Global analysis of a continuum model for pulse-coupled oscillators

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We consider phase oscillators on the circle interacting through an impulsive instantaneous coupling. In contrast with previous studies on related pulse-coupled models, global stability results are obtained in the continuum limit. For the nonlinear transport equation governing the evolution of the oscillators, we propose (under technical assumptions) a global Lyapunov function which is induced by a total variation distance between quantile densities. The monotone time evolution of the Lyapunov function completely characterizes the dichotomic behavior of the oscillators: either the oscillators converge in finite time to a synchronous state or they asymptotically converge to an asynchronous state uniformly spread on the circle. The results of the present paper apply to various popular phase oscillators models (e.g. the well-known leaky integrate-and-fire model) and provide a novel approach for the (global) analysis of pulse-coupled oscillators.

Keywords: phase oscillators, impulsive coupling, synchronization, partial differential equations, transport equation, global stability, Lyapunov function, total variation distance

I. INTRODUCTION

Networks of interacting agents are omnipresent in natural [4, 22, 33] as well as in artificial systems [6, 10, 27]. In spite of their apparent simplicity, they may exhibit rich and complex ensemble behaviors [28] and have led to intense research during the last few decades. In this context, coupled phase oscillators are generic models of paramount importance when studying the collective behaviors of a large collection of systems [34].

Phase oscillators appear as reductions or approximations of (realistic) dynamical models. They are obtained through the computation of a phase sensitivity function, or phase response curve (PRC) [22, 34], which characterizes the phase sensitivity of an oscillator to an external perturbation, such as the influence of the surrounding oscillators in the network. Since phase oscillators are characterized by a one-dimensional state-space $S^1(0, 2\pi)$, they are more amenable to a formal mathematical study of the collective behaviors even though the nonlinear interactions between oscillators often yield mathematical puzzles [12, 32].

Within the network, oscillators interact through a nonlinear coupling. In most studied models, the coupling has a permanent influence on the network. However, in many situations encountered in biology or physics, the oscillators influence the network only during a tiny fraction of their cycle (e.g. yeast cell dynamics [2]). It is particularly so when the interconnection between the agents consists in the emission of fast pulses (spiking neurons [7], cardiac pacemaker cells [21], earthquakes dynamics [18], etc.). In this paper, we consider the limit of an impulsive and instantaneous coupling, where a (pulse-coupled) oscillator interacts with the network only when its phase is equal to a given value. For a particular type of oscillator, the model is linked to the impulsive Peskin model [13, 21] and corresponds to the fast coupling limit of a pulse-coupled model investigated in [1, 31].

We introduce a Lyapunov function whose monotone time evolution forces convergence of the solution either to perfect synchrony or to an asynchronous state uniformly spread on the circle. The proposed Lyapunov function has the interpretation of a total variation distance between (quantile) density functions. Modulo technical conditions detailed in the paper, we show that the time evolution of this Lyapunov function is governed by the derivative of the sensitivity function, resulting in a global convergence analysis for monotone sensitivity functions.

To the best of the authors knowledge, a global analysis of the continuous pulse-coupled model is novel. The use of a total variation distance as a strict Lyapunov function for a nonlinear transport equation seems also new and specific to the impulsive nature of the coupling (the results in [24, 29] suggest that total variation distance is useless for systems of conservation laws). The existence of a global Lyapunov function for pulse-coupled phase models opens new avenues for an analysis of pulse-coupled models similar to the celebrated Kuramoto model [12, 26].

The paper is organized as follows. In Section II, the model of pulse-coupled oscillators is presented. Considering the thermodynamic limit, a nonlinear partial differential equation (PDE) dictates the time evolution of the continuum of oscillators. The asymptotic behavior of the solutions is investigated in Section III which underlines the dichotomic

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behavior of the system. Section IV presents our main result and proposes a Lyapunov function based on the total variation distance. Section V is devoted to the convergence analysis of oscillators with a monotone sensitivity function. In Section VI the results are applied to general models of oscillators. The paper closes with a discussion (Section VII) and some concluding remarks (Section VIII).

II. A CONTINUITY EQUATION FOR PULSE-COUPLED PHASE DENSITY MODELS

The model studied in this paper considers the time evolution of a density, which represents a continuum of phase oscillators. When considering an instantaneous impulsive coupling, the PDE is characterized by a nonlinear coupling term.

A. Phase model

An isolated phase oscillator is characterized by a single phase \( \theta(t) \) which evolves on the circle with a constant natural velocity, that is, \( \dot{\theta} = \omega > 0, \theta \in S^1(0, 2\pi) \). Under the influence of an external input \( u(t) \), the phase dynamics of the oscillator write

\[
\dot{\theta} = \omega + Z(\theta) u(t) \equiv v(\theta, t),
\]

(1)

where the function \( Z \in C^1([0, 2\pi]) \) is the phase sensitivity function of the oscillator. This function, also called phase response curve (PRC), can be computed from a general high-dimensional state model possessing a stable limit cycle [25, 34], in which case the phase dynamics (1) represent a one-dimensional reduced model valid in the neighborhood of the limit cycle. Examples will be considered in Section VI.

B. Phase density equation

In the thermodynamic limit \((N \to \infty)\) of a large number of \(N\) oscillators, it is of interest to model the distribution of (identical) oscillators on \(S^1(0, 2\pi)\) by a phase density function

\[
\rho(\theta, t) \geq 0 \in C^0(L^1(0, 2\pi); \mathbb{R}^+) ,
\]

with the quantity \( \rho(\theta, t)d\theta \) being the fraction of oscillators with a phase between \( \theta \) and \( \theta + d\theta \) at time \( t \). The density is nonnegative and satisfies the normalization

\[
\int_0^{2\pi} \rho(\theta, t) d\theta = 1 \quad \forall t.
\]

It can also be interpreted as a probability density of solutions of (1) for a random initial condition.

The time evolution of the density obeys the well-known continuity equation

\[
\frac{\partial}{\partial t} \rho(\theta, t) = -\frac{\partial}{\partial \theta} [v(\theta, t) \rho(\theta, t)] .
\]

(2)

In the above equation, the function

\[
v(\theta, t) \rho(\theta, t) \equiv J(\theta, t)
\]

(3)

is the (nonnegative) flux \( J(\theta, t) \geq 0 \in C^0(L^1(0, 2\pi); \mathbb{R}^+) \). The quantity \( J(\theta, t)dt \) represents the fraction of oscillators flowing through phase \( \theta \) between time \( t \) and \( t + dt \). Since the phase \( \theta \) is defined on \( S^1(0, 2\pi) \equiv \mathbb{R} \mod 2\pi \), the flux must satisfy the boundary conditions

\[
J(0, t) = J(2\pi, t) \equiv J_0(t) \quad \forall t .
\]

(4)

For the sake of simplicity, we use in the sequel the notation \( J_0 \) to denote the boundary flux (4).
C. Impulsive coupling

We consider that the oscillators are pulse-coupled: each oscillator influences the dynamics of the other oscillators only when it satisfies the (instantaneous) phase condition \( \theta = 0 \). The coupling is then impulsive and the external input \( u(t) = u^{\text{imp}}(t) \) is thereby proportional to the flux at \( \theta = 0 \), that is, \( u^{\text{imp}}(t) = K J_0(t) \). The coupling strength \( K \) is a constant and its sign determines the coupling nature. If \( K > 0 \), the coupling is excitatory and if \( K < 0 \), the coupling is inhibitory. The phase dynamics now write

\[
\dot{\theta} = v(\theta, t) = \omega + Z(\theta) K J_0(t)
\]

and, at \( \theta = 0 \), one has the implicit relationship

\[
J_0(t) = v(0, t) \rho(0, t) = [\omega + K Z(0) J_0(t)] \rho(0, t).
\]

The input is then rewritten as the feedback term

\[
u^{\text{imp}}(t) = K J_0(t) = K \frac{\omega \rho(0, t)}{1 - K Z(0) \rho(0, t)}.
\]

The continuity equation thus leads to the nonlinear PDE for the density

\[
\frac{\partial \rho(\theta, t)}{\partial t} = -\omega \frac{\partial \rho(\theta, t)}{\partial \theta} - \frac{K \omega \rho(0, t)}{1 - K Z(0) \rho(0, t)} \frac{\partial}{\partial \theta} [Z(\theta) \rho(\theta, t)].
\]

The boundary condition is expressed in terms of density as

\[
\frac{\rho(0, t)}{1 - K Z(0) \rho(0, t)} = \frac{\rho(2\pi, t)}{1 - K Z(2\pi) \rho(2\pi, t)} \left( = \frac{J_0(t)}{\omega} \right).
\]

The reader will notice that, in contrast to the periodicity condition on the flux, no periodicity is assumed on the sensitivity function \( Z(\theta) \), and therefore on the density \( \rho(\theta, t) \). (In particular, \( \rho(0, t) \neq \rho(2\pi, t) \) if \( Z(0) \neq Z(2\pi) \).

III. A DICHTOMIC BEHAVIOR

We consider the solutions of the PDE with boundary conditions and we put emphasis on the coupling term, which induces a dichotomic behavior of the oscillators.

