Simulation-based discrete optimization of stochastic discrete event systems subject to non closed form constraints

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Abstract—This paper addresses the discrete optimization of stochastic discrete event systems for which both the performance function and the constraint function are non known but can be evaluated by simulation and the solution space is either finite or unbounded. Our method is based on random search in a neighborhood structure called the most promising area proposed in [7] and a moving observation area. The simulation budget is allocated dynamically to promising solutions. Simulation-based constraints are taken into account in an augmented performance function via an increasing penalty factor. We prove that under some assumptions, the algorithm converges with probability 1 to a set of true local optimal solutions. These assumptions are restrictive and difficult to verify but we hope that the encouraging numerical results would motivate future research exploiting ideas of this paper.

Index Terms—Simulation based optimization, discrete event systems, most promising area, observation area, non closed-form constraints

I. INTRODUCTION

Discrete optimization frequently arises in the design and operation of real-life stochastic discrete event systems. These problems are often difficult due to the complexity of real-life discrete event systems, their intricate dynamics and omnipresence of random disturbances. For some discrete event systems, analytical solutions can be derived under some simplification and often restrictive assumptions and, in this case, the related discrete optimization problem becomes a combinatorial optimization problem that can be solved with usual discrete optimization techniques. Unfortunately, most real-life stochastic discrete event systems do not have closed-form solutions and discrete event simulation is usually the only resort for performance evaluation.

Recently, Discrete Optimization via Simulation (DOvS) techniques have been developed to solve stochastic optimization problems for which the optimality is measured based on simulation results and the decision variables are discrete (see [3, 13, 14] for surveys). Discrete optimization problems with small size solutions space can be efficiently solved using ranking and selection methods [4]. If discrete event simulation is taken as a black box for performance evaluation, some general meta-heuristics for deterministic combinatorial optimization such as tabu search and genetic algorithms can be applied directly. However such straightforward applications neglect major barriers of discrete optimization via simulation including the long computation time for simulation and the noise of performance evaluation using simulation. This has motivated development of simulation-based optimization methods for large or unbounded solution space that takes into account issues related to simulation budget and estimation errors. These methods include but are not limited to ordinal optimization [5, 6], simulated annealing [1], nested partitions [12] and random search in a most promising area [7, 8].

The results presented in this paper are closely related to the COMPASS method (Convergent Optimization via Most-Promising-Area Stochastic Search) proposed in [7] for optimization problems for which the performance measures are estimated via a stochastic discrete-event simulation and the decision variables are integer ordered. It is based on random search in a neighborhood structure called the most promising area. It was proved that COMPASS algorithm converges to a set of local optimal solutions with probability 1 under the assumption that unlimited number of simulation observations is performed for each visited solution. It was extended in [8], which sets up a general framework of locally convergent random-search algorithms that ensures the convergence of the algorithms under some conditions. Convergence rates and a central limit theorem were also established.

In this paper we consider discrete optimization problems for which both the performance function and some constraint functions need to be evaluated by simulation. Such constraints will be called non closed-form constraints. Both finite and unbounded solution space cases will be considered. Our optimization approach combines the most promising area concept of [7] with two new concepts: the moving observation area and an augmented performance function. A finite moving observation area around the current sample optimum is used to identify the exploration area. It allows us to concentrate the simulation budget on solutions near the current sample optimum and avoid spending simulation budget on visited but uninteresting solutions that are far away. We believe that this could help improve the convergence to local optimum solutions and quicker convergence than COMPASS algorithm.

Manuscript received September 18, 2007.
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in [7] was observed in the extended numerical experiments conducted in [10, 11]. Discrete optimization with non closed-form constraints poses additional technical barriers due the need for feasibility assessment. We introduce a penalty cost function to combine the simulation-based constraints into an augmented objective function. The optimization algorithms are proved to converge to true local optimum under some conditions.

Although most simulation-based optimization techniques can be extended to address problems considered in this paper, to the best of our knowledge, non closed-form constraints are hardly explicitly addressed in discrete simulation-based optimization methods. One exception is the constraint ordinal optimization approach proposed in [9] that replaces non closed-form constraints by some order constraints. As shown in this paper, non closed-form constraints pose additional difficulties in the investigation of the convergence properties of the simulation-based optimization algorithms. Our theoretical results are obtained under some assumptions that are restrictive and difficult to verify. However we hope that the encouraging numerical results given in [10, 11] would motivate future research exploiting ideas of this paper.

