On one semidiscrete Galerkin method for a generalized time-dependent 2D Schrödinger equation

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\textbf{A B S T R A C T}

An initial–boundary value problem for a generalized 2D Schrödinger equation in a rectangular domain is considered. Approximate solutions of the form \( c_1(x_1, t)\chi_1(x_1, x_2) + \cdots + c_N(x_1, t)\chi_N(x_1, x_2) \) are treated, where \( \chi_1, \ldots, \chi_N \) are the first \( N \) eigenfunctions of a 1D eigenvalue problem in \( x_2 \) depending parametrically on \( x_1 \) and \( c_1, \ldots, c_N \) are coefficients to be defined; they are of interest for nuclear physics problems. The corresponding semidiscrete Galerkin approximate problem is stated and analyzed. Uniform-in-time error bounds of arbitrarily high orders \( O(N^{-\theta} \log N) \) in \( L^2 \) and \( O(N^{-\theta-1} \log^{1/2} N) \) in \( H^2, \theta > 1 \), are proved.

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1. Introduction

The description of large amplitude collective motion in atomic nuclei such as nuclear fission remains one of the most challenging problems in contemporary nuclear physics. Under specific physical assumptions, in particular in the low energy regime, the problem is amenable to the resolution of the time-dependent problem for a Schrödinger–like equation written in the space of \( M \) so-called collective variables \( q = (q_1, q_2, \ldots, q_M) \) \cite{1-3}:

\[
\frac{\partial v}{\partial t}(q, t) = \mathcal{H}^{(M)}(q, t), \quad \mathcal{H}^{(M)} := -\sum_{i,j=1}^{M} \frac{\partial}{\partial q_i} \left( B_{ij}(q) \frac{\partial}{\partial q_j} \right) + V(q),
\]

where the inertia coefficients \( B_{ij}(q) \) and the potential \( V(q) \) are known functions, for an appropriate initial condition \( v(q, t = 0) = v_0(q) \) (\( t \) is the imaginary unit). This kind of variable coefficient hamiltonian has also been used to analyze the problem of multidimensional collective tunneling \cite{4,5}.

In spite of the fact that there exist a lot of numerical methods for solving the problem, physical insights into the solutions can be gained by attempting to expand them in a set of stationary functions which solve the corresponding eigenvalue problem in a reduced collective space of dimension \( M' < M \). This technique has been employed in a simple case in \cite{5}. In the particular but practically important case \( M = 2 \) and \( M' = 1 \) such a method amounts to analysis techniques exploiting finite expansions of the form

\[
v^{(N)}(x_1, x_2, t) = c_1(x_1, t)\chi_1(x_1, x_2) + \cdots + c_N(x_1, t)\chi_N(x_1, x_2).
\]  \hspace{1cm} (1)

Here the basis functions \( \chi_1, \ldots, \chi_N \) are the first \( N \) eigenfunctions of a 1D eigenvalue problem in \( x_2 \) depending parametrically on \( x_1 \) and \( c_1, \ldots, c_N \) are corresponding coefficients. Advantages of this form are that physical meaning can be assigned to

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2. An initial–boundary value problem and an auxiliary 1D eigenvalue one

1. We consider the initial–boundary value problem for a generalized time-dependent Schrödinger equation

$$i\rho D_t \psi = \mathcal{H}\psi := -\sum_{i,j=1}^2 D_i (k_{ij} D_j \psi) + V \psi \quad \text{in } \Omega \times \mathbb{R}^+,$$

$$\psi|_{\partial \Omega \times \mathbb{R}^+} = 0, \quad \psi|_{t=0} = \psi^0(x) \quad \text{on } \Omega := I_1 \times I_2 := (0, X_1) \times (0, X_2).$$

The coefficients are real and satisfy $\rho, k_{ij} \in L^\infty(\Omega), V \in L^p(\Omega)$ for some $p > 1$. Hereafter we exploit standard (generally complex) Lebesgue, Sobolev and related function spaces. Moreover, the matrix $(k_{ij})_{i,j=1}^2$ is symmetric and positive definite uniformly in $\Omega$ and we also have $V \geq 0$ and $\rho(x) \geq \delta > 0$ in $\Omega$.