A. Asymptotic behavior

Without coupling \( (K = 0) \), the last term of \( \partial Z/\partial \theta \) disappears and the PDE is a standard transport equation. Its solution is a rigid translation of the initial density \( \rho(0, t) = \rho_0(\theta) \) with a constant velocity \( \omega \), that is, \( \rho(\theta, t) = \rho_0([\theta - \omega t] \mod 2\pi) \). In this case, any solution is periodic (with period \( 2\pi/\omega \)) and the system is marginally stable.

When the oscillators are coupled, the last term of \( \partial Z/\partial \theta \) has the interpretation of a “spatial feedback” which modifies the transport equation. Under the influence of the coupling, the velocity depends on both time and phase. The density is thereby “stretched” or “compressed”. This is illustrated when computing the total time derivative along a characteristic curve \( \Theta(t) \) defined by \( \dot{\Theta} = v(\Theta(t), t) \), that is

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \theta} v(\Theta(t), t) = \frac{\partial \rho}{\partial \theta} |_{\theta = \Theta(t)} J_0(t) K \frac{dZ}{d\theta} |_{\theta = \Theta(t)},
\]

where and have been used. The total derivative shows that the density is modified on a characteristic curve whenever the sensitivity function \( Z \) is not constant. In addition to the rigid translation, the density undergoes a nonlinear transformation, possibly leading to asymptotic convergence to a particular density function corresponding to a particular stationary organization of the oscillators.

The total derivative gives clear insight that the sign of the derivative \( K dZ/d\theta \) is of first importance. In fact, the sign of \( K dZ/d\theta \) will dictate the dichotomy studied in the present paper. The condition \( K dZ/d\theta < 0 \), or \( K Z(\theta) \) being a monotone decreasing function, will be shown to enforce convergence to a uniform flux on \( S^1([0, 2\pi]) \), corresponding to the maximal spreading of the oscillators on the circle (Figure). In contrast, the reverse condition
Figure 1: Time-evolution of the density (left) and flux through $\theta = 0$ (right) for a monotone increasing sensitivity function ($\omega = 4.128$, $Z(\theta) = 1.97 \exp(0.485 \theta)$). With an inhibitory coupling $K = -0.1 < 0$, the function $K Z$ is monotone decreasing. (a) The density converges towards a stationary solution $\rho^*$ and (b) the flux $J_0(t) = J_0(t)$ tends to a constant value $J^* \approx 0.53$.

Figure 2: With the same sensitivity function as in Fig. 1 but with an excitatory coupling $K = 0.1 > 0$, the function $K Z$ is monotone increasing. (a) The density converges towards a synchronous solution and (b) the flux $J_0(t) = J_0(t)$ tends to a Dirac function.

$K dZ/d\theta > 0$, or $K Z(\theta)$ being a monotone increasing function, will be shown to enforce convergence to a delta-like flux characterized by the synchronization of all the oscillators (Figure 2).

The limit case between the two situations corresponds to a constant sensitivity function, that is, $K dZ/d\theta = 0$. The total derivative (10) is equal to zero, as in the uncoupled case. As a matter of fact, the velocity $v(t) = \omega + K Z J_0(t)$ does not depend on the phase and the density thereby undergoes a rigid translation. Any solution evolves periodically and the stationary solution is marginally stable.

### B. Stationary asynchronous state

The simplest stationary solution of the PDE (2) is characterized by a constant flux $J(\theta, t) = J^*$, which means that the oscillators cross any phase $\theta \in [0, 2\pi]$ with the same constant rate. In this situation, the oscillators spread over $S^1(0, 2\pi)$ to make the flux uniform: it is an asynchronous state [1, 31]. This uniform distribution is the continuous counterpart of the splay state, in which a finite number of oscillators are evenly spread over the circle $S^1(0, 2\pi)$ [15]. From (9), the stationary density for the asynchronous state must satisfy

$$\rho^*(\theta) = \frac{J^*}{\omega + K Z(\theta) J^*}.$$  (11)
(This density could also be made uniform by a proper change of variable under which the velocity \( \dot{\theta} = \omega + K Z(\theta) J^* \) becomes constant.)

The stationary asynchronous state thus exists if there exists a value \( J^* > 0 \) so that the stationary density is nonnegative

\[
\frac{J^*}{\omega + K Z(\theta) J^*} \geq 0, \quad \forall \theta \in [0, 2\pi]
\]  

(12)

and normalized

\[
\int_0^{2\pi} \frac{J^*}{\omega + K Z(\theta) J^*} d\theta = 1.
\]  

(13)

The following proposition gives necessary and sufficient conditions to ensure the existence and uniqueness of the stationary solution.

**Proposition 1.** A stationary flux \( J^* > 0 \) verifying the conditions (12) and (13) exists if and only if the inequality

\[
\lim_{r' \to r} \frac{r}{r'} \int_0^{2\pi} \frac{1}{K Z(\theta) + r'} d\theta > 1
\]  

(14)

is satisfied with

\[
r \triangleq \begin{cases} 
0 & \text{if } K Z(\theta) \geq 0 \quad \forall \theta \in [0, 2\pi], \\
\min_{\theta \in [0, 2\pi]} \{K Z(\theta)\} & \text{otherwise}.
\end{cases}
\]

Moreover, the solution is unique when it exists.

**Proof.** Inequality (12) implies that the velocity \( \omega + K Z(\theta) J^* \) is strictly positive, so that

\[
J^* \in \mathcal{I} \triangleq \left( 0, \lim_{r' \to r} r' / r \right).
\]

The function

\[
I(J) = \int_0^{2\pi} \frac{J}{\omega + K Z(\theta) J} d\theta
\]

satisfies \( I(0) = 0 \), is continuous on \( \mathcal{I} \), and is strictly increasing on \( \mathcal{I} \) since

\[
\frac{dI}{dJ} = \int_0^{2\pi} \frac{\omega}{[\omega + K Z(\theta) J]^2} d\theta > 0 \quad \forall J \in \mathcal{I}.
\]

As a consequence, the equation \( I(J) = 1 \), which is equivalent to condition (13), has a (unique) solution \( J^* \in \mathcal{I} \) if and only if

\[
\lim_{r' \to r} \int_0^{2\pi} \frac{1}{K Z(\theta) + r'} d\theta > 1,
\]

which concludes the proof.

In the case \( r = 0 \), an interpretation of condition (14) is that a stationary asynchronous state exists provided that the coupling feedback does not accelerate the oscillators at will.

**IV. A STRICT LYAPUNOV FUNCTION INDUCED BY THE TOTAL VARIATION DISTANCE**

The total variation distance between quantile densities will be proposed as a Lyapunov function for the PDE (8) with the boundary condition (9). Although [24] shows that the total variation distance is constant for most of the systems of conservation laws, we highlight the relevance of the distance for the PDE (8) since the monotone time variation of the Lyapunov function is triggered by the nonlinear coupling term.
A. Quantile density

Given a density function $\rho : [0, 2\pi] \mapsto \mathbb{R}^+$, the cumulative density function $P(\theta) : [0, 2\pi] \mapsto I \triangleq [0, 1]$ is defined as

$$P(\theta) = \int_0^\theta \rho(\theta') \, d\theta'.$$

The quantile function $Q : I \mapsto [0, 2\pi]$, which is widely used in statistics \[20\], is the inverse cumulative density function, that is,

$$Q(x) = P^{-1}(x) = \inf\{\theta | P(\theta) \geq x\}.$$

The quantile density function $q : I \mapsto \mathbb{R}^+$, also called sparsity function, is the derivative of the quantile function (Figure 3). This implies the relation between the density function and the quantile density function

$$q(x) = \frac{dQ}{dx} = \frac{1}{\rho(Q(x))}.$$

(15)

In order to avoid some ill-defined cases, the condition $\rho > 0$ must be satisfied on $[0, 2\pi]$.

Figure 3: The density function $\rho(\theta)$ (left) has a cumulative density $P(\theta)$ (center). The quantile function $Q(x) = P^{-1}(x)$ is the cumulative density function of the quantile density function $q(x)$ (right).

For the density of oscillators (at a given time $t$), the variable $x \in I$ is interpreted as an oscillator index, assuming that the oscillators are continuously labeled on the interval $I$. The quantile function attributes a phase $\theta = Q(x) \in [0, 2\pi]$ to each oscillator with index $x$, and $Q(0) = 0$, $Q(1) = 2\pi$ for any time $t > 0$. The quantile density expresses the increase of phase per unit increase of oscillator index. The reader will notice that, as the oscillators density $\rho(\theta, t)$ depends on time in the model (2), the associated quantile function and quantile density function also depend on time and are then rigorously defined as the two-variable functions $Q(x, t)$ and $q(x, t)$. But for notational convenience, the time variable is omitted when clear for the context. In addition, we denote the quantile function and the quantile density associated to the stationary solution (11) by $Q^*$ and $q^*$ respectively.