The remainder of the paper is organized as follows. Section 2 introduces DoVs problems with non closed-form constraints and presents the optimization algorithms. Section 3 establishes formally the convergence properties.

II. DISCRETE OPTIMIZATION WITH NON-CLOSED FORM CONSTRAINTS

A. Problem setting

The problems considered in this paper are discrete optimization problems of the following form:

$$\min_{x \in \Theta} E[G(x, w)]$$

subject to

$$E[F(x, w)] > \alpha$$

where $\Theta = \Phi \cap \mathbb{Z}^d$ is the solutions space, $\Phi$ is a closed set in $\mathbb{R}^d$, and $\mathbb{Z}^d$ is the set of d-dimensional integer vectors. $w$ represents all random uncontrollable factors. For the sake of simplicity, the constraint function is assumed to be a scalar function. Extension to multiple non-closed form constraints is immediate and all results of this paper hold for the general case.

We assume that closed-form expressions for both the cost function $E[G(x, w)]$ and the constraint function $E[F(x, w)]$ are not always available. However both $E[G(x, w)]$ and $E[F(x, w)]$ can be estimated by sample mean of repetitive simulation observations of $G(x, w)$ and $F(x, w)$. Let $g(x) = E[G(x, w)]$ and $f(x) = E[F(x, w)]$.

We call the problem defined above a Discrete Optimization via Simulation (DoVs) problem with non closed form constraints. The problem is said to be fully constrained if $\Phi$ is a compact set, i.e. $\Theta$ is a finite set. Otherwise, the problem is said to be partially constrained or unconstrained.

When the closed-form expression for $f(x)$ is available, an equivalent DoVs formulation without constraint (2) can be obtained by introducing the constraint (2), i.e. $f(x) > \alpha$, in the definition of $\Phi$. For this reason, the DoVs of the following form: $\min_{x \in \Theta} E[G(x, w)]$ is called DoVs with closed-form constraints. A general framework to solve this class of DoVs problems was proposed in [7, 8].

B. Fully constrained discrete optimization with non closed-form constraints

The COMPASS algorithm proposed in [7] is adapted to take into account non closed-form constraints. The new ingredient is the introduction of a penalty cost function to combine the simulation-based constraints into an augmented objective function. The COMPASS algorithm of [7] is summarized hereafter by taking into account the non closed-form constraints.

More precisely, the algorithm does not evaluate all solutions $x \in \Theta$. Solutions to consider are generated progressively. Let $V_i$ be the set of all solutions visited through iteration $k$. Further, the simulation budget, i.e. the number of simulation replications also called observations of each solution $x$, is allocated dynamically. Let $N_k(x)$ be the total number of observations allocated to solution $x$ up to iteration $k$. Let $\overline{x}_k(x)$ and $\overline{F}_k(x)$ denote the sample mean of all observations $G_i(x, w)$ and $F_i(x, w)$, $i = 1, \ldots, k$ of $G(x, w)$ and $F(x, w)$ until iteration $k$ (i.e. $x \in V_k$):

$$\overline{G}_k(x) = \frac{1}{N_k(x)} \sum_{i=1}^{N_k(x)} G_i(x, w)$$

$$\overline{F}_k(x) = \frac{1}{N_k(x)} \sum_{i=1}^{N_k(x)} F_i(x, w)$$

In order to take into account the non-closed form constraints, a penalty of estimated constraint function is added to the cost function and the sample optimum is determined according to the following augmented cost function:

$$H_k(x) = \overline{G}_k(x) + S_k(x)(\alpha - \overline{F}_k(x))^+$$

where $S_k(x)$ is the penalty factor for constraint violation, $(x)^+ = \max(0, x)$.

The algorithm focuses on sample optimum $x_k^* = \arg \min_{x \in V_k} H_k(x)$ and samples randomly additional solutions to consider in a so-called most promising area:

$$C_k = \{ x \in \Theta : \|x - x_k^*\| < 3 \forall y \in V_k \text{ and } y \neq x_k^* \}$$

where $\|x - y\|$ denotes the Euclidean distance between $x$ and $y$.