By definition, a weak solution to this IBVP with the properties $\psi \in C(\mathbb{R}^+; H_0^1(\Omega))$ and $D_t(\rho \psi) \in C(\mathbb{R}^+; H^{-1}(\Omega))$ satisfies the integral identity

$$\langle D_t(\rho \psi)(\cdot, t), \varphi(\cdot) \rangle_{\mathcal{L}_2} = \mathcal{L}_2(\psi(\cdot, t), \varphi(\cdot)) \quad \text{for any } \varphi \in H_0^1(\Omega) \quad \text{and } t \geq 0,$$

with the bounded and positive definite on $H_0^1(\Omega) \times H_0^1(\Omega)$ sesquilinear form

$$\mathcal{L}_2(w, \varphi) := \int_{\Omega} \left( \sum_{i,j=1}^2 k_{ij} D_i w \cdot D_j \varphi^* + V w \varphi^* \right) \, dx,$$

together with the initial condition $\psi|_{t=0} = \psi^0 \in H_0^1(\Omega)$. Hereafter $(\cdot, \cdot)$ is the duality bracket on $H^{-1}(U) \times H_0^1(U)$, the abbreviations like $D_t w \cdot u = (D_t w) u$ allow us to avoid extra brackets and $\varphi^*$ is the complex conjugate of $\varphi$.

This solution exists and is unique. Moreover, the following conservation laws and bound hold:

$$\|\sqrt{\rho} \psi(t, \cdot)\|_{\mathcal{L}_2(\Omega)} = \|\sqrt{\rho} \psi_0\|_{\mathcal{L}_2(\Omega)}, \quad \|\psi(t, \cdot)\|_{\mathcal{L}_2(\Omega)} = \|\psi_0\|_{\mathcal{L}_2(\Omega)} \quad \text{for any } t \geq 0,$$

$$\sup_{t \geq 0} \|D_t(\rho \psi)(\cdot, t)\|_{H^{-1}(\Omega)} \leq K\|\psi_0\|_{\mathcal{L}_2(\Omega)},$$

where $\|w\|_{\mathcal{L}_2(\Omega)} := \|\mathcal{L}_2(w, w)\|^{1/2}$ is the equivalent energy norm in $H_0^1(\Omega)$. The existence and the conservation laws can be proved, for example, by the Fourier method using expansions with respect to the system of eigenfunctions of the corresponding eigenvalue problem; the expansion for $\psi$ converges in $H_0^1(\Omega)$ uniformly in $t \geq 0$. The uniqueness can be proved by the energy method.

2. To solve the IBVP (2) and (3), we introduce, for physical reasons, an auxiliary 1D eigenvalue problem with respect to $x_2$ depending on a parameter $x_1 \in I_1$:

$$\mathcal{H}_2 \chi := -D_2 (k_0 D_2 \chi) + V_0 \chi = \alpha \rho_0 \chi \quad \text{on } I_2, \quad \chi|_{x_2=0, x_1} = 0,$$

where the coefficients $k_0, V_0$ and $\rho_0$ are real and such that $k_0, \rho_0 \in L^\infty(\Omega), V_0 \in \tilde{L}^{1,\infty}(\Omega)$ and we also have $k_0 \geq \delta, V_0 \geq 0$ and $\rho_0 \geq \delta$. Hereafter $\tilde{L}^{1,\infty}(\Omega)$ is the anisotropic Lebesgue space equipped with the norm $\|w(x_1, x_2)\|_{\tilde{L}^{1,\infty}(\Omega)} := \|w(x_1, x_2)\|_{L^{1,\infty}(\Omega)} \leq \|w\|_{\mathcal{L}_2(\Omega)}, t = 1, 2$.

According to well-known results, for almost all (a.a.) $x_1 \in I_1$, the problem has a sequence of eigenvalues that are real and such that $0 < \alpha_1(x_1) < \cdots < \alpha_\ell(x_1) < \cdots, \alpha_\ell(x_1) \to \infty$ as $\ell \to \infty$, and the corresponding real eigenfunctions $\{\chi_\ell(x_1, \cdot)\}_{\ell=1}^\infty$ form the orthonormal basis in $L^2(I_2)$ with the weight $\rho_0(x_1, \cdot)$:

$$\int_{I_2} (\chi_k(x_1, \cdot) \rho_0(x_1, x_2) \, dx_2 = 0 \quad \text{for any } k \neq \ell, \quad \int_{I_2} (\chi_\ell^2 \rho_0) (x_1, x_2) \, dx_2 = 1 \quad \text{for any } k.$$
Proposition 1. For comparison, notice that below we omit the symbol $H_2$.