B. Total variation distance

Our main result shows that the total variation distance, which is a common distance in probability theory, leads to a Lyapunov function for the PDE (8). The total variation distance between two random variables corresponds to the $L^1$-distance between the corresponding density functions (see \[5\] for further details). In this paper, we choose as Lyapunov function the total variation distance between the quantile density functions, that is,

$$V(\rho) = \|q - q^*\|_{L^1} = \int_0^1 \left| \frac{\partial Q}{\partial x} - \frac{dQ^*}{dx} \right| \, dx \quad \forall \rho > 0 \in C^0([0, 2\pi]).$$

(16)

In particular, $V(\rho) = 0 \iff q = q^*$ a.e. $\iff \rho = \rho^*$ a.e.

In order to remove the integral from the expression (16), we define the critical points $x^{(i)}_c$, for $i = 1, \ldots, N_c$, as the values verifying

$$q(x^{(i)}_c) = q^*(x^{(i)}_c) \quad \text{or} \quad \left. \frac{\partial Q}{\partial x} \right|_{x^{(i)}_c} = \left. \frac{dQ^*}{dx} \right|_{x^{(i)}_c}$$

(17)
Introducing the analog notation to (17), the critical phase \( i \) between continuous analog of (21) and is directly inspired from the discrete case of finite populations.

The (discrete) total variation distance between (20) and a stationary configuration oscillators is described by the vector \( N \) by the that is, \( (\cdot) \), \( \partial Q / \partial x \) is defined as the values verifying \( [x_c^{(i-1)}, x_c^{(i)}] \) with, by convention, \( x_c^{(0)} = 0 < x_c^{(i)} < x_c^{(N_c+1)} = 1 \), for \( i = 2, \ldots, N_c \). Without loss of generality, one has assumed that \( \partial Q / \partial x \leq \partial Q^* / \partial x \) for \( x \in [0, x_c^{(1)}] \). With this notation, the Lyapunov function (16) is rewritten as

\[
\mathcal{V}(\rho) = \sum_{i=1}^{N_c+1} (-1)^i \int_{x_c^{(i-1)}}^{x_c^{(i)}} \left( \frac{\partial Q^*}{\partial x} - \frac{\partial Q}{\partial x} \right) dx = 2 \sum_{i=1}^{N_c} (-1)^i \left( Q(x_c^{(i)}) - Q^*(x_c^{(i)}) \right),
\]

since \( Q(0) = Q^*(0) = 0 \) and \( Q(1) = Q^*(1) = 2\pi \). The total variation distance is then the sum of the highest (and the lowest) value differences between the two quantile functions \( Q \) and \( Q^* \) (Figure 4). Our main result shows that this quantity evolves monotonically in time along the trajectory of (8).

Figure 4: The Lyapunov function (19) is the total variation distance between two quantile density functions, which is equal to the sum \( \mathcal{V} = 2 \sum_i \delta^{(i)} \) of the highest value differences \( |Q - Q^*| \).

Finite number of oscillators. If the number \( N \) of oscillators is finite, the continuous quantile function is replaced by the \( N \)-quantiles \( Q_k^{(N)} \), \( k = 1, \ldots, N-1 \). When an oscillator crosses \( \theta = 2\pi \), the snapshot configuration of the other oscillators is described by the vector

\[
[\theta_1 \cdots \theta_{N-1}] = \left[ Q_1^{(N)} \cdots Q_{N-1}^{(N)} \right] \triangleq Q^{(N)}.
\]

The (discrete) total variation distance between (20) and a stationary configuration \( Q^{(N)} \) leads to a Lyapunov function

\[
\mathcal{V}(N) = \left| Q_1^{(N)} - Q_1^{(N)} \right| + \sum_{k=2}^{N-1} \left| Q_k^{(N)} - Q_{k-1}^{(N)} \right| - \left( Q_k^{(N)} - Q_{k-1}^{(N)} \right) + \left| Q_{N-1}^{(N)} - Q_{N-1}^{(N)} \right|.
\]

This Lyapunov function has been used in (15) to prove the global stability of splay configurations (clustering) in finite populations of Peskin oscillators (see Section VII A). The Lyapunov function (19) proposed in this paper is the continuous analog of (21) and is directly inspired from the discrete case of finite populations.

Density function vs. quantile density. The Lyapunov function (16) is induced by the total variation distance between quantile densities. An alternative choice would be the total variation distance between density functions, that is,

\[
\mathcal{V}_1(\rho) = \int_0^{2\pi} |\rho - \rho^*| d\theta \quad \forall \rho \geq 0 \in C^0([0, 2\pi]).
\]

Introducing the analog notation to (17), the critical phases \( \theta_c^{(i)} \), for \( i = 1, \ldots, N_c \), are defined as the values verifying

\[
\rho(\theta_c^{(i)}, t) = \rho^*(\theta_c^{(i)}, t)
\]

where

\[
(\cdot)^* \geq (\cdot)
\]

for some \( x \in (x_c^{(i-1)}, x_c^{(i)}) \)
Theorem 1. Let \( \theta \) along the solutions of (8)-(9). Our main result is summarized in the following theorem. 

\[
(-1)^i \rho(\theta) \geq (-1)^i \rho^*(\theta) \quad \forall \theta \in [\theta^{(i-1)}_c, \theta^{(i)}_c] \\
(-1)^i \rho(\theta) > (-1)^i \rho^*(\theta) \quad \text{for some } \theta \in (\theta^{(i-1)}_c, \theta^{(i)}_c)
\]

with, by convention, \( \theta^{(0)}_c = 0 < \theta^{(i-1)}_c < \theta^{(i)}_c < \theta^{(N_c+1)}_c = 2\pi \), for \( i = 2, \ldots, N_c \). Without loss of generality, one has assumed that \( \rho(0) \leq \rho^*(0) \) for \( \theta \in [0, \theta^{(1)}_c) \). The Lyapunov function writes

\[
V_1(\rho) = \sum_{i=1}^{N_c+1} (-1)^i \int_{\theta^{(i-1)}_c}^{\theta^{(i)}_c} (\rho - \rho^*) d\theta
\]

and its time derivative is given by

\[
\dot{V}_1 = \sum_{i=1}^{N_c+1} (-1)^i \int_{\theta^{(i-1)}_c}^{\theta^{(i)}_c} \frac{\partial \rho}{\partial t} d\theta = \sum_{i=1}^{N_c+1} (-1)^i \left[ J(\theta^{(i-1)}_c) - J(\theta^{(i)}_c) \right],
\]

(24)

where we used (2). A simple argument shows that the Lyapunov function cannot be strictly decreasing along the solutions of (8). Indeed, for any density satisfying \( \rho(0) = \rho^*(0) \), one must also have \( J_0 = J^*(0) = J^* \) since it follows from (9) that there is a bijection between the values \( \rho(0) \) and \( J_0 \). But then, (3) and (23) imply that \( J(\theta^{(i)}_c) = J^*(\theta^{(i)}_c) = J^* \) and the derivative (24) leads to \( \dot{V}_1 = 0 \). This argument shows that a direct application of the total variation distance on density functions does not lead to a good candidate Lyapunov function for the PDE (8). A key point is to apply the total variation distance on quantile functions instead.

C. A global Lyapunov function

The monotonicity properties of the sensitivity function lead to a monotone time evolution of the Lyapunov function along the solutions of (8)-(9). Our main result is summarized in the following theorem.

Theorem 1. Let \( \rho(\theta, t) \in C^0(L^1(0,2\pi), \mathbb{R}^+) \) be a strictly positive solution of (8)-(9). If the stationary density (17) exists and if either \( d^2 Z/d\theta^2 \geq 0 \forall \theta \in [0,2\pi] \) or \( d^2 Z/d\theta^2 \leq 0 \forall \theta \in [0,2\pi] \), then the Lyapunov functional (16) satisfies

\[
J(0,t) \min_{\theta \in [0,2\pi]} \left( K \frac{dZ}{d\theta} \right) V(\rho) \leq \dot{V}(\rho) \leq J(0,t) \max_{\theta \in [0,2\pi]} \left( K \frac{dZ}{d\theta} \right) V(\rho).
\]

(25)

Proof. The Lyapunov functional is well-defined for all time \( t \in \mathbb{R}^+ \) such that \( 0 < \rho(x,t) < \infty \). For the sake of simplicity, the time variable \( t \) is omitted in the sequel. Furthermore, one supposes

\[
\left. \frac{\partial Q}{\partial x} \right|_0 \leq \left. \frac{dQ^*}{dx} \right|_0
\]

without loss of generality, since the proof for the other case follows on similar lines. Under this assumption, the critical points (17) satisfy (18) and the Lyapunov function is given by (19). If \( \rho \neq \rho^* \), the quantile densities \( q \) and \( q^* \) do not coincide and there exists at least one critical point.