At each iteration, a Simulation Allocation Rule (SAR) is used to determine the additional number $a_k(x)$ of observations to allocate to solution $x$ at iteration $k$. As a result,

$$N_k(x) = \sum_{i=0}^{k} a_i(x)$$

is the total number of observations on
solution \( x \) until iteration \( k \).

**Algorithm 1** (Fully constrained DOvS with non closed-form constraints)

Step 1: Initialization
1. Set iteration count \( k = 0 \).
2. Fix an initial solution \( x_0 \in \Theta \), set \( V_0 = \{ x_0 \} \), \( x^*_0 = x_0 \) and \( C_0 = \Theta \).
3. Determine \( a_0(x_0) \) according to a Simulation Allocation Rule (SAR).
4. Perform \( a_0(x_0) \) simulation replications for \( x_0 \), set \( N_0(x_0) = a(x_0) \), calculate \( G_0(x_0) \) and \( F_0(x_0) \).

Step 2: Let \( k = k + 1 \). Sample new solutions and allocate additional simulation budget

2.1. Sample \( m \) solutions \( x_{k1}, x_{k2}, \ldots, x_{km} \) from \( C_{k-1} \).
2.2. Let \( V_k = V_{k-1} \cup \{ x_{k1}, x_{k2}, \ldots, x_{km} \} \).
2.3. Determine \( a_k(x) \) according to the SAR for all \( x \in V_k \).
2.4. For all \( x \in V_k \), perform \( a_k(x) \) observations, then update \( N_k(x) \), \( G_k(x) \) and \( F_k(x) \).

Step 3: Determine the new sample optimum \( x^*_k \), update the most promising area \( C_k \), and go to Step 2.

**C. Partially constrained discrete optimization with non closed-form constraints**

The direct application of Algorithm 1 to this case is impossible as the most promising area is unbounded. In this subsection, Algorithm 1 is extended by introducing the concept of moving observation area from which new solutions are sampled.

At each iteration of the optimization process, the observation area \( D \) is defined as the cube around the current sample optimum \( x^* \). Thus,

\[
D = \prod_{i=1}^{d} [x^{*(i)} - q, x^{*(i)} + q]
\]

where \( q > 0 \) is the width of the observation area. Note that the observation area \( D \) is a moving but finite observation window during the optimization process. This is different from the COMPASS algorithms of [7, 8] that use an observation hyperbox around some given point and covering all visited solutions. The observation hyperbox grows as the algorithm progresses. With the moving observation area, the simulation budget is devoted to the best sampled optimum \( x^* \) and to solutions that are in the nearby area of \( x^* \) instead of those that are far away from \( x^* \). We hope to be able to speed up the convergence in terms of the total number of simulation observations to true local optimal solutions of DOvS.

**Algorithm 2** (Partially constrained DOvS with non closed-form constraints)

Identical to Algorithm 1 with Step 2.1 replaced by :

2.1’. Sample \( m \) solutions \( x_{k1}, x_{k2}, \ldots, x_{km} \) from \( C_{k-1} \cap D_{k-1} \) where the observation area is \( D_k = \prod_{i=1}^{d} [x^{*(i)} - q, x^{*(i)} + q] \).

**III. CONVERGENCE OF DOvS ALGORITHMS**

In this section, we prove that under some assumptions the above algorithms converge to a set of local optimums defined as follows:

\[
M = \{ x \in \Theta : f(x) > \alpha \text{ and } (g(x) \leq y \text{ or } f(x) \leq \alpha) \}
\]

where \( NH(x) = \{ y \in \Theta : \| x - y \| \leq 1 \} \) is the local neighborhood of \( x \). By definition, \( NH(x) \) includes \( x \). Further, a solution \( x \) which satisfies the constraint and whose neighbors \( y \in NH(x) \) do not satisfy the constraint, is a local optimum regardless the value of the objective function.

The following assumptions are needed for the convergence proofs of the DOvS algorithms.

