ax + (4cA^3 + 2aA^2B + 3cA^3B^2)t)) \times e^{i(\beta A(-x - (4cA^3 + 2aA^2B + 3cA^3B^2)t) - \omega x)}, \\
u_4(x, t) &= ib_1 (\tan(Ax + (-8cA^3 + 2aA^2B + 3cA^3B^2)t) - \cot(Ax + (-8cA^3 + 2aA^2B + 3cA^3B^2)t)) \times e^{i(\beta A(-x - (8cA^3 - 2aA^2B + 3cA^3B^2)t) - \omega x)},
\end{align*}
\]}

5.2. Using sine–cosine function method

In this section, the sine–cosine method is applied to the GNLS equation solution. Substituting Eqs. (3.6) and (3.8) with \( n = 1 \) into Eq. (5.4) and rewriting the equation in terms of cosine function gives
\[
\begin{align*}
&cx^2 \lambda^2 \mu^2 \cos(\mu \zeta) \theta - cx^2 \lambda^2 \mu^2 \cos(\mu \zeta) \theta - \lambda v \cos(\mu \zeta) \theta + 2i \alpha x \beta \cos(\mu \zeta) \theta \\
&- 3icx^2 \mu^2 \cos(\mu \zeta) \theta + d \lambda^2 \cos(\mu \zeta) \theta - C = 0.
\end{align*}
\]
Balancing the terms of the cosine functions in Eq. (5.30), we have
\[
3\beta = \beta - 2 \quad \Rightarrow \quad \beta = -1.
\]
Substituting Eq. (5.31) into Eq. (5.4) and equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equations:
\[
\begin{align*}
&\cos^0(\mu \zeta) : \quad -C = 0, \\
&\cos^{-1}(\mu \zeta) : \quad -iv + 2i \alpha x \beta - 3icx^2 \beta - cx^2 \lambda \mu^2 = 0, \\
&\cos^{-3}(\mu \zeta) : \quad 2cx^2 \lambda^2 \mu^2 + dx^3 = 0.
\end{align*}
\]
Solving 5.32, 5.33 and 5.34 equations by Maple, we obtain:
\[
\begin{align*}
C &= 0, & v &= 2ax \beta - 3cx^2 \beta^2 - cx^2 \mu^2, & \lambda &= \lambda, & d &= -\frac{2cx^2 \mu^2}{\lambda^2}, & \mu &= \mu, & c &= c, & a &= a, & \beta &= \beta, & \alpha &= \alpha,
\end{align*}
\]
The results in Eq. (5.35) can be easily obtained if we use the sine method, Eq. (3.5), as well. Combining Eqs. (5.35) and (3.6), the following exact solutions will be obtained:
\[
\begin{align*}
\psi(x, t) &= \frac{\lambda}{\cos(\mu \zeta(x - (2ax \beta - 3cx^2 \beta^2 - cx^2 \mu^2)t))}, \\
or,
\psi(x, t) &= \frac{\lambda e^{i(\mu \zeta(x - (2ax \beta - 3cx^2 \beta^2 - cx^2 \mu^2)t) - \omega x))}}{\cos(\mu \zeta(x - (2ax \beta - 3cx^2 \beta^2 - cx^2 \mu^2)t)))},
\end{align*}
\]
Eq. (5.37a) satisfies Eq. (1.1).
In addition, using Eqs. (5.20) and (5.21) into Eq. (5.37a), we have:

\[ u(x,t) = \frac{1}{2} \sec h \left( \mu (x + (2aAB + 3cA^2B^2 - cA^2 \mu^2) t) \right) \times e^{iBA_{(x+(-2aAB-3cA^2B^2+cA^2\mu^2)t-v_{w}x})}, \]  

(5.37b)

which is the exact kink-shaped solitary wave solution of GNLS equation.

5.3. Using Exp-Function method

In this section, the Exp-Function method is applied to solve the GNLS equation. To determine the values of \( c \) and \( p \), we balance the linear term of the highest order \( \psi''(\xi) \) with the highest order nonlinear term \( \psi^3(\xi) \) in Eq. (4.1). We have:

\[ \psi''(\xi) = \frac{c_1 \exp(3p + c)}{c_2 \exp[4p_2] + \cdots}, \]

(5.38)

\[ \psi^3(\xi) = \frac{c_3 \exp[3c + \cdots] \exp(p \xi)}{c_4 \exp[3p_2] + \cdots} = \frac{c_3 \exp(3c + p)}{c_4 \exp[4p_2] + \cdots}, \]

(5.39)

where \( c_i \) are determined coefficients only for simplicity. Balancing the highest order of Exp-Function in Eqs. (5.38) and (5.39) gives

\[ 3p + c = p + 3c. \]  

(5.40)

So

\[ p = c. \]  

(5.41)

Similarly, to determine values of \( d \) and \( q \), the linear term of lowest order in Eq. (4.1) is balanced:

\[ \psi''(\xi) = \frac{\cdots + d_1 \exp[-(3q + d)]}{\cdots + d_2 \exp[-4q \xi]}, \]

(5.42)

And

\[ \psi^3 = \frac{\cdots + d_3 \exp[-(q + 3d)]}{\cdots + d_4 \exp[-4d \xi]}, \]