Next, the time derivative writes as

\[
\dot{V}(\rho) = 2 \sum_{i=1}^{N_c} (-1)^i \left. \frac{\partial Q}{\partial t} \right|_{\theta^{(i)}_c}.
\]

(26)

Differentiating the expression \( \theta \equiv Q[P(\theta,t),t] \) with respect to time \( t \) leads to

\[
0 = \frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} \frac{\partial P}{\partial t}
\]

or

\[
\frac{\partial Q}{\partial t} = - \frac{\partial P}{\partial t} \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial t} \frac{1}{\rho(Q(x))},
\]

where

\[
\rho(Q(x)) \equiv \rho \left( \frac{Q}{\rho^*} \right)
\]

and

\[
\rho^*(Q(x)) \equiv \rho^* \left( \frac{Q}{\rho^*} \right)
\]
It follows that one gets \( \theta \) with \( J \) given (15). Furthermore, \( \dot{\theta} = \int_{0}^{\theta} \frac{\partial \rho}{\partial t} d\theta' = \int_{0}^{\theta} \frac{\partial J}{\partial \theta} d\theta' = J_{0} - J(\theta) \) and the expression (26) becomes

\[
\dot{V}(\rho) = 2 \sum_{i=1}^{N_{c}} (-1)^{i} J_{i} \left[ Q(x_{c}^{(i)}) - J_{0} \rho[Q(x_{c}^{(i)})] \right].
\]

Since it follows from (15) that the critical values satisfy \( \rho[Q(x_{c}^{(i)})] = \rho^*[Q^*(x_{c}^{(i)})] \), \( i = 1, \ldots, N_{c} \), one obtains, given (3), (5), and (11),

\[
\dot{V}(\rho) = 2 \sum_{i=1}^{N_{c}} (-1)^{i} \left[ \psi[Q(x_{c}^{(i)})] - J_{0} \rho^*[Q^*(x_{c}^{(i)})] \right] = 2 \sum_{i=1}^{N_{c}} (-1)^{i} \left\{ \omega + K Z[Q(x_{c}^{(i)})]J_{0} - \frac{\omega J_{0}}{J^*} - K Z[Q^*(x_{c}^{(i)})]J_{0} \right\}. \tag{28}
\]

The boundary condition (29) yields a monotone relationship between the values \( \rho(0) \) and \( \rho(2\pi) \), that is, \( \rho(0) > \rho^*(0) \) if and only if \( \rho(2\pi) > \rho^*(2\pi) \). Apart from the case \( \rho(0) = \rho^*(0) \), \( \rho(2\pi) = \rho^*(2\pi) \), the number \( N_{c} \) of critical points, which verify (27), is even owing to the continuity of \( \rho \) and \( Q \). Consequently, the terms \((-1)^{i} [\omega - \omega J_{0}/J^*] \) in (28) cancel each other. In the particular case \( \rho(0) = \rho^*(0) \), \( N_{c} \) is not necessarily even but it follows from (29) that \( J_{0} = J^*(0) = J^* \) and the above-mentioned terms are still removed. As a consequence, one obtains

\[
\dot{V}(\rho) = 2 J_{0} \sum_{i=1}^{N_{c}} (-1)^{i} K \left[ Z[Q(x_{c}^{(i)})] - Z[Q^*(x_{c}^{(i)})] \right] \]

\[
= 2 J_{0} \sum_{i=1}^{N_{c}} (-1)^{i} K \left. \frac{dZ}{d\theta} \right|_{\theta_{i}} \left[ Q(x_{c}^{(i)}) - Q^*(x_{c}^{(i)}) \right] \]

\[
\triangleq 2 J_{0} \sum_{i=1}^{N_{c}} T^{(i)}, \tag{29}
\]

with \( \theta_{i} \in [Q(x_{c}^{(i)}), Q^*(x_{c}^{(i)})] \) or \( \theta_{i} \in [Q^*(x_{c}^{(i)}), Q(x_{c}^{(i)})] \) and where the second equality is obtained through the mean value theorem.

It remains to consider separately each term \( T^{(i)} \) in the sum (29). In the sequel, we consider for the sake of simplicity the case \( K d^{2}Z/d\theta^{2} \geq 0 \). Denoting \((-1)^{i} [Q(x_{c}^{(i)}) - Q^*(x_{c}^{(i)})] \) by \( \Delta^{(i)} Q \), we distinguish two cases: \( \Delta^{(i)} Q > 0 \) and \( \Delta^{(i)} Q \leq 0 \).

In the case \( \Delta^{(i)} Q > 0 \), one has

\[
T^{(i)} = K \left. \frac{dZ}{d\theta} \right|_{\theta_{i}} \Delta^{(i)} Q \leq \max_{\theta_{i} \in [0,2\pi]} \left( K \left. \frac{dZ}{d\theta} \right|_{\theta_{i}} \right) \Delta^{(i)} Q. \tag{30}
\]

and

\[
T^{(i)} = K \left. \frac{dZ}{d\theta} \right|_{\theta_{i}} \Delta^{(i)} Q \geq \min_{\theta_{i} \in [0,2\pi]} \left( K \left. \frac{dZ}{d\theta} \right|_{\theta_{i}} \right) \Delta^{(i)} Q. \tag{31}
\]

In the case \( \Delta^{(i)} Q \leq 0 \), we need to consider the addition of the term \( T^{(i)} \) with the term \( T^{(i-1)} \) or \( T^{(i+1)} \). By (18), one gets

\[
\Delta^{(i-1)} Q < \Delta^{(i)} Q \leq 0, \quad \tag{32}
\]

\[
\Delta^{(i+1)} Q < \Delta^{(i)} Q \leq 0. \tag{33}
\]

It follows that

\[
T^{(i-1)} + T^{(i)} = K \left. \frac{dZ}{d\theta} \right|_{\theta_{i-1}} \Delta^{(i-1)} Q + K \left. \frac{dZ}{d\theta} \right|_{\theta_{i}} \Delta^{(i)} Q \leq K \left. \frac{dZ}{d\theta} \right|_{\theta_{i}} \left( \Delta^{(i-1)} Q + \Delta^{(i)} Q \right), \tag{34}
\]
The assumption $K d^2 Z/d\theta^2 \geq 0$ implies $K dZ/d\theta|_{\theta_{i-1}} \leq K dZ/d\theta|_{\theta_i}$, with $\theta_{i-1} \leq \theta_i$. The above inequality then follows from (32). In addition, (32) also implies that the right hand in inequality (34) is the multiplication of $K dZ/d\theta$ with the positive quantity $\Delta^{(i-1)} Q + \Delta^{(i)} Q$. Hence, (33) can be rewritten as

$$
T^{(i-1)} + T^{(i)} \leq \max_{\theta \in [0,2\pi]} \left( K \frac{dZ}{d\theta} \right) \left( \Delta^{(i-1)} Q + \Delta^{(i)} Q \right).
$$

Similarly, considering the addition of the terms $T^{(i)}$ and $T^{(i+1)}$ and using (33), one obtains

$$
T^{(i)} + T^{(i+1)} \geq \min_{\theta \in [0,2\pi]} \left( K \frac{dZ}{d\theta} \right) \left( \Delta^{(i)} Q + \Delta^{(i+1)} Q \right).
$$

Next, the inequalities (30) and (35) lead to

$$
\dot{V}(\rho) = 2 J_0 \sum_{i=1}^{N_e} T^{(i)} \leq J_0 \max_{\theta \in [0,2\pi]} \left( K \frac{dZ}{d\theta} \right) V(\rho).
$$

In the situation 2, that is, $\Delta^{(i)} Q \leq 0$, the two terms $T^{(i)}$ and $T^{(i-1)}$ are considered together. The additional $T^{(i-1)}$ itself corresponds to the situation 1, that is, $\Delta^{(i-1)} Q > 0$, and does not need to be associated in turn with another term. In addition, there is no boundary problem since the term $T^{(1)}$ also corresponds to the situation 1, given (18), and $Q(0) = Q^*(0) = 0$.

Similarly, the inequalities (31) and (36) lead to

$$
\dot{V}(\rho) = 2 J_0 \sum_{i=1}^{N_e} T^{(i)} \geq J_0 \min_{\theta \in [0,2\pi]} \left( K \frac{dZ}{d\theta} \right) V(\rho).
$$

In the case $K d^2 Z/d\theta^2 \leq 0$, the inequalities (35) and (36) are reversed, that is, the sum $T^{(i-1)} + T^{(i)}$ has a lower-bound and the sum $T^{(i)} + T^{(i+1)}$ has an upper-bound. Hence, the inequalities (37) and (38) still hold, which completes the proof.

V. CONVERGENCE ANALYSIS FOR MONOTONE SENSITIVITY FUNCTIONS

The result of Theorem 1 has a strong implication in the case of monotone sensitivity functions. It implies that the Lyapunov function $V(\rho)$ has a monotone time evolution if the sensitivity function is monotone. In this situation, the Lyapunov function either converges to a lower bound or to an upper bound. These two bounds correspond to the two particular behaviors which characterize the dichotomy. They are given by

$$
0 \leq V(\rho) \leq \|q\|_{L^1} + \|q^*\|_{L^1} = Q(1) - Q(0) + Q^*(1) - Q^*(2\pi) = 4\pi.
$$

At the lower bound, the function is equal to zero if and only if the density corresponds to the asynchronous (stationary) density $\rho^*$. On the other hand, the Lyapunov function tends to the upper bound $4\pi$ if the density $\rho$ tends to a Dirac function (synchronization).

A. Exponential convergence to the asynchronous state

Theorem 1 will be used to study convergence to the asynchronous state for monotone decreasing functions $K Z(\theta)$. However, in order to apply (23) along the solutions, we need to show independently that the flux $J_0(t)$ remains strictly positive and bounded for all time. This condition will restrict the set of admissible initial conditions. In particular, the initial conditions have to ensure that $0 < J_0 < \infty$ when any oscillator crosses $\theta = 2\pi$ for the first time. Formally, we consider the characteristic curves $\Theta(\theta)$ defined by $\Theta(\theta) = v_1(\Theta(t), \theta)$, $\Theta(0) = \theta \in [0,2\pi]$, $\Theta(2\pi) = 2\pi$ (Figure 5).