**Assumption 1:**

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N} G_i(x, w) = g(x) = 1,
\]

\[\forall x \in \Theta.\]

**Assumption 2:**

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N} F_i(x, w) = f(x) = 1,
\]

\[\forall x \in \Theta.\]

**Assumption 3:** \( N_k(x) \to \infty \Rightarrow S_k(x) \to \infty \), \( \forall x \in \Theta. \)

**Assumption 4:** The SAR guarantees that \( a_k(x) \geq 1 \), for all \( x \in V_k \). In addition, with probability \( \beta > 0 \), \( a_k(x) \geq 1 \) for each \( x \in D_k \cap NH(x^*_k) \cap V_k \).

**Assumption 5:** \( x_0 \) is a feasible solution.

**Assumption 6:** \( \lim_{k \to \infty} N_k(x_0) = +\infty \).

**Assumption 7:** There is no solution \( x \in \Phi \) such that \( f(x) = \alpha \).

**Assumption 8:** In Step 2.1 or Step 2.1’, (i) the solution sampling is made independently for different iterations \( k \) and (ii) for each iteration \( k \), any solution \( x \) in the sampling area, i.e., \( x \in C_{k-1} \) in algorithm 1 and \( x \in C_{k-1} \cap D_{k-1} \) in algorithm 2, is sampled with a probability of at least \( \gamma \) for some \( \gamma > 0 \).

**Remark 1:** Assumptions 1-2 are standard assumptions for simulated-based optimization and are met by most simulation algorithms. Assumption 3 is needed to ensure the convergence to feasible solutions. The penalty factor can be chosen either deterministically or stochastically. Assumption 8 can be easily satisfied as our sampling area is always bounded whether the solution space is bounded or not. The uniform sampling scheme proposed in [7] can be used if the DOvS problem is defined on a polyhedron \( \Phi \).

**Remark 2:** Assumption 4 ensures that every new solution is simulated at least once and that, with some positive probability, additional simulation budget is allocated to the solutions which are in the neighborhood of the last optimum sample \( x^*_{k-1} \) including \( x^*_{k-1} \). This assumption is very weak for optimization algorithms defined in this paper. It is significantly weaker than related assumption made in [7].
which requires additional simulation runs for all solutions in $V^\circ$. However, it is similar to related assumption made in [8] brought to our attention during the review of this paper.

**Remark 3:** Assumption 5 is restrictive and difficult to check. In practice, it seems reasonable for most experimental systems for which a feasible benchmark solution, often the current solution, does exist. If such a solution is not available, the problem is solved in two phases by starting with the maximization of $f(x)$.

**Remark 4:** Assumption 7 is impossible to check a priori. We describe here one practical approach to apply the optimization algorithms and results of this paper. If the DOVs algorithms ends with a solution of $x^*$ such that the estimation $F_k(x^*)$ of $f(x^*)$ is very close to $x$, then we replace $x$ with $x + \epsilon$, where $\epsilon$ is a small positive number such that $F_k(x^*) > x - \epsilon$ and re-run the DOVs algorithms.

**Lemma 1:** Under Assumptions 1-3 and 7, if $\lim_{k \to \infty} N_k(x) = \infty$, then $\lim_{k \to \infty} H_k(x) = \infty$ if $f(x) < \alpha$ and $H_k(x) = g(x)$ if $f(x) > \alpha$.

Proof. Let $\epsilon = |f(x) - \alpha|/2$. According to Assumption 7, $\epsilon > 0$. Furthermore, by Assumption 2, there exists $K > 0$ such that $|f(x) - F_k(x)| < \epsilon$ for all $k > K$. (For the sake of simplicity, the expression “w.p. 1” will be omitted in the following proofs). Hence, $F_k(x) - \alpha < \epsilon$ if $f(x) > \alpha$ and $F_k(x) - \alpha < \epsilon$ if $f(x) < \alpha$. As a result, $f(x) > \alpha$, by Assumption 1,

$$\lim_{k \to \infty} H_k(x) = \lim_{k \to \infty} \left[ G_k(x) + S_k(x)(\alpha - F_k(x)) \right] = \lim_{k \to \infty} [G_k(x)] = g(x)$$

If $f(x) < \alpha$, Assumption 3 leads to $\lim_{k \to \infty} H_k(x) = \lim_{k \to \infty} [G_k(x) + S_k(x)] = \infty$. \hfill $\Box$

**Lemma 2:** Under Assumptions 1-7, w.p. 1, any solution $x \in \Theta$ such that $x_k^* \in \mathcal{F}_k(x)$ i.o. (infinitely often) is a feasible solution, i.e. $f(x) < \alpha$.