(5.43)

where \( d_i \) are determined coefficients only for simplicity. Balancing the lowest order of Exp-Function in Eqs. (5.42) and (5.43), we have

\[ -(3q + d) = -(q + 3d). \]  

(5.44)

So

\[ q = d. \]  

(5.45)

5.3.1. Case 1: \( p = c = 1, d = q = 1 \)

According to case 1, Eq. (4.1) reduces to

\[ \psi(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \]

(5.46)

Substituting Eq. (5.46) into Eq. (5.4) and using Maple, we have

\[ \frac{1}{A} \sum_{j=-3}^{3} C_j e^{i\xi} = 0, \]

(5.47)

where

\[ A = (b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi))^3 \]

(5.48)

and \( C_n \) are coefficients of \( \exp(n \xi) \). The coefficients of \( \exp(n \xi) \) must be zero, therefore we have

\[ C_{-3} = -C_b^2 + 2ax/a_{-1}b_2^2 - 3cA^2\beta^2a_{-1}b_{-1} + da_{-1} - va_{-1}b_{-1} = 0, \]

\[ C_{-2} = 2ax/a_1b_2^2 + cA^2a_0b_2 - 2va_1b_0b_{-1} - 3cA^2\beta^2a_0b_{-1} + 4ax/a_{-1}b_0b_{-1} \]

\[ -cA^2a_{-1}b_{-1} - va_0b_2^2 - 3da_{-1}a_0 - 3Cb_0b^2 - 6cA^2\beta^2a_{-1}b_0b_{-1} = 0, \]

\[ C_{-1} = 4cA^2\beta^2a_1b_1 - 4a_1b_0b_{-1} - va_{-1}b_2^2 - 2va_1b_1b_{-1} - cA^2a_{-1}b_1 - 2a_1b_0b_{-1} + 2ax/a_{-1}b_0b_{-1} \]

\[ -6cA^2\beta^2a_{-1}b_1b_{-1} + 4ax/a_1b_2 - 2ax/a_{-1}b_2 - 2va_1b_0b_{-1} - 6cA^2\beta^2a_0b_{-1} \]

\[ + 4ax/a_0b_{-1} - 3Cb_0b_{-1} - 3Cb_1b^2 + 3da_{-1}a_0^2 + 4cA^2a_1b_{-1}^2 - va_{-1}b_0^2 \]

\[ -3cA^2\beta^2a_{-1}b_0^2 - 3cA^2\beta^2a_{-1}b_{-1}^2 + 3da_{-1}a_0^2 = 0, \]
Substituting Eq. (5.53) into (5.46) gives the following solution:

\[ a_1 = \frac{b_1}{a_1}, \quad a_0 = 0, \quad a_{-1} = \frac{b_0(a_1 b_0 + a_0 b_1)}{8a_1 b_1}, \quad b_1 = b_1, \quad b_0 = b_0, \quad b_{-1} = \frac{b_0(a_1 b_0 + a_0 b_1)}{8a_1 b_1}, \quad \alpha = \alpha, \quad \beta = \beta, \]

\[ c = \frac{b_1 C(-a_0 b_0 - a_0 b_1) 2\alpha^2 a_1 (a_1 b_0 + a_0 b_1) + d \cdot -b_1^2 C b_0}{a_1^2 (a_1 b_0 + a_0 b_1)}, \]

\[ v = \frac{2\alpha x a_0 b_0 + 2\alpha x a_0 b_0 + 2b_1^2 C C^2 b_0 - 2b_1 C a_1 b_0 - 2C b_1 a_1 - a_0 b_1 C}{a_1 (a_1 b_0 + a_0 b_1)} \]

\[ (5.50) \]

where \( a_0, a_1, b_0, \) and \( b_1 \) are arbitrary constant parameters which are determined according to the boundary/initial conditions. Substituting these results into (5.46), we obtain the following generalized solitary solutions of Eq. (5.4): \n
\[ \psi(\xi) = \frac{a_1 \exp(\xi) a_0 + \frac{b_0(a_1 b_0 + a_0 b_1)}{8a_1 b_1 \exp(\xi)}}{b_1 \exp(\xi) + b_0 + \frac{b_0(a_1 b_0 + a_0 b_1)}{8a_1 b_1 \exp(\xi)}}, \]

\[ (5.51) \]

or

\[ u(\xi) = \frac{a_1 \exp(\xi) a_0 + \frac{b_0(a_1 b_0 + a_0 b_1)}{8a_1 b_1 \exp(\xi)}}{b_1 \exp(\xi) + b_0 + \frac{b_0(a_1 b_0 + a_0 b_1)}{8a_1 b_1 \exp(\xi)}} \times e^{i(\beta \xi - \omega t + \phi)}, \]

\[ (5.52) \]

where

\[ \xi = \alpha \left( x - \frac{2\alpha x a_0 b_0 + 2\alpha x a_0 b_0 + 2b_1^2 C C^2 b_0 - 2b_1 C a_1 b_0 - 2C b_1 a_1 - a_0 b_1 C}{a_1 (a_1 b_0 + a_0 b_1)} \right). \]

\[ (5.53) \]