The (strictly positive) initial density $\rho(\theta,0) \equiv \rho_0(\theta) > 0$ must be such that the value of the flux at the intersection of the characteristic curves with $\theta = 2\pi$ is strictly positive and bounded, that is

$$
\rho_0(\theta) > 0 \Rightarrow \quad 0 < J[\Theta(\theta) = 2\pi, \theta] = J_0(\theta) < \infty \quad \forall \theta \in [0,2\pi].
$$

Condition (39) is the condition that is imposed on the initial conditions to ensure that the flux satisfies $0 < J_0(t) < \infty$ for $t \in [0,\bar{t}]$. But if the initial condition satisfies (39), then the flux satisfies $0 < J_0(t) < \infty$ for all time and Theorem 1 can be applied to prove the exponential decreasing of the Lyapunov function. The result is summarized in the following proposition.
Figure 5: The characteristic curve $\Theta_v(t)$, defined by $\Theta_v = v(\Theta_v(t), t)$, with the initial value $\Theta_v(0) = \theta$, reaches $2\pi$ at time $t_\theta$.

**Proposition 2.** Consider the transport PDE (8)-(9) and assume that the sensitivity function $Z(\theta)$ is such that (i) the stationary density (11) exists, (ii) $K dZ/d\theta < 0 \forall \theta \in [0, 2\pi]$, and (iii) either $d^2 Z/d\theta^2 \geq 0 \forall \theta \in [0, 2\pi]$ or $d^2 Z/d\theta^2 \leq 0 \forall \theta \in [0, 2\pi]$. Then all solutions $\rho(\theta, t) \in C^0(L^1(0, 2\pi), \mathbb{R}^+)$ with an initial condition satisfying (39) exponentially converge to the asynchronous state and the Lyapunov functional (12) is exponentially decreasing along them.

**Proof.** One considers a characteristic curve $\Theta(t)$, with $\Theta(t) = 0$ and $\Theta(t) = 2\pi$. Solving the total derivative equation (10) on $\Theta(t)$ yields

$$
\rho(0, t) \exp \left( - \int_{t}^{t_0} J_0(t) K \left. \frac{dZ}{d\theta} \right|_{\Theta(t)} dt \right) = \rho(2\pi, t).
$$

Next, using (3) and expressing the integral in the space variable $\Theta$ along the characteristic curve leads to

$$
\frac{J_0(t)}{\omega + K Z(0) J_0(t)} \exp \left( - \int_{0}^{2\pi} \frac{J_0}{\omega + K Z(\Theta)} J_0 \left. \frac{dZ}{d\theta} \right|_{\Theta} d\Theta \right) \leq \frac{J_0(t)}{\omega + K Z(2\pi) J_0(t)}.
$$

If the flux is constant on $[\theta, 2\pi]$, then $J_0(t) = J_0 = J_M$, with $J_m \triangleq \min_{r \in [\theta, 2\pi]} J_0(r')$ and $J_M \triangleq \max_{r \in [\theta, 2\pi]} J_0(r')$. Otherwise, since $K dZ/d\theta$ is negative, replacing $J_0$ in the integral of the above expression by the bounds $J_m$ and $J_M$ and computing the integral gives the strict inequalities

$$
\frac{J_0(t)}{\omega + K Z(0) J_m} \leq \frac{J_0(t)}{\omega + K Z(2\pi) J_m} < \frac{J_0(t)}{\omega + K Z(2\pi) J_0(t)} \leq \frac{J_0(t)}{\omega + K Z(0) J_M}.
$$

By definition, $J_m \leq J_0(t) \leq J_M$ and one obtains

$$
\frac{J_m}{\omega + K Z(2\pi) J_m} < \frac{J_0(t)}{\omega + K Z(2\pi) J_0(t)} < \frac{J_M}{\omega + K Z(2\pi) J_M},
$$

or equivalently

$$
J_m < J_0(t) < J_M.
$$

Within any interval $[\theta, 2\pi]$, the flux remains either constant or satisfies (10). By induction, there holds

$$
J_{\min} \leq J_0(t) \leq J_{\max} \quad \forall t > 0,
$$

with $J_{\min} \triangleq \min_{r \in [0, 2\pi]} J_0(r')$ and $J_{\max} \triangleq \max_{r \in [0, 2\pi]} J_0(r')$. In addition, the initial condition (39) implies that $J_{\min} > 0$ and $J_{\max} < \infty$.

Since $\rho_0 > 0$, the exponential evolution of the density along (all) the characteristic curves implies that the density remains strictly positive for all $t > 0$. The results of Theorem 1 are then applied and yields

$$
J_{\min} \min_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \mathcal{V}(\rho) \leq \dot{V}(\rho) \leq J_{\max} \max_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \mathcal{V}(\rho),
$$

or, equivalently,

$$
- J_{\max} \max_{\theta \in [0, 2\pi]} \left| K \frac{dZ}{d\theta} \right| \mathcal{V}(\rho) \leq \dot{V}(\rho) \leq - J_{\min} \min_{\theta \in [0, 2\pi]} \left| K \frac{dZ}{d\theta} \right| \mathcal{V}(\rho) \leq 0
$$

since $K dZ/d\theta < 0$. It follows that the Lyapunov function is exponentially decreasing, which concludes the proof.

$\square$
Proposition 2 is a strong result showing that, provided that the function $KZ(\theta)$ is decreasing, the solution $\rho(.,t)$ remains, for all time, in a particular set of functions $\{\rho|V(\rho) < C\}$, with the constant $C > 0$. Inside this set, the solution eventually converges at exponential rate towards the stationary solution $\rho^*$, corresponding to $V(\rho^*) = 0$.

Condition (39) is rather weak for decreasing functions $KZ(\theta)$, as shown by the following proposition.

**Proposition 3.** Assume that $K dZ/d\theta < 0 \quad \forall \theta \in [0,2\pi]$. If $KZ(2\pi) \leq 0$, then (39) is always satisfied. If $KZ(2\pi) > 0$, then (39) is satisfied if

$$\rho_0(\theta) < \frac{1}{KZ(\theta)} \quad \forall \theta \in [0,2\pi].$$

(41)

**Proof.** Case $KZ(2\pi) \leq 0$. Using the boundary condition (10), the condition $0 < J_0(\tilde{t}_\theta) < \infty$ is turned into a condition on $\rho(2\pi,\tilde{t}_\theta)$ and (39) is equivalent to

$$\rho_0(\theta) > 0 \quad \Rightarrow \quad \rho(2\pi,\tilde{t}_\theta) > 0 \quad \forall \theta \in [0,2\pi].$$

(42)

For $\theta = 2\pi$, condition (42) is satisfied, since $\tilde{t}_{2\pi} = 0$. Next, we proceed by induction on $\theta$: given $\tilde{\theta}$ and assuming that (42) is satisfied for all $\theta' > \tilde{\theta}$, we prove that (42) also holds at $\theta = \tilde{\theta}$. Hence, the total derivative equation (10) is well-defined on the characteristic curve $\Theta(t)$ since $0 < J_0(t') < \infty$ for all $t' < \tilde{t}_\theta$. Solving (10) along the characteristic curve yields

$$\rho(2\pi,\tilde{t}_\theta) = \rho_0(\tilde{\theta}) \exp \left( - \int_{\tilde{\theta}}^{\tilde{t}_\theta} J_0(t') K \frac{dZ(t')}{d\theta} d\theta \right).$$

(43)

and $\rho_0(\tilde{\theta}) > 0$ implies $\rho(2\pi,\tilde{t}_\theta) > 0$. Condition (39), equivalent to (42), is then always satisfied.

Case $KZ(2\pi) > 0$. Using (10), the condition $0 < J_0(\tilde{t}_\theta) < \infty$ is turned into a condition on $\rho(2\pi,\tilde{t}_\theta)$ and (39) is equivalent to

$$\rho_0(\theta) > 0 \quad \Rightarrow \quad 0 < \rho(2\pi,\tilde{t}_\theta) < \frac{1}{KZ(2\pi)} \quad \forall \theta \in [0,2\pi].$$

(44)

The strict positivity condition $\rho_0(\theta) > 0$ always implies $\rho(2\pi,\tilde{t}_\theta) > 0$, as in the former case. Hence, we focus on the additional upper bound on the density $\rho(2\pi,\tilde{t}_\theta)$. For $\theta = 2\pi$, condition (44) implies (44), since $\tilde{t}_{2\pi} = 0$. Next, we proceed by induction on $\theta$: given $\tilde{\theta}$ and assuming that (44) is satisfied for all $\theta' > \tilde{\theta}$, we prove that (44) also holds at $\theta = \tilde{\theta}$ (provided that condition (44) is satisfied). Using (44) with condition (44) leads to

$$\rho(2\pi,\tilde{t}_\theta) < \frac{1}{KZ(\theta)} \exp \left( - \int_{\tilde{\theta}}^{\tilde{t}_\theta} J_0(t') K \frac{dZ(t')}{d\theta} d\theta \right).$$

Expressing the integral in the space variable along the characteristic curve yields

$$\rho(2\pi,\tilde{t}_\theta) < \frac{1}{KZ(\theta)} \exp \left( - \int_{\tilde{\theta}}^{2\pi} J_0(\omega) K \frac{dZ(\omega)}{d\theta} d\theta \right).$$

Since $K dZ/d\theta$ is negative, the flux $J_0$ can be replaced by its maximal value, that is, $J_0 \to \infty$ and the above inequality leads to

$$\rho(2\pi,\tilde{t}_\theta) < \frac{1}{KZ(\theta)} \exp \left( - \int_{\tilde{\theta}}^{2\pi} \frac{1}{Z(\theta)} d\theta \right).$$

(45)

The relation (45) is well-defined since $Z(\theta) \neq 0$ for all $\theta \in [0,2\pi]$. Finally, computing the integral in (45) implies that $\rho(2\pi,\tilde{t}_\theta) < 1/[KZ(2\pi)]$. Condition (39), equivalent to (44), is then satisfied. This concludes the proof.