Proof. Assume that there exists a solution such that $x_k^* = x$ i.o. and $f(x) < \alpha$. From Assumption 4, $N_k(x) \to \infty$ as $k$ increases. From Lemma 1, w.p. 1, $\lim_{k \to \infty} H_k(x) = \infty$. Similarly, by Assumptions 5-6 and Lemma 1, w.p. 1, $\lim_{k \to \infty} H_k(x_0) = g(x_0)$. The above two relations imply that, w.p. 1, $x$ can only be sample optimum a finite number of times which contradicts $x_k^* = x$ i.o. and concludes the proof. \hfill $\Box$

**Theorem 1:** If Assumptions 1-8 are satisfied and if $\Theta$ is finite, then the sequence $x_k^*$ generated by Algorithm 1 converges w.p. 1 to the set $M$ in the sense that w.p. 1, there exists $K > 0$ such that $x_k^* \in M$, for all $k > K$.

**Theorem 2:** Under Assumptions 1-7 and if $\Theta$ is finite, w.p. 1,

$$\lim_{k \to \infty} g(x_k) = \min_{x \in U_\circ} g(x)$$

where $U_\circ$ is the set of feasible solutions for which an unlimited number of simulations is performed in Algorithm 1.

**Proof of Theorem 1:** From the finiteness of $\Theta$, the theorem holds if $P\left[ \forall x_k^* \in M \text{ i.o.} \right] = 0$ that we prove by contradiction. If $x_k \in M$ i.o., there exists an $x \in \Theta$ and $x \not\in M$ such that $x = x_k$ i.o. From Lemma 2, $x$ is a feasible solution. Since $x \not\in M$, there exists $y \not\in NH(x) \cap \Theta$ such that $g(y) < g(x)$ and $f(y) > \alpha$. Since $\|y - x\| = 1 \leq \|y - z\|$ for all $z \neq y$, $y \in C_k$ if $x = x_k^*$ and $y \not\in V_k$.

According to Assumption 8,

$$P\left[ \forall x_k^* \in M \text{ i.o.} \right] = 0$$

Since $x = x_k^*$ i.o., there exists $K > 0$ such that $y \in U_\circ \forall k \geq K$.

By Assumption 4, $x = x_k^*$ i.o. implies $\lim_{k \to \infty} N_k(x) = \infty$. Since both $x$ and $y$ are feasible and $g(y) < g(x)$, from Lemma 1, there exists an integer $K_2$ such that $H_k(y) > H_k(y)$ for all $k \geq K_2$. As a result, $x$ can be sample optimum (i.e. $x = x_k^*$) only a finite number of times. This is a contradiction. Therefore, $P\left[ \forall x_k^* \in M \text{ i.o.} \right] = 0$ which concludes the proof. \hfill $\Box$

**Proof of Theorem 2:** Since the set $\Theta$ of solutions is finite, there exists a constant $K_1 > 0$ such that for all $k \geq K_1$, $x_k^*$ belongs to the set of solutions that are sample optimum infinitely often. By Lemma 2, $x_k^* \in U_\circ \forall k \geq K_1$.

Let $S = \left\{ x \in U_\circ : g(x) = \min_{y \in U_\circ} g(y) \right\}$. As $U_\circ$ is nonempty and finite, $\epsilon = \frac{1}{2} \min_{y \in U_\circ} g(y) - \min_{y \in U_\circ} g(y) > 0$. Since $U_\circ(x)$ tends to infinity for all $x \in U_\circ$, by Lemma 1, there exists a constant $K_2 > K_1$ such that for all $k \geq K_2$, $H_k(x) < \min_{y \in U_\circ} g(y) + \epsilon$, $\forall x \in S$ and $H_k(x) > \min_{y \in U_\circ} g(y) + \epsilon$, $\forall x \in U_\circ \setminus S$. This implies that $x_k^* \in S \forall k \geq K_2$ and concludes the proof. \hfill $\Box$

**Remark 5:** For DOVS with closed-form constraints, i.e. DOVS without constraint (2), Theorems 1 and 2 hold under only Assumptions 1, 4 and 8. Lemma 1 is just Assumption 1, Lemma 2 trivially holds. The proofs of Theorems 1 and 2 are the same. For this special case, although Theorems 1-2 are similar to related results of [7, 8], our proofs are significantly simpler than those of [7, 8].