Substituting Eq. (5.53) into (5.46) gives the following solution:

\[ \psi(\xi) = \frac{a_1 b_1 e^{i} + a_{-1} b_1 (\xi + 8b_0 b_1)}{b_1 e^{i} + b_0 + b_{-1} e^{i}}, \]

\[ (5.54) \]

or

\[ u(\xi) = \frac{a_1 b_1 e^{i} + a_{-1} b_1 (\xi + 8b_0 b_1)}{b_1 e^{i} + b_0 + b_{-1} e^{i}} \times e^{i(\beta \xi - \omega t + \phi)}, \]

\[ (5.55) \]

where

\[ \xi = \alpha \left( x - \frac{b_0 \xi C - 8b_1 b_{-1} C - 6C b_0^2 + 24C \beta^2 b_1 b_{-1} + 16\alpha x a_1 b_1}{8a_{-1} b_1} \right). \]

These are exact solutions of generalized nonlinear Schrödinger (GNLS) equation.
5.3.2. Case 2: \( p = c = 2, \ d = q = 2 \)

As mentioned above, the values of \( c \) and \( d \) can be freely chosen, we set \( p = c = 2 \) and \( d = q = 2 \), then the trial function, Eq. (4.1), is

\[
\psi(\zeta) = a_2 \exp(2\zeta) + a_1 \exp(\zeta) + a_0 + a_{-1} \exp(-\zeta) + a_{-2} \exp(-2\zeta)
\]

\[
b_2 \exp(2\zeta) + b_1 \exp(\zeta) + b_0 + b_{-1} \exp(-\zeta) + b_{-2} \exp(-2\zeta)
\]

By the same operation illustrated above, we obtain:

5.3.2.1. Case 2.1.

\[
a_2 = -\frac{C b_0^2}{8a^2}, \quad a_1 = 0, \quad a_0 = -\frac{b_0(a^2 + C)}{a^2}, \quad a_{-1} = 0, \quad a_{-2} = a_2, \quad b_2 = -\frac{C b_0^2}{8a^2}, \quad b_1 = 0, \quad b_0 = b_0, \quad b_{-1} = 0.
\]

\[
b_{-2} = 1, \quad a = a, \quad c = -\frac{2a^2 + C}{4a^2}, \quad d = d, \quad \alpha = \alpha, \quad \beta = \beta, \quad v = \frac{-4c + 4a^2 + 6b^2a^2 + 3b^2C + 8ax/a_2}{4a_2}.
\]

Substituting Eq. (5.57) into (5.56) gives the following solution:

\[
\psi(\zeta) = \frac{-C b_0^2}{8a^2}e^{2\zeta} - \frac{b_0(a^2 + C)}{a^2}e^{\zeta} + a_2 e^{-2\zeta},
\]

or,

\[
u(\zeta) = \frac{-C b_0^2}{8a^2}e^{2\zeta} + b_0 + e^{-2\zeta} \times e^{i(\zeta^2 - \omega t + x_0)}
\]

where

\[
\zeta = \alpha \left( x - \frac{-4c + 4a^2 + 6b^2a^2 + 3b^2C + 8ax/a_2}{4a_2}t \right)
\]

5.3.2.2. Case 2.2.

\[
a_2 = \frac{b_0(a_0b_{-2} + a_2b_0)}{8b^2}, \quad a_1 = 0, \quad a_0 = a_0, \quad a_{-1} = 0, \quad a_{-2} = a_2, \quad b_2 = \frac{b_0(a_0b_{-2} + a_2b_0)}{8a_2b_{-2}}, \quad b_1 = 0, \quad b_0 = b_0,
\]

\[
b_{-1} = 0, \quad b_{-2} = b_{-2}, \quad a = a, \quad \alpha = \alpha, \quad \beta = \beta, \quad c = -\frac{C b_0^2(a_0b_{-2} - a_2b_0)}{4a_2b^2(a_0b_{-2} + a_2b_0)}, \quad d = -\frac{b_0}{a^2}(a_0b_{-2} + a_2b_0),
\]

\[
v = \frac{-4a_0b^2C - 8Cb_0b_{-2}a_2 + 3Cb_2b^2a_0 - 3Cb_2b^2a_{-2}b_0 + 8ax/a_0b_{-2}a_2 + 8ax/a_2b_0}{4a_2(a_0b_{-2} + a_2b_0)}.
\]

Substituting Eq. (5.60) into (5.56) gives the following solution:

\[
\psi(\zeta) = \frac{b_0(a_0b_{-2} + a_2b_0)}{8b^2}e^{2\zeta} + a_0 + a_2 e^{-2\zeta},
\]

or,

\[
u(\zeta) = \frac{b_0(a_0b_{-2} + a_2b_0)}{8b^2}e^{2\zeta} + a_0 + a_2 e^{-2\zeta} \times e^{i(\zeta^2 - \omega t + x_0)}
\]

where

\[
\zeta = \alpha \left( x - \frac{-4a_0b^2C - 8Cb_0b_{-2}a_2 + 3Cb_2b^2a_0 - 3Cb_2b^2a_{-2}b_0 + 8ax/a_0b_{-2}a_2 + 8ax/a_2b_0}{4a_2(a_0b_{-2} + a_2b_0)}t \right)
\]