When the conditions of Proposition 3 are not satisfied, condition (39) may fail to hold, in which case the flux has a finite escape time to infinity. In the case $KZ(2\pi) > 0$, this occurs when the density value $\rho(2\pi,t)$ approaches the critical value

$$\rho(2\pi,t) = \frac{1}{KZ(2\pi)}. $$

(46)

so that a high value of $J_0$ increases the velocity (and thus the flux) through the coupling. For $\rho(2\pi,t)$ satisfying (46), the flux is high enough to trigger a finite escape time phenomenon through the positive feedback between the flux and the velocity. This phenomenon is linked to the absorption phenomenon observed in a related model for a finite number of oscillators (see Section VII A).
B. Finite time convergence to a synchronous state

For increasing functions \( K Z(\theta) \), Theorem \( \ref{lem:Z} \) implies that the Lyapunov function \( \ref{lyapunov} \) is strictly increasing and a synchronous behavior is observed in finite time, for any initial condition. Either the flux \( J_0(t) \) becomes infinite in finite time or the density \( \rho(0, t) \) becomes infinite in finite time. The finite time convergence is established in Proposition \( \ref{prop:finite_time_convergence} \). As a preliminary to this result, we need the following lemma.

**Lemma 1.** The Lyapunov function \( \ref{lyapunov} \) satisfies

\[
\mathcal{V} = \|q - q^*\|_{L^1} \leq 4\pi - 2q_{\text{min}}
\]

with

\[
q_{\text{min}} = \min \left( \min_{x \in [0,1]} q(x), \ min_{x \in [0,1]} q^*(x) \right).
\]

**Proof.** Given \( \ref{lyapunov} \), the Lyapunov function writes

\[
\mathcal{V} = \sum_{i=1}^{N_e+1} (-1)^i \int_{x_c^{(i-1)}}^{x_c^{(i)}} (q - q^*) \, dx.
\]

The highest value is obtained when \( q = q_{\text{min}} \) for \( x \in [x_c^{(i-1)}, x_c^{(i)}], \) with \( i \) odd, and \( q^* = q_{\text{min}} \) for \( x \in [x_c^{(i-1)}, x_c^{(i)}], \) with \( i \) even. This leads to the inequality

\[
\mathcal{V} \leq \sum_{i=1}^{N_e+1} \int_{x_c^{(i-1)}}^{x_c^{(i)}} q \, dx - q_{\text{min}} \sum_{i=1}^{N_e+1} (x_c^{(i)} - x_c^{(i-1)}) + \sum_{i=1}^{N_e+1} \int_{x_c^{(i-1)}}^{x_c^{(i)}} q^* \, dx - q_{\text{min}} \sum_{i=1}^{N_e+1} (x_c^{(i)} - x_c^{(i-1)}).
\]

Moreover, one has

\[
\sum_{i=1}^{N_e+1} \int_{x_c^{(i-1)}}^{x_c^{(i)}} q \, dx + q_{\text{min}} \sum_{i=1}^{N_e+1} (x_c^{(i)} - x_c^{(i-1)}) \leq \int_0^1 q \, dx = Q(1) - Q(0) = 2\pi,
\]

and an equivalent expression also holds for \( q^* \). Next, injecting \( \ref{Z} \) in inequality \( \ref{V} \) yields

\[
\mathcal{V} \leq 2\pi - 2q_{\text{min}} \sum_{i=1}^{N_e+1} (x_c^{(i)} - x_c^{(i-1)}) + 2\pi - 2q_{\text{min}} \sum_{i=1}^{N_e+1} (x_c^{(i)} - x_c^{(i-1)})
\]

or

\[
\mathcal{V} \leq 4\pi - 2q_{\text{min}}(x_c^{(N_e+1)} - x_c^{(0)}) = 4\pi - 2q_{\text{min}}.
\]

With the help of Theorem \( \ref{lem:Z} \) and of the preceding lemma, the following proposition establishes the finite time convergence to the synchronous state.

**Proposition 4.** Consider the transport PDE \( \ref{transport} \) and assume that the sensitivity function \( Z(\theta) \) is such that (i) the stationary density \( \ref{stationary} \) exists, (ii) \( K \frac{dZ}{d\theta} > 0 \) \( \forall \theta \in [0, 2\pi] \), and (iii) either \( \frac{d^2Z}{d\theta^2} \geq 0 \) \( \forall \theta \in [0, 2\pi] \) or \( \frac{d^2Z}{d\theta^2} \leq 0 \) \( \forall \theta \in [0, 2\pi] \). Then all solutions converge in finite time to a synchronous state. That is, if \( K Z(0) \geq 0 \), the flux verifies \( J(0, t_{\text{fin}}) = \infty \), or if \( K Z(0) < 0 \), the density verifies \( \rho(0,t_{\text{fin}}) = \infty \), with \( t_{\text{fin}} < \infty \).

**Proof.** An infinite flux \( J_0(t) \) or an infinite density \( \rho(0, t) \) is obtained when the density \( \rho(2\pi, t) \) reaches a critical value. If \( K Z(0) \geq 0 \), the value \( K Z(2\pi) \) is positive since \( K Z \) is increasing. Hence, the flux \( J_0 \) becomes infinite when the density \( \rho(2\pi, t) \) exceeds the critical value \( \ref{critical} \). If \( K Z(0) < 0 \), the velocity \( \ref{velocity} \) at \( \theta = 0 \) is equal to zero when the flux reaches the value \( J_0(t) = \omega/K Z(0) \) or equivalently, given \( \ref{critical} \), when the density reaches the value

\[
\rho(2\pi, t) = \frac{1}{K Z(2\pi) - K Z(0)}.
\]
With a velocity equal to zero at $\theta = 0$, the relationship \([3]\) implies that the density is infinite at $\theta = 0$. (If $K Z(0) < 0$ along with $K Z(2\pi) > 0$, the reader will notice that the value \([46]\) has no importance, since \([49]\) is greater than \([44]\).)

Next, we show that the density $\rho(2\pi, t)$ must necessarily reach the critical value \([40]\) or \([49]\) in finite time $t_{\text{fin}}$. Let consider a characteristic curve $\Theta(t)$, with $\Theta(0) = 0$ and $\Theta(2\pi) = 2\pi$ and assume that $\rho(\theta, t) > 0 \in C^0(L^1(0, 2\pi), [L, T])$. Applying Theorem 1 and integrating (25), one has

$$\min_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \int_0^T J_0(t) \, dt \leq \int_0^T \frac{1}{V} \, dV \leq \max_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \int_0^T J_0(t) \, dt .$$

Since $[L, T]$ is the time interval corresponding to the complete evolution of an oscillator from $\theta = 0$ to $\theta = 2\pi$, the integral of the flux $J_0$ in the above equation is equal to one and it follows that

$$\exp \left[ \min_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \right] \leq \frac{\int_0^T \exp(V(\theta)) \, d\theta}{\int_0^T \exp(V(\theta)) \, d\theta} \leq \exp \left[ \max_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \right] .$$

The condition $K dZ/d\theta > 0$ implies that the Lyapunov function strictly increases within the time interval $[L, T]$. Considering $n$ successive intervals $[0, T_1], \ldots, [T_{n-1}, T_n]$, with $T_{i+1} = T_i$, one obtains

$$\frac{\int_0^{T_n} \exp(V(\theta)) \, d\theta}{\int_0^{T_n} \exp(V(\theta)) \, d\theta} \geq \exp \left[ n \min_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \right] .$$

Hence, a given value $V > V(0)$ is reached within at most $n_{\text{max}}$ time intervals, with

$$n_{\text{max}} \leq \frac{\log \left[V(0)/\int_0^{T_n} \exp(V(\theta)) \, d\theta\right]}{\min_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right)} .$$