We now consider the convergence of Algorithm 2 for optimization of partially constrained DOVS problems with non closed-form constraints.

The following additional assumption is needed.

**Assumption 9:** For the given starting point $x_0$, there exists a compact finite set $\Pi$ (not necessarily known) and a positive
constant \( \delta > 0 \) such that \( x_0 \in \Pi \cap \Theta \) and 
\[ G(x, w) \geq g(x_0) + \delta \] for any \( x \in \Pi \cap \Theta \) where \( \Pi \) is the complementary set of \( \Pi \).

**Remark 6**: This assumption ensures the existence of finite optimal solution. It is more restrictive than related assumption of \([7, 8]\) that only requires \( g(x) \geq g(x_0) + \delta \) for any \( x \in \Pi \cap \Theta \). Focus now on the case \( G(x, w) \geq 0 \) which holds for most real-life applications. Assumption 9 directly holds if the cost function \( G(x, w) \) contains deterministic terms that go to infinity as \( x \) goes to infinity. Many applications have this property as \( x \) typically corresponds to resources capacities and \( G(x, w) \) contains resource capacity related cost. Even if Assumption 9 does not hold directly, under the assumption of the existence of a finite optimal solution, we can approximate \( G(x, w) \) by 
\[ U(x, w) = G(x, w) + \varepsilon \| x - x_0 \| \] where \( \varepsilon > 0 \) is small enough such that \( g(x) + \varepsilon \| x - x_0 \| \) is a good approximation of \( g(x) \) in the area of interest. The new sample cost function \( U(x, w) \) satisfies Assumption 9 as \( G(x, w) \geq 0 \). More generally, if the error term \( \varepsilon \| x - x_0 \| \) of the cost function is too large for the solution determined by Algorithm 2, we can reduce \( \varepsilon \) accordingly and re-run Algorithm 2.

Let \( \Pi = \Pi^d \cap \Pi^b \) be the compact finite set considered in Assumption 9, where \( \Pi^d \) and \( \Pi^b \), \( i = 1, \ldots, d \) are the (not necessarily known) lower and upper bounds.

**Theorem 3**: Under Assumptions 1-9, the sequence \( \{ x_k \}_{k=0}^\infty \) generated by Algorithm 2 converges w.p. 1 to the set \( M \) in the sense that w.p. 1, there exists \( K > 0 \) such that \( x_k \in M \cap \Pi \), for all \( k \geq K \).

**Theorem 4**: Under assumptions 1-7 and 9, w.p. 1,
\[ \lim_{k \to \infty} g(x_k^*) = \min_{x \in \Pi} g(x) \]
where \( U_x \) is the set of feasible solutions for which an unlimited number of simulations is performed in algorithm 2.

**Proof of Theorem 3**: Note that Lemma 1 and 2 still hold. For any sequence \( \{ V_0, V_1, \ldots \} \) generated by Algorithm 2, since \( V_k \subseteq V_{k+1} \subseteq \Theta \), \( \forall k \geq 0 \), \( V_\infty = \bigcup_{k=0}^\infty V_k \) exists and \( V_\infty \subseteq \Theta \). From Assumption 1 and assumptions 5-6, there exists a constant \( K_1 \geq 0 \) such that for all \( x \in V_{\infty} \cap \Pi \), 
\[ H_k(x_0) = g(x_0) + \delta \] for any \( x \in \Pi \cap \Theta \) and \( V_{\infty} \). Thus, \( H_k(x_0) = g(x_0) + \delta \) for all \( k \geq K_1 \) and \( \forall x \in \Pi \cap \Theta \). Therefore, \( x_k^* \in \Pi \) for all \( k \geq K_1 \), which implies \( V_k \subseteq V_{K_1} \cup \Pi^* \), where 
\[ \Pi^* = \Pi_{i=1}^d \left[ b^{(i)}(0), b^{(i)}(0) \right] \] and \( \Pi^* = \Pi_{i=1}^d \left[ b^{(i)}(0), b^{(i)}(0) \right] \). Since \( V_{K_1} \) is finite, i.e. \( V_{K_1} \) is finite, i.e.
\[ \left| V_{K_1} \right| < \infty \] and \( \left| \Pi^* \right| < \infty \), the set \( V_x \) is finite, i.e. \( \left| V_x \right| < \infty \).