Let $\mathcal{P}_N: L^2(\Omega) \to S_N^0$ and $\mathcal{P}_N^{(1)}: H_2^0(\Omega) \to S_N$ be projectors such that

$$\int_\Omega (w - \mathcal{P}_N w) \varphi \rho_0 \, dx = 0, \quad \mathcal{L}_\varphi (w - \mathcal{P}_N^{(1)} w, \varphi) = 0 \quad \text{for any } \varphi \in S_N.$$

To study the approximation properties of $\mathcal{P}_N$ and $\mathcal{P}_N^{(1)}$, we define a family of complex Hilbert spaces:

$$H^{0,\theta}(\Omega) := \left\{ w \in L^2(\Omega): \|w\|_{H^{0,\theta}(\Omega)} := \left[ \int_\Omega \left( \sum_{\ell=0}^N \alpha_\ell^2 \|\tilde{w}_\ell(x)\|^2 \right) \|w_\ell\|^2 \, dx \right]^{1/2} \right\}$$

for $\theta \geq 0$ (in the spirit of [10]). The spaces $H^{0,\theta}(\Omega)$ strictly enlarge as $\theta$ decreases. Notice that

$$\|w\|_{\tilde{H}^{0,\theta}(\Omega)} = \|\sqrt{\rho_0} w\|_{\tilde{H}^{0,\theta}(\Omega)} \quad \text{and} \quad \|w\|_{H^{0,\theta}(\Omega)} = \|w\|_{\tilde{H}^{0,\theta}(\Omega)}$$

$$\quad \text{and} \quad \|w\|_{H^{0,\theta}(\Omega)} = \|w\|_{H^{0,\theta}(\Omega)},$$

and $H^{0,\theta}(\Omega) = L^2(\Omega)$ and $H^{0,1}(\Omega) = \{ w \in L^2(\Omega): \|w\| = \|w_1 \times \omega\| \}$ by the collections of elements. Next, the space $H^{0,2}(\Omega)$ consists in functions $w \in H^{0,1}(\Omega)$ having a derivative $D_2(w) \in L^1(\Omega)$ and such that $\mathcal{H}_0 w \in L^2(\Omega)$, with the norm

$$\|w\|_{H^{0,2}(\Omega)} = \|\rho_0^{-1/2} \mathcal{H}_0 w\|_{\tilde{H}^{0,2}(\Omega)}.$$ 

The last two conditions on $w$ are reduced to the simple condition $D_2w \in L^2(\Omega)$ provided that $D_2k_0, V_0 \in L^2(\Omega)$. Moreover, $H^{0,\theta}(\Omega)$ for $\theta > 2$ is the space of functions $w \in H^{0,2}(\Omega)$ such that $\rho_0^{-1} H_0 w \in H^{0,\theta-2}(\Omega)$ and

$$\|w\|_{H^{0,\theta}(\Omega)} = \|\rho_0^{-1} \mathcal{H}_0 w\|_{H^{0,\theta-2}(\Omega)}.$$

We define one more family of (real) Hilbert spaces, for $\theta \geq 0$ and $a.a. x_1 \in I_1$:

$$H^{0,\theta}_x(\Omega) := \left\{ \xi \in L^2(\Omega): \|\xi\|_{H^{0,\theta}_x(\Omega)} := \left[ \int_\Omega \left( \sum_{\ell=0}^N \alpha_\ell^2 \|\tilde{\xi}_\ell(x_1)\|^2 \right) \|\xi\|^2 \, dx \right]^{1/2} \right\}, \quad \tilde{\xi}_\ell := \int_\Omega \xi \tilde{w}_\ell(x_1) \rho_0 \, dx_2;$$

below we omit the symbol $[x_1]$ for brevity. The spaces have properties similar to those listed above; in particular, $H^{0,\theta}(I_2) = L^2(\Omega)$ and $H^{0,1}(I_2) = H_2^0(I_2)$ up to equivalence of norms (uniformly in $x_1 \in I_1$) and

$$\|\xi\|_{H^{0,\theta}(I_2)} = \|\sqrt{\rho_0} \xi\|_{L^2(I_2)}.$$

We introduce the regularity condition

$$D_2k_0, D_1\rho_0 \in L^\infty(\Omega), \quad D_2k_0^{(1)}, V_0, D_1V_0, D_2\rho_0^{(1)} \in L^2(\Omega),$$

where $k_0^{(1)} := (D_1k_0)/k_0$ and $\rho_0^{(1)} := (D_1\rho_0)/\rho_0$.