Since the time length of each interval $[T_i, T_{i+1}]$, $i = 1, \ldots, n_{\text{max}}$, is finite, any value $V < 4\pi$ is reached in finite time. By Lemma 4 $V = V(0)$ implies that $q_{\text{min}} \leq (4\pi - V)/2$. When considering values $V$ close to $4\pi$, the minimum of the quantile density $q$ reaches in finite time any given value close to zero. Given (45), this implies that the maximum of the density $\rho_M$ reaches in finite time any given value (provided that the critical value (40) or (49) is not already reached). In particular, the value

$$\rho_M = \rho_c \exp \left[ \max_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \right] ,$$

(50)

with $\rho_c$ denoting the critical value (40) or (49), is reached in finite time. Then, the maximum value $\rho_M$ (obtained at $\theta_M$ at time $t_M$) decreases along the characteristic curve $\Theta(t)$, with $\Theta(t_M) = \theta_M$. Solving the total derivative equation (10), the variation along the characteristic curve is bounded:

$$\rho(2\pi, t_{\text{fin}}) = \rho_M \exp \left( - \int_{t_M}^{t_{\text{fin}}} J_0(t') \left. \frac{dZ}{d\theta} \right|_{\theta(t')} \, dt' \right) \geq \rho_M \exp \left( - \max_{\theta \in [0, 2\pi]} \left( K \frac{dZ}{d\theta} \right) \right) ,$$

where the inequality is obtained since the integral of $J_0$ is less than one. With $\rho_M$ given by (50), the density exceeds the critical value in finite time $t_{\text{fin}}$, which concludes the proof. \( \square \)

Proposition 3 shows that oscillators with monotone increasing functions $K Z$ converge to a synchronous behavior corresponding to a solution with a finite escape time to infinity. If the coupling is excitatory ($0 \leq K Z(0) < K Z(2\pi)$), the flux $J_0$ is infinite at time $t_{\text{fin}}$, as well as the velocity $v(2\pi, t_{\text{fin}})$. This behavior is the phenomenon described at the end of Section 7.1 for particular (ill-defined) initial conditions. However, there is a difference for increasing functions $K Z$: the finite time explosion of solutions is observed with any initial condition. If the coupling is inhibitory ($K Z(0) < 0$), the velocity $v(0, t)$ reaches the value zero at time $t_{\text{fin}}$ and the oscillators accumulate at $\theta = 0$, yielding an infinite density $\rho(0, t_{\text{fin}})$.

VI. APPLICATIONS

There is an abundant literature (e.g. [3, 31]) on analytical and numerical computations of sensitivity functions $Z(\theta)$, also called phase response curves (PRC). Integrate-and-fire oscillators [11, 14] and oscillators on a limit cycle close to a homoclinic bifurcation [23] are characterized by a monotone sensitivity function. Under an instantaneous impulsive coupling, all the results of this paper directly apply to those models.
A. Integrate-and-fire oscillators

Integrate-and-fire models are well-known models commonly used in computational neuroscience. Integrate-and-fire oscillators are characterized by the one-dimensional phase dynamics

$$\dot{z} = F(z) + u^{\text{imp}}(t), \quad \text{with } F(z) > 0 \quad \forall z \in [\underline{z}, \overline{z}],$$

(51)

where the impulsive coupling term $u^{\text{imp}}(t)$ is given by (7). Whenever the state reaches the upper threshold $\overline{z}$, it is instantaneously reset to the lower threshold $\underline{z}$ and the oscillator is said to fire. In the case of neurons, the coupling models the effect of the action potential of a firing neuron on the other neurons.

When setting $\theta = 0$ at $z = \underline{z}$ and $\theta = 2\pi$ at $z = \overline{z}$, the sensitivity function expresses as (see [3] for instance)

$$Z(\theta) = \frac{d\theta}{dz} = \frac{\omega}{F[z(\theta)]}.$$

The results of Proposition 1 are applied to this model. For an excitatory coupling $K > 0$, the condition $K Z(\theta) \geq 0 \forall \theta \in [0, 2\pi]$ holds so that $r = 0$. Then, the condition (14) of Proposition 1 writes

$$\int_{0}^{1} \frac{1}{K Z(\theta)} d\theta = \int_{\underline{z}}^{\overline{z}} \frac{1}{K} dz > 1,$$

or equivalently

$$K < \overline{z} - \underline{z}.$$

This upper bound on the coupling strength leads to a more intuitive interpretation of the condition (14) in Proposition 1. Through the time integration of (51), we notice that the whole population passing across $\theta = 0$ induces (through the coupling) a global state increase equal to $K$. As a consequence, setting $K \geq \overline{z} - \underline{z}$ leads to an infinite velocity of the oscillators and the asynchronous state cannot exist.

The results of Proposition 2 and Proposition 4 apply to the most popular integrate-and-fire model, that is, the leaky integrate-and-fire model [14], characterized by

$$F(z) = R + S z > 0, \quad \text{with } z \in [0, 1].$$

The corresponding sensitivity function is given by

$$Z(\theta) = \frac{\omega}{R} \exp \left[ -\frac{S\theta}{\omega} \right]$$

with

$$\omega = 2\pi S \left[ \ln \left( \frac{R + S}{R} \right) \right]^{-1}.$$

The sensitivity function has a curvature with constant sign and the function $K Z$ is monotone. The parameters $K S > 0$ correspond to $K \frac{dZ}{d\theta} < 0$ (Figure 6) and the assumptions of Proposition 2 are satisfied. On the other hand, $K S < 0$ leads to $K \frac{dZ}{d\theta} > 0$ and the assumptions of Proposition 4 are satisfied. The results for this important model are summarized in the following proposition.

**Proposition 5.** For a continuum of leaky integrate-and-fire oscillators with an instantaneous impulsive coupling, the asynchronous state exists provided that $-\infty < K < 1$. If $K S > 0$, the Lyapunov function (16) is strictly decreasing and the oscillators exponentially converge to the asynchronous state. If $K S < 0$, the Lyapunov function is strictly increasing and the oscillators converge to a synchronous state in finite time.

**Proof.** The result follows from the results of Proposition 1, Proposition 2 and Proposition 4.

B. Limit cycle close to a homoclinic bifurcation

The results presented in the paper also apply to oscillators characterized by a limit cycle close to a homoclinic bifurcation. Homoclinic bifurcation occurs when there exists, for a given parameter value, a homoclinic orbit to a saddle point with real eigenvalues. At the bifurcation, a limit cycle appears. Assuming that there is a single unstable eigenvalue $\lambda_u$ such that $\lambda_u < |\lambda_s|$, with $\lambda_s$ being the stable eigenvalues, the limit cycle is stable [8]. Since the
unstable direction is slower than the other (stable) directions, the sensitivity function is approximatively computed and expresses as (see [3])

$$Z(\theta) = C \omega \exp\left(\frac{2\pi \lambda u}{\omega}\right) \exp\left(-\frac{\lambda u}{\omega} \theta\right),$$

where $C > 0$ is a model-dependent constant. The sensitivity function $Z$ is strictly decreasing and has a positive curvature. It follows that such oscillators, interacting through an excitatory impulsive coupling ($K > 0$), verify the hypothesis of Proposition 2. Oscillators near a homoclinic bifurcation exponentially converge towards the asynchronous state if the coupling is excitatory, provided that the asynchronous state exists. By Proposition 4, the asynchronous state exists if and only if

$$-\infty < K < \frac{1}{C\lambda u} \left[1 - \exp\left(-\frac{2\pi \lambda u}{\omega}\right)\right].$$

For an inhibitory coupling ($K < 0$), Proposition 4 implies that the oscillators reach a synchronous state in finite time.

In the popular Morris-Lecar model [17], which is among the most widely used conductance-based models in computational neuroscience, a homoclinic bifurcation can occur for low external currents [23, 30]. The behavior of pulse-coupled Morris-Lecar oscillators (close to the bifurcation point) is then characterized by the results developed in the present paper.

VII. DISCUSSION

In the sequel, we explore connections between the analysis of the present paper with earlier results for the popular Peskin and Kuramoto models. We also discuss the relevance of an instantaneous impulsive coupling and the role of the assumptions on the phase response curve.

A. A link with the Peskin model

First derived to model cardiac pacemaker cells, Peskin oscillators are leaky integrate-and-fire oscillators with a discrete impulsive coupling [21]. Whenever an oscillator fires, the state variable of any other oscillator is instantaneously increased by a positive constant value $\epsilon$. The continuous model studied in the present paper, with the leaky integrate-and-fire dynamics (see Section VIA)

$$\dot{z} = R + Sz + u^\text{imp}(t) \quad \text{for } z \in [0,1]$$

and the impulsive coupling $u^\text{imp}(t)$ given by (7), is the continuous limit ($N \to \infty$) of the Peskin model. When the number of oscillators is finite, the density is a series of weighted Dirac functions and, for $N$ distinct oscillators, expresses as

$$\rho(\theta, t) = \frac{1}{N} \sum_{i=1}^{N} \delta[\theta - \theta_i(t)],$$

Figure 6: The function $K Z(\theta)$ has a curvature with constant sign. When $KS > 0$, the function $K Z$ is decreasing and the conditions of Proposition 2 are satisfied. When $KS < 0$, the function $K Z$ is increasing and the conditions of Proposition 4 are satisfied.
where each $\theta_i(t)$ is the respective phase of each oscillator \[ \text{[13]} \]. The flux at $\theta = 0$ also writes

$$J_0(t) = \frac{1}{N} \sum_{k=0}^{\infty} \delta(t - t_k^*) ,$$

with an oscillator crossing $\theta = 0$ at each (firing) time $t_k^*$. Then the coupling is given by

$$a^{\text{imp}}(t) = KJ_0(t) = \frac{K}{N} \sum_{k=0}^{\infty} \delta(t - t_k^*) , \quad \text{for} \ x \in [0, 1].$$

The coupling is realized by the transmission of $\delta$-like pulses between the oscillators. The pulses increase the state by a finite amount $K/N$ and the dynamical behavior therefore corresponds to the discrete Peskin model, with $\epsilon = K/N$. The leaky dynamics \[ \text{[62]} \], characterized by an instantaneous coupling and applied to a continuum of oscillators, are a good approximation of the Peskin model, provided that the number of oscillators is large enough.