Since \( x_k^* \in \Pi \) for all \( k \geq K_1 \), the theorem holds if \( P \left[ x_k^* \in M \cap \Pi \right] = 0 \) that we prove by contradiction. If \( x_k^* \notin M \cap \Pi \) i.o., there exists \( x \in \Pi \) and \( x \in M \) such that \( x = x_k^* \) i.o. The remaining proof is the same as that of Theorem 1.

**Proof of Theorem 4**: From the proof of Theorem 3, there exists \( K_1 \geq 0 \) such that \( V_k \subseteq V_{K_1} \cup \Pi^* \) for all \( k \geq K_1 \), where \( \Pi^* = \Pi_{i=1}^d \left[ b^{(i)}(0), b^{(i)}(0) \right] \). As a result, \( U_x \) is a nonempty finite set and \( x_0 \in U_x \). The remaining proof is similar to that of Theorem 2 with \( \Theta \) replaced by the finite set \( V_{K_1} \cup \Pi^* \).

**IV. CONCLUSION**

In this paper we proposed a simulation based algorithm for optimizing stochastic discrete event systems where both the performance measure and some constraints are estimated by simulation. Our approach combines most promising area concept proposed in \([7]\) with two new concepts: the moving observation area and an augmented cost function. The non-closed form constraints are taken into account in an augmented performance function by means of an increasing penalty factor. Moving observation area is introduced to naturally concentrate the simulation budget on solutions in the nearby area of the current sample optimum. Convergence properties are established under reasonable conditions and with very simple proofs.

The algorithms proposed in this paper have been applied to the optimization of a multi-stage serial production-distribution system controlled by base stock policies (see \([11]\)). The algorithms were first used to determine the optimum value of base stock levels in order to minimize the inventory holding costs and demand backlogging cost. The proposed algorithms were shown to converge much faster than the COMPASS algorithms of \([7]\). The algorithms were then applied to the minimization of inventory holding costs subject to fill rate constraints. The algorithms converge to fairly good feasible solutions in a surprisingly small number of simulation observations, in less than 1000 simulation observations for a 5-stage system. Our algorithms have also been applied in \([10]\) for optimization of various control strategies of serial supply chains. Quick convergence was observed and the resulting solutions follow properties of optimal solutions observed in other supply chain studies.

The weak conditions on the SAR, the solution sampling scheme and the selection of the observation areas (Assumptions 4, 6, 8) allow non uniform sampling of the most
promising area and provide flexibility in implementing the simulation-based optimization algorithms. The moving observation area, sampling scheme and the SAR allow different optimization strategies. A small observation area and an SAR that assigns simulation budgets only to neighbor solutions of the current optimum are expect to have quicker convergence and might lead to a bad local optimum. A combination of a large observation area and an SAR that assigns evenly the simulation budgets to all solutions in the most promising area tends to have slower convergence but might lead to better solutions. The optimum selection of the observation area, the SAR and the sampling scheme has a profound impact on the efficiency of the simulation-based optimization algorithms and is an interesting research direction.

Assumptions 5, 7 and 9 are restrictive technical conditions that are needed in this paper to establish the convergence properties. The relaxation of these assumptions requires novel proof techniques for convergence studies of simulation-based optimization with non closed-form constraints.

Combining results of this paper with novel concepts in simulation-based optimization is another interesting future research direction. As the exploration-exploitation principle used in re-enforced learning, combining solution sampling in the whole solution space and sampling in the most promising area and moving observation area could improve the probability of convergence to global optimum. Recent Ordinal Optimization methods such as OCBA method in [2] could improve the simulation budget allocation and improve the convergence rate.

**REFERENCES**