5.3.2.3. Case 2.3

\[
a_2 = a_2, \quad a_1 = 0, \quad a_0 = -\frac{b_0(a^2 + C)}{a^2}, \quad a_{-1} = 0, \quad a_{-2} = \frac{C b_0^2}{8a^2}, \quad b_2 = 1, \quad b_1 = 0, \quad b_0 = b_0, \quad b_{-1} = 0,
\]

\[
b_{-2} = \frac{C b_0^2}{8a^2}, \quad a = a, \quad \alpha = \alpha, \quad \beta = \beta, \quad c = -\frac{2a^2 + C}{4a^2}, \quad d = d, \quad v = \frac{8ax/a_2 - 4c + 4a^2 + 6b^2a^2 + 3b^2C}{4a_2}.
\]

5.3.2.4. Case 2.4

\[
a_2 = a_2, \quad a_1 = 0, \quad a_0 = -\frac{b_0(a^2 + C)}{a^2}, \quad a_{-1} = 0, \quad a_{-2} = \frac{C b_0^2}{8a^2}, \quad b_2 = 1, \quad b_1 = 0, \quad b_0 = b_0, \quad b_{-1} = 0,
\]

\[
b_{-2} = \frac{C b_0^2}{8a^2}, \quad a = a, \quad \alpha = \alpha, \quad \beta = \beta, \quad c = -\frac{2a^2 + C}{4a^2}, \quad d = d, \quad v = \frac{8ax/a_2 - 4c + 4a^2 + 6b^2a^2 + 3b^2C}{4a_2}.
\]
Substituting Eq. (5.63) into (5.56) gives the following solution:

$$
\psi(\xi) = \frac{a_2 e^{-2 \xi} - \frac{b_0 (a_1^2 + \bar{C})}{a_1^2}}{e^{2 \xi} + b_0 - \frac{C b_0}{8a_2^2} e^{-2 \xi}},
$$

or,

$$
u(\xi) = \frac{a_2 e^{-2 \xi} - \frac{b_0 (a_1^2 + \bar{C})}{a_1^2}}{e^{2 \xi} + b_0 - \frac{C b_0}{8a_2^2} e^{-2 \xi}} \times e^{(\bar{C}-\omega t+x_0)},
$$

where

$$
\xi = \alpha \left( x - 8a_2 \beta a_1 - 4a_1^2 + 6\beta^2 a_1^2 + 3\beta^2 C \right) t.
$$

5.3.2.4. Case 2.4

$$
a_2 = \frac{b_1(a_1b_0 + a_0b_1)}{8b_0^2}, \quad a_1 = a_1, \quad a_0 = a_0, \quad a_1 = 0, \quad a_2 = 0, \quad b_2 = \frac{b_1(a_1b_0 + a_0b_1)}{8a_0b_0}, \quad b_1 = b_1, \quad b_0 = b_0,
$$

$$
b_1 = 0, \quad b_2 = 0, \quad a = a, \quad \alpha = \alpha, \quad \beta = \beta, \quad c = -\frac{C b_0(a_1b_0 - a_0b_1)}{2a_0(a_1b_0 + a_0b_1)}, \quad d = -\frac{b_1^2 C b_1}{a_0^2(a_1b_0 + a_0b_1)},
$$

$$
\nu = -\frac{a_1 b_1^2 C}{2C b_1 b_0 a_0} + 3C b_0^2 \beta^2 a_1 - 3C b_0^2 \beta^2 b_1 + 2a\alpha^2 a_0 a_1 b_0 + 2a\alpha^2 a_0^2 b_1}{a_0(a_1b_0 + a_0b_1)}.
$$

Substituting Eq. (5.66) into Eq. (5.56) gives the following solution:

$$
\psi(\xi) = \frac{b_1(a_1b_0 + a_0b_1)}{8b_0^2} e^{2 \xi} + a_1 e^\xi + a_0
$$

or,

$$
u(\xi) = \frac{b_1(a_1b_0 + a_0b_1)}{8b_0^2} e^{2 \xi} + b_1 e^\xi + a_0 \times e^{(\bar{C}-\omega t+x_0)},
$$

where

$$
\xi = \alpha \left( x - \frac{a_1 b_1^2 C}{2C b_1 b_0 a_0} + 3C b_0^2 \beta^2 a_1 - 3C b_0^2 \beta^2 b_1 + 2a\alpha^2 a_0 a_1 b_0 + 2a\alpha^2 a_0^2 b_1}{a_0(a_1b_0 + a_0b_1)} t.
$$

5.3.2.5. Case 2.5

$$
a_2 = a_2, \quad a_1 = a_1, \quad a_0 = 0, \quad a_1 = \frac{3a_1^2}{8a_2^2}, \quad a_2 = \frac{9a_1^4}{64a_2^2}, \quad b_2 = -\frac{2b_0 a_1^2}{3a_1^2}, \quad b_1 = 0, \quad b_0 = b_0, \quad b_1 = 0,
$$

$$
b_2 = -\frac{3b_0 a_1^2}{32a_2^2}, \quad \alpha = \alpha, \quad \beta = \beta, \quad a = a, \quad d = \frac{2C b_0 a_1^2}{9a_1^2}, \quad c = \frac{C b_0 a_2}{3a_1^2 \alpha^2}, \quad v = \frac{7C b_0 a_2 + 12a\alpha^2 a_0^2 + 6C b_0 a_2}{6a_1^2}.
$$