The link between the Peskin model and the continuous model considered in this paper is reinforced by a similar dichotomic behavior. A phase-locked behavior, observed with Peskin oscillators, is the discrete counterpart of the asynchronous state. At each firing, the $N$ Peskin oscillators are characterized by phase-locked phases $0 = \theta_1^* < \cdots < \theta_N^* < 2\pi$ so that they fire in a periodical way. In particular, the value $\theta_N^*$ can be well-approximated by the stationary flux $J^*$. The asymptotic behavior of the Peskin model is in agreement with the asymptotic behavior of the continuous model. The phase-locked behavior is described in \[ \text{[15]} \], where the global stability is shown for $S > 0$, or equivalently $\epsilon S > 0$. As shown in Section \[ \text{[IV B]} \] the Lyapunov function used in \[ \text{[15]} \] is the finite counterpart of the total variation distance studied here. When $S < 0$, or equivalently $\epsilon S < 0$, it is shown in \[ \text{[16]} \] that the Peskin oscillators achieve perfect synchrony for almost all initial conditions. The conditions are identical to the conditions of Proposition \[ \text{[5]} \].

In the (finite) Peskin model, the firing of one oscillator can trigger the firing of the second oscillator. The two oscillators then fire together and subsequently evolve as a single oscillator when the state difference between two oscillators is small enough (that is, less than $\epsilon$). This phenomenon is called an absorption. In the continuous model, the corresponding phenomenon is a finite escape time of the flux and arises when the upper-bound condition $\text{[16]}$ on the initial density is violated. Moreover, the condition which ensures the existence of the asynchronous state is $K < 1$ (see section \[ \text{[IV A]} \]). In the Peskin model, we recover the same condition $\epsilon = K/N < 1/N$. Since the average state difference between the oscillators is $1/N$, at least one absorption must occur when $\epsilon > 1/N$ and a phase-locked behavior of $N$ distinct oscillators cannot exist.

### B. Comparison with the Kuramoto model

The well-known Kuramoto model \[ \text{[12, 26]} \] is an approximation of weakly pulse-coupled oscillators with a sinusoidal sensitivity function. Indeed, it is shown in \[ \text{[13]} \], for a finite number of oscillators, that time-averaging the action of a weak impulsive coupling leads to the Kuramoto model. The “permanent” coupling of the Kuramoto model is then the time-averaged approximation of an impulsive coupling. It is therefore not surprising that (identical) Kuramoto oscillators are also characterized by a dichotomic behavior. This leads to an evident parallel between the behavior of Kuramoto oscillators and the behavior of pulse-coupled oscillators with a monotonically increasing sensitivity function. With a positive coupling strength (excitatory coupling), the oscillators asymptotically achieve perfect synchrony for almost every initial conditions, in the two models. With a negative coupling strength (inhibitory coupling), they adopt the opposite behavior, that is, a desynchronized (incoherent) state corresponding to the stationary asynchronous state.

The $L^2$-norm Lyapunov functional proposed in the present paper is to be contrasted with the $L^2$-norm Lyapunov functional of the Kuramoto model. For a continuum of (identical) Kuramoto oscillators, a Lyapunov functional for the continuity equation is the so-called order parameter, which is derived from the inner product of the $L^2$ Hilbert space. Indeed, the functional

$$0 \leq V(\rho) = \|\langle \rho, e^{i\theta}\rangle\|_2^2 = \left\| \int_0^{2\pi} \rho(\theta) e^{i\theta} d\theta \right\|^2 \leq 1$$

is strictly increasing (resp. decreasing) when the coupling is excitatory (resp. inhibitory).

### C. Instantaneous impulsive coupling

In the present paper, the coupling is assumed to be instantaneous, since a given oscillator influences the other oscillators only at the very instant when $\theta = 0$. Other models consider that the coupling is not instantaneous. For
instance, [1, 31] considered leaky integrate-and-fire oscillators with a coupling
\[ u(t) = K \int_{-\infty}^{t} \alpha^2(t-\tau) e^{\alpha(t-\tau)} J_0(\alpha \tau) \, d\tau. \] (53)

In case of a finite number of oscillators, the coupling is realized by means of a series of smoothed finite-width pulses corresponding to \( \alpha \)-functions \( \alpha^2/N \exp[\alpha(t)] \), instead of a series of \( \delta \)-like pulses. However, when \( \alpha \to \infty \), the coupling (53) is equivalent to the impulsive coupling (7) since the \( \alpha \)-functions tend to Dirac functions. The model investigated in the present paper is recovered and thereby appears to be the limit case (instantaneous coupling) of the model studied in [1, 31, 35] (non-instantaneous coupling).

Considering an instantaneous impulsive coupling is nevertheless relevant for many realistic systems characterized by negligible transmission time delays. In addition, the global analysis in the present paper does not seem to easily extend to the coupling (53). To the authors' knowledge, the convergence results obtained for non-instantaneous coupling are only local and based on linearization.

It is shown in [35] that some high frequency modes are not captured by the thermodynamic limit \( (N \to \infty) \) for the model (53), so that the stability properties are not preserved through the continuum approximation. The ratio \( \alpha/N \), which should be preserved, actually tends to zero when \( N \to \infty \). The case of an instantaneous coupling is characterized by the ratio \( \alpha/N = \infty \), so that the ratio remains constant in the thermodynamic limit. It turns out that, in this case, the thermodynamic limit is a correct approximation and the obtained stability results are in agreement with the behavior of \( N \) finite oscillators, as seen in Section VII A. This reinforces the interest of the instantaneous coupling.

D. The sensitivity function of oscillators

The results of the present paper only apply to strictly monotone sensitivity functions. As a consequence of the monotonicity, the sensitivity functions must be characterized by \( Z(0) \neq Z(2\pi) \) and therefore satisfy no periodicity condition. Sensitivity functions corresponding to phase response curves computed on a limit cycle are however periodic, unless the limit cycle is discontinuous. For instance, such a discontinuity is artificially created in the integrate-and-fire model (Section VII A). In other cases, the time spent on a part of the limit cycle is negligible with respect to the time spent on the rest of the cycle. In good approximation, the fast part can be replaced by a discontinuity, as in [3] for a limit cycle close to a homoclinic bifurcation (Section VII B). Relaxation models with separated time scales (such as FitzHugh-Nagumo oscillators, Van der Pol oscillators, ...), as well as spiking oscillators, are thereby characterized in good approximation by non-periodic sensitivity functions.

The hypothesis of Theorem 1 also imposes a mild assumption on the second derivative of the sensitivity function, which is a sufficient condition to ensure the strict variation of the Lyapunov function. The fulfillment of the condition is observed with several models of oscillators and can depend on the model parameters. When considering the quantity directly induced by the total variation distance between the densities themselves (22), it follows from (24) that the monotone variation of the Lyapunov function can be proved without the second derivative assumption. However, one has observed for such a function that its variation is not strict and that it remains constant when the boundary values of the density and the stationary density are equal.

VIII. CONCLUSION

In the present paper, an instantaneous impulsive coupling has been proposed to model the interaction between phase oscillators. The network is characterized by a dichotomic behavior, which emphasizes a strong parallel with Kuramoto phase oscillators. Two opposite behaviors are observed: either the oscillators achieve perfect synchrony (synchronous state) or they uniformly spread over \( S^1(0, 2\pi) \) (asynchronous state).

When studying the thermodynamic limit of the model, the population is represented by a continuous density. In this framework, a necessary and sufficient condition ensures existence and uniqueness of the stationary solution (asynchronous state).

The density evolves according to a transport equation with a nonlinear spatial feedback term due to the coupling. The main result of the paper proposes a Lyapunov function for the global convergence analysis of this transport equation. The proposed Lyapunov function is the total variation distance between quantile density functions. In addition, the two extreme values of the distance correspond to the two steady-state behaviors of the system, that is, the synchronous and asynchronous states. The stability results obtained for general phase oscillators are applied to particular models of importance (e.g. leaky integrate-and-fire model, Morris-Lecar model).

The main result of the paper applies under monotonicity assumptions on the oscillators sensitivity to the coupling. The time evolution of the proposed Lyapunov function is no longer monotone when those assumptions fail, even
though the observed dichotomy behavior seems more general. This restriction raises some open questions about the generalization of the Lyapunov function and about the use of the total variation distance for a larger class of transport equations.

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