Substituting Eq. (5.69) into (5.56) gives the following solution:

$$
\psi(\xi) = \frac{a_2 e^{2 \xi} + a_1 e^\xi - \frac{3a_1^2}{8a_2^2} e^{-\xi} + \frac{9a_1^4}{64a_2^2} e^{-2 \xi}}{-\frac{2b_0 a_1^2}{3a_1^2} e^{2 \xi} + b_0 - \frac{3b_0 a_1^2}{32a_2^2} e^{-2 \xi}},
$$

or,

$$
u(\xi) = \frac{a_2 e^{2 \xi} + a_1 e^\xi - \frac{3a_1^2}{8a_2^2} e^{-\xi} + \frac{9a_1^4}{64a_2^2} e^{-2 \xi}}{-\frac{2b_0 a_1^2}{3a_1^2} e^{2 \xi} + b_0 - \frac{3b_0 a_1^2}{32a_2^2} e^{-2 \xi}} \times e^{(\bar{C}-\omega t+x_0)},
$$
where
\[
\zeta = \left( x - \frac{7C_b a_2 + 12ax\beta a_1^2 + 6Cb_0a_2b^2}{6a_1^2} \right).
\]

5.3.2.6. Case 2.6.
\[
a_2 = a_2, \quad a_1 = a_1, \quad a_0 = \frac{(4da_3^3 + 3Cb_b)da_3a_1^2}{4(2da_2^3 +Cb_b)^2}, \quad a_{-1} = -\frac{Ca_2b_2a_1d}{8(2da_2^3 +Cb_b)^2}, \quad a_{-2} = \frac{a_2^2C^2b_0^3a_1^4a_2^2}{64(2da_2^3 +Cb_b)^4}, \quad b_2 = b_2.
\]
\[
b_0 = \frac{-da_2a_1b_2(4da_3^3 +Cb_b)}{4(2da_2^3 +Cb_b)^2}, \quad b_{-1} = 0, \quad b_{-2} = \frac{C^2b_0^3a_1^4a_2^2}{64(2da_2^3 +Cb_b)^4}, \quad \alpha = \alpha, \quad \beta = \beta, \quad a = a.
\]
\[
c = -\frac{2da_3^3 +Cb_b^3}{x^2a_2b_2^2}, \quad d = d, \quad \nu = \frac{I}{x^2a_2b_2^2} \frac{da_3^3 -Cb_b^3 + 2ax\beta a_2b_2^3 + 6\beta^2da_3^3 + 3\beta^2Cb_b^3}{b_2a_2}.
\]

Substituting Eq. (5.72) into (5.56) gives the following solution:
\[
\psi(\zeta) = \left( a_2e^{2\zeta} + a_1e^{\zeta} + \frac{(4da_3^3 + 3Cb_b)da_3a_1^2}{4(2da_2^3 +Cb_b)^2} \frac{Ca_2b_2a_1^4d}{8(2da_2^3 +Cb_b)^2} e^{-\zeta} + \frac{a_2^2C^2b_0^3a_1^4a_2^2}{64(2da_2^3 +Cb_b)^4} e^{-2\zeta} \right)
\]
\[
\times \left( b_2e^{2\zeta} - \frac{-da_2a_1b_2(4da_3^3 +Cb_b)}{4(2da_2^3 +Cb_b)^2} + \frac{C^2b_0^3a_1^4a_2^2}{64(2da_2^3 +Cb_b)^4} e^{-2\zeta} \right),
\]
(5.73)

or,
\[
\psi(\zeta) = \left[ \left( a_2e^{2\zeta} + a_1e^{\zeta} + \frac{(4da_3^3 + 3Cb_b)da_3a_1^2}{4(2da_2^3 +Cb_b)^2} \frac{Ca_2b_2a_1^4d}{8(2da_2^3 +Cb_b)^2} e^{-\zeta} + \frac{a_2^2C^2b_0^3a_1^4a_2^2}{64(2da_2^3 +Cb_b)^4} e^{-2\zeta} \right)
\]
\[
\times \left( b_2e^{2\zeta} - \frac{-da_2a_1b_2(4da_3^3 +Cb_b)}{4(2da_2^3 +Cb_b)^2} + \frac{C^2b_0^3a_1^4a_2^2}{64(2da_2^3 +Cb_b)^4} e^{-2\zeta} \right) \right] \times e^{i(\beta t - \omega t)}.
\]
(5.74)

where
\[
\zeta = \left( x - \frac{da_3^3 -Cb_b^3 + 2ax\beta a_2b_2^3 + 6\beta^2da_3^3 + 3\beta^2Cb_b^3}{b_2a_2} \right).
\]

These are exact solutions of generalized nonlinear Schrödinger (GNLS) equation.

5.3.3. Case 3: \( p = c = 3, \ d = q = 1 \)

As mentioned above, the values of \( c \) and \( d \) can be selected without limits. Setting \( p = c = 3 \) and \( d = q = 1 \), then the trial function, Eq. (4.1), changes to
\[
\psi(\zeta) = \frac{a_3e^{3\zeta} + a_2e^{2\zeta} + a_1e^{\zeta} + a_0 + a_{-1}e^{(-\zeta)}}{b_3e^{3\zeta} + b_2e^{2\zeta} + b_1e^{\zeta} + b_{-1}e^{(-\zeta)}}
\]
(5.75)

By the same approach as demonstrated above, we obtain:

5.3.3.1. Case 3.1.
\[
a_3 = \frac{b_2(a_2b_1 + a_1b_2)}{8b_1^2}, \quad a_2 = a_2, \quad a_1 = a_1, \quad a_0 = 0, \quad a_{-1} = 0, \quad b_1 = \frac{b_2(a_2b_1 + a_1b_2)}{8a_1b_1}, \quad b_2 = b_2, \quad b_1 = b_1,
\]
\[
b_0 = 0, \quad b_{-1} = 0, \quad a = a, \quad c = \frac{Cb_1(a_2b_1 - a_1b_2)}{a_2b_2(a_2b_1 + a_1b_2)}, \quad d = -\frac{b_1}{a_2b_1(a_2b_1 + a_1b_2)}, \quad \alpha = \alpha, \quad \beta = \beta.
\]
\[
v = \frac{3Cb_1^2a_2 - 3Cb_1\beta^2a_1b_2 - a_2b_1^2C - 2Cb_1b_3a_1 + 2ax\beta a_2b_1 + 2ax\beta a_2b_1}{a_1(a_2b_1 + a_1b_2)}.
\]
(5.76)

Substituting Eq. (5.76) into Eq. (5.75) gives the following solution:
\[
\psi(\zeta) = \frac{b_2(a_2b_1 + a_1b_2)}{8b_1^2} e^{3\zeta} + a_2e^{2\zeta} + a_1e^{\zeta}
\]
(5.77)

or,
\[
\psi(\zeta) = \frac{b_2(a_2b_1 + a_1b_2)}{8b_1^2} e^{3\zeta} + a_2e^{2\zeta} + a_1e^{\zeta} e^{i(\beta t - \omega t)}
\]
(5.78)
where
\[
\xi = \alpha \left( x - \frac{3Cb_1 \beta^2 a_2 - 3Cb_1 \beta^2 a_1 b_2 - a_2 b_1^2 C - 2Cb_1 a_1 + 2a_x \beta a_1 a_2 b_1 + 2a_x \beta a_1 b_2 t}{a_1(a_2 b_1 + a_1 b_2)} \right).
\]

5.3.3.2 Case 3.2

\[
a_3 = \frac{b_1(b_1 a_1 + b_1 a_1)}{8b_3}, \quad a_2 = 0, \quad a_1 = a_1, \quad a_0 = 0, \quad a_1 = a_1, \quad b_1 = \frac{b_1(b_1 a_1 + b_1 a_1)}{8a_3}, \quad b_2 = 0, \quad b_1 = b_1,
\]

\[
b_0 = 0, \quad b_1 = b_1, \quad a = a, \quad c = \frac{Cb_1(b_1 a_1 - b_1 a_1)}{4 \beta^2 a_1(b_1 a_1 + b_1 a_1)}, \quad d = -\frac{b_1^2 Cb_1}{a_1^2(b_1 a_1 + b_1 a_1)}, \quad \alpha = \alpha, \quad \beta = \beta,
\]

\[
v = \frac{-4a_1 Cb_1^2 - 8Ca_1 b_1 a_1 + 3C_1 \beta^2 a_1 - 3Cb_1 \beta^2 a_1 a_1 + 8a_x \beta a_1 a_1 a_1 + 8a_x \beta a_1 b_1 a_1}{4a_1(b_1 a_1 + b_1 a_1)}.
\]

Substituting Eq. (5.79) into (5.75) gives the following solution:

\[
\psi(\xi) = \frac{b_1(b_1 a_1 + b_1 a_1)}{8b_3} e^{\beta \xi} + a_1 e^{\beta \xi} + a_1 e^{-\beta \xi},
\]

or,

\[
u(\xi) = \frac{b_1(b_1 a_1 + b_1 a_1)}{8a_3} e^{\beta \xi} + b_1 e^{\beta \xi} + b_1 e^{-\beta \xi} \times e^{(\beta - \omega \xi)}
\]

where

\[
\xi = \alpha \left( x - \frac{-4a_1 Cb_1^2 - 8Ca_1 b_1 a_1 + 3C_1 \beta^2 a_1 - 3Cb_1 \beta^2 a_1 a_1 + 8a_x \beta a_1 a_1 a_1 + 8a_x \beta a_1 b_1 a_1}{4a_1(b_1 a_1 + b_1 a_1)} \right)
\]

Cases 3.1 and 3.2 are the same as case 2.4 and case 2.2, respectively. Therefore, we conclude that Eq. (5.75) is equal to Eq. (5.56).

5.3.4. Case 4: \( p = c = 3, d = q = 3 \)

In this case we consider \( p = c = 3 \) and \( d = q = 3 \). Then the trial function, Eq. (4.1), is presented as:

\[
\psi(\xi) = \frac{a_1 e^{3\xi} + a_1 e^{2\xi} + a_1 e^{\xi} + a_1 e^{-\xi} + a_1 e^{-2\xi} + a_1 e^{-3\xi}}{b_1 e^{3\xi} + b_1 e^{2\xi} + b_1 e^{\xi} + b_1 e^{-\xi} + b_1 e^{-2\xi} + b_1 e^{-3\xi}}.
\]

By the same approach illustrated above, we obtain:

5.3.4.1 Case 4.1

\[
a_3 = a_3, \quad a_2 = 0, \quad a_1 = 0, \quad a_0 = 0, \quad a_1 = a_1, \quad a_2 = 0, \quad a_3 = \frac{b_0(a_1 b_0 + a_0 b_1)}{8b_3}, \quad b_1 = \frac{b_0(a_1 b_0 + a_0 b_1)}{8b_3}, \quad b_2 = 0, \quad b_1 = 0,
\]

\[
b_0 = b_0, \quad b_1 = b_0, \quad b_2 = 0, \quad b_3 = \frac{b_0(a_1 b_0 + a_0 b_0)}{8b_3}, \quad a = a, \quad c = \frac{Cb_1(a_1 b_0 + a_0 b_0)}{9a_3}, \quad d = -\frac{b_1^2 b_0 C}{a_1(a_1 b_0 + a_0 b_1)}.
\]

\[
\alpha = \alpha, \quad \beta = \beta,
\]

\[
v = \frac{-6Cb_1 a_1 b_3 + 3b_1^2 a_3 C - b_1^2 \beta^2 a_0 + b_1 C \beta^2 a_3 b_0 - 6a_x \beta b_0 a_1^2 - 6a_x \beta b_3 a_1 a_3}{3a_3(a_1 b_0 + a_0 b_3)}.
\]

Substituting Eq. (5.83) into (5.82) gives the following solution:

\[
\psi(\xi) = \frac{a_1 e^{3\xi} + a_0 + \frac{b_0(a_1 b_0 + a_0 b_1)}{8b_3} e^{-3\xi}}{b_1 e^{3\xi} + b_0 + \frac{b_0(a_1 b_0 + a_0 b_1)}{8b_3} e^{-3\xi}},
\]

or,

\[
\psi(\xi) = \frac{a_1 e^{3\xi} + a_0 + \frac{b_0(a_1 b_0 + a_0 b_1)}{8b_3} e^{-3\xi}}{b_1 e^{3\xi} + b_0 + \frac{b_0(a_1 b_0 + a_0 b_1)}{8b_3} e^{-3\xi}} \times e^{(\beta - \omega \xi)}.
\]

where

\[
\xi = \alpha \left( x - \frac{-6Cb_1 a_1 b_3 + 3b_1^2 a_3 C - b_1^2 \beta^2 a_0 + b_1 C \beta^2 a_3 b_0 - 6a_x \beta b_0 a_1^2 - 6a_x \beta b_3 a_1 a_3}{3a_3(a_1 b_0 + a_0 b_3)} \right).
\]
5.3.4.2. Case 4.2

\[ a_3 = -\frac{a_1^2}{216a_3}, \quad a_2 = 0, \quad a_1 = -\frac{a_1^2}{6a_3}, \quad a_0 = 0, \quad a_{-1} = a_{-1}, \quad a_{-2} = 0, \quad a_{-3} = a_{-3}, \quad b_2 = \frac{2c_2^2a_1^3}{81a_3^2c}, \quad b_2 = 0, \]

\[ b_1 = 0, \quad b_0 = 0, \quad b_{-1} = 0, \quad b_{-2} = 0, \quad b_{-3} = -\frac{16c_2^2a_3}{3c}, \quad a = a, \quad c = c, \quad d = \frac{512c_2^5}{27c^2}, \quad \alpha = \alpha, \quad \beta = \beta, \]

\[ v = -\alpha(-2a\beta - 6c\alpha + 3c\beta^2). \quad (5.86) \]

Substituting Eq. (5.86) into (5.82) gives the following solution:

\[ \psi(\xi) = -\frac{\alpha^3}{216\alpha^3}a_1^3 - \frac{\alpha^3}{6a_3}e^\xi + a_{-1}e^{-\xi} + a_3e^{-3\xi}; \]

\[ \frac{2c_2^2a_1^3}{81a_3^2c}e^{3\xi} - \frac{16c_2^2a_3}{3c}e^{-3\xi}; \]

\[ \frac{2c_2^2a_1^3}{81a_3^2c}e^{3\xi} - \frac{16c_2^2a_3}{3c}e^{-3\xi}. \quad (5.87) \]

or,

\[ U(\xi) = -\frac{\alpha^3}{216\alpha^3}a_1^3 - \frac{\alpha^3}{6a_3}e^\xi + a_{-1}e^{-\xi} + a_3e^{-3\xi}; \]

\[ \frac{2c_2^2a_1^3}{81a_3^2c}e^{3\xi} - \frac{16c_2^2a_3}{3c}e^{-3\xi} \times e^{(j(\xi - \omega t))x}. \quad (5.88) \]

where

\[ \xi = \alpha(x - \alpha(-2a\beta - 6c\alpha + 3c\beta^2)t). \]

5.3.4.3. Case 4.3

\[ a_1 = \frac{b_0a_1^3c(c^2x^4a_0^3 + 2Cc_2^2b_0a_0 + C^2b_0^2)}{32c^3x^6a_0^6}, \quad a_2 = -\frac{a_1^2(-4c_2^2x^4a_0^3 - Cc_2^2b_0a_0 + 3C^2b_0^2)}{16c_2^2x^4a_0^3}, \quad a_1 = a_1, \quad a_0 = a_0, \quad a_{-1} = 0, \]

\[ a_{-2} = 0, \quad a_{-3} = 0, \quad b_1 = \frac{b_0a_1^3c(c^2x^4a_0^3 + 2Cc_2^2b_0a_0 + C^2b_0^2)}{32c^3x^6a_0^6}, \quad b_2 = -\frac{b_0a_1^3(4c_2^2x^4a_0^3 + 7Cc_2^2b_0a_0 + 3C^2b_0^2)}{16c_2^2x^4a_0^3}, \]

\[ b_1 = 1, \quad b_0 = b_0, \quad b_{-1} = 0, \quad b_{-2} = 0, \quad b_{-3} = 0, \quad a = a, \quad c = c, \quad d = \frac{b_0^2(Cb_0 + cx^2a_0)}{2a_0}, \quad \alpha = \alpha, \quad \beta = \beta, \]

\[ v = -\frac{cx^2a_0 + 6cx^2\beta^2a_0 - 4ax\beta a_0 + 3Cb_0}{2a_0}. \quad (5.89) \]

Substituting Eq. (5.89) into (5.82) gives the following solution:

\[ \psi(\xi) = \left( \frac{b_0a_1^3c(c^2x^4a_0^3 + 2Cc_2^2b_0a_0 + C^2b_0^2)}{32c^3x^6a_0^6} e^{3\xi} - \frac{a_1^2(-4c_2^2x^4a_0^3 - Cc_2^2b_0a_0 + 3C^2b_0^2)}{16c_2^2x^4a_0^3} e^{2\xi} + a_1e^{\xi} + a_0 \right) \]

\[ \times \left( \frac{b_0a_1^3c(c^2x^4a_0^3 + 2Cc_2^2b_0a_0 + C^2b_0^2)}{32c^3x^6a_0^6} e^{3\xi} - \frac{b_0a_1^3(4c_2^2x^4a_0^3 + 7Cc_2^2b_0a_0 + 3C^2b_0^2)}{16c_2^2x^4a_0^3} e^{2\xi} + b_0 \right). \quad (5.90) \]

or,

\[ U(\xi) = \left( \frac{b_0a_1^3c(c^2x^4a_0^3 + 2Cc_2^2b_0a_0 + C^2b_0^2)}{32c^3x^6a_0^6} e^{3\xi} - \frac{a_1^2(-4c_2^2x^4a_0^3 - Cc_2^2b_0a_0 + 3C^2b_0^2)}{16c_2^2x^4a_0^3} e^{2\xi} + a_1e^{\xi} + a_0 \right) \times e^{(j(\xi - \omega t))x} \]

\[ \times \left( \frac{b_0a_1^3c(c^2x^4a_0^3 + 2Cc_2^2b_0a_0 + C^2b_0^2)}{32c^3x^6a_0^6} e^{3\xi} - \frac{b_0a_1^3(4c_2^2x^4a_0^3 + 7Cc_2^2b_0a_0 + 3C^2b_0^2)}{16c_2^2x^4a_0^3} e^{2\xi} + b_0 \right). \quad (5.91) \]

where

\[ \xi = \alpha \left( x - \frac{cx^2a_0 + 6cx^2\beta^2a_0 - 4ax\beta a_0 + 3Cb_0}{2a_0} t \right). \]

These are other exact solutions of generalized nonlinear Schrödinger (GNLS) equation.

5.3.5. Case 5: \( p = c = 4, \quad d = q = 2 \)

Finally, we consider \( p = c = 4 \) and \( d = q = 2 \) then the trial function, Eq. (4.1), changes to

\[ \psi(\xi) = \frac{a_4e^{4(\xi)} + a_3e^{3(\xi)} + a_2e^{2(\xi)} + a_1e^{(\xi)} + a_0 + a_{-1}e^{(-1(\xi)} + a_{-2}e^{(-2(\xi)}}}{b_4e^{4(\xi)} + b_3e^{3(\xi)} + b_2e^{2(\xi)} + b_1e^{(\xi)} + b_0 + b_{-1}e^{(-1(\xi)} + b_{-2}e^{(-2(\xi)}}. \quad (5.92) \]
We rewrite Eq. (5.92) in the following form:

$$
\psi(\xi) = \frac{a_1e^{2(\xi)} + a_2e^{\xi} + a_3 + a_4e^{-\xi} + a_5e^{-2(\xi)}}{b_1e^{\xi} + b_2e^{2(\xi)} + b_3e^{3(\xi)} + b_4e^{4(\xi)} + b_5e^{-\xi} + b_6e^{-2(\xi)} + b_7e^{-3(\xi)}}
$$

(5.93)

which has the same form as Eq. (5.82). Therefore, Case 5 is equivalent to Case 4.

6. Conclusion

In this study, Extended tanh-function, sine–cosine and Exp-Function methods were applied in order to find the exact solutions of the generalized nonlinear Schrödinger (GNLS) equation with a source. The first two procedures can be considered as simple methods for solving some of nonlinear partial differential equations without resort to any symbolic computation and the obtained results are very concise. On the other hand, it can be concluded that, in comparison with the other methods, the Exp-Function method is a very powerful and effective technique in finding the exact solutions for a wide range of problems. All the presented cases in this paper have shown that the results of the last method are not strongly dependent on the chosen values of $c$ and $d$.

The next attempts of authors are to focus on solution and physical description of PDEs with various initial and boundary conditions using advanced and exact procedures like Extended tanh–coth, Sine–cosine and Exp-Function methods and eventually to compare their results.

References