Let $\mathcal{H}_0 \chi := -D_2(D_1k_0 \cdot D_2\chi) + D_1V_0 \cdot \chi$. We introduce one more regularity condition, for some $\beta \geq 0$:

$$\rho_0^{-1} \mathcal{H}_0 \chi \|\tilde{w}_\ell\|_{L^2(I_2)} \leq K_{1,\beta} \alpha_\ell^{1/2+1}, \quad \|\rho_0^{-1} \mathcal{H}_0 \chi \|_{H^{0,\theta}(\Omega)} \leq K_{1,\beta} \alpha_\ell^{1/2+1}, \text{ for any } \ell \geq 1.$$

For comparison, notice that $\rho_0^{-1} \mathcal{H}_0 \chi \|\tilde{w}_\ell\|_{L^2(I_2)} = \alpha_\ell \|\tilde{w}_\ell\|_{H^{0,\theta}(\Omega)} = \alpha_\ell^{1/2+1}$, for any $\beta \geq 0$ and $\ell \geq 1$.

Let $0 \leq \beta_0 < \beta_1$. If one of the inequalities is valid for $\beta = \beta_0, \beta_1$, then this is valid for any $0 \leq \beta_0 < \beta \leq \beta_1$ (with $K_{1,\beta} = K_{1,\beta_0} K_{1,\beta_1}$). Therefore one may consider only integer values of $\beta$. In particular, condition (10) holds for $\beta = 0$ under regularity condition (9).

We introduce the following regularity and spectral gap conditions:

$$D_1\alpha_k \in L^1(I_1), \quad D_1\chi_k(x_1, \cdot) \in H^0_2(I_2), \quad \sqrt{\alpha_k(x_1)} - \sqrt{\alpha_k(x_1)} \geq \delta(\ell - k), \quad \text{for any } 1 \leq k < \ell, \text{ on } I_1.$$
(ii) Under additional assumption \( w|_{x_1=0,x_2} = 0 \) and regularity condition
\[
\kappa_{ij}, D_k \kappa_{ij} \in L^\infty(\Omega) \quad \text{for any } i, j, \quad V \in L^2(\Omega), \tag{13}
\]
the following \( L^2(\Omega) \)-error bound holds as well:
\[
\|w - P_{N}^{(1)} w\|_{L^2(\Omega)} \leq KN^{-\theta} \log N \left( \|w\|_{W^{\theta,\theta}(\Omega)} + \|D_1 w\|_{W^{\theta-1,\theta}(\Omega)} \right). \tag{14}
\]
Hereafter quantities \( K \) can depend on \( \theta \) but are independent of \( N \), and \( N \geq 2 \) in error bounds.

Notice that, for Claim 2 (ii), bound (12) easily implies a similar one for \( P_{N}^{(1)} \) replacing \( P_{N} \) and bound (14) follows from this on applying the well-known so-called Nitsche trick.

3. A semidiscrete Galerkin method

Our semidiscrete Galerkin method for the IBVP (2) and (3) exploits an approximate solution \( y^{(N)}(\cdot, t) \in S_{N} \) for any \( t \geq 0 \), more precisely, of the form
\[
y^{(N)}(x_1, x_2, t) = \sum_{\ell=1}^{N} c_{\ell}(x_1, t) \chi_{\ell}(x_1, x_2) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{15}
\]
with the vector-function of the coefficients \( c := (c_1, \ldots, c_N)^T \) such that
\[
c \in C\left( \mathbb{R}^+; \left[ H^1_{0}(I_1) \right]^N \right), \quad \text{and } \quad D_t (Mc) \in C\left( \mathbb{R}^+; \left[ H^{-1}(I_1) \right]^N \right), \tag{16}
\]
where the \( N \times N \) mass matrix \( M \) has the entries, for any \( 1 \leq k \leq N \) and \( 1 \leq \ell \leq N \),
\[
m_{k\ell} = \int_{I_1} \chi_{k} \chi_{\ell} \rho \, dx \quad \text{on } I_1, \quad \|m_{k\ell}\|_{L^\infty(I_1)} \leq K\|\chi_{k}\|_{L^2(I_1)} \|\chi_{\ell}\|_{H^1(I_1)}.
\]
We seek an approximate solution satisfying the integral identity
\[
t D_t \int_{\Omega} \rho y^{(N)}(\varphi) \, dx = \mathcal{L}_N \left( y^{(N)}(\cdot, t), \varphi(\cdot) \right) \quad \text{for any } \varphi \in S_N \text{ and } t \geq 0 \tag{17}
\]
(compare with (4)), together with the initial condition
\[
y^{(N)}|_{t=0} = y^{(N),0} := \sum_{\ell=1}^{N} c_{\ell}^0 \chi_{\ell} \in S_N, \tag{18}
\]
where \( y^{(N),0} \) is a given approximation of \( \psi^0 \), with a vector-function of the coefficients \( e^0 := (e_1^0, \ldots, e_N^0)^T \).

Proposition 2. Properties (16) imply that
\[
y^{(N)} \in C(\mathbb{R}^+; H^0_{0}(\Omega)), \quad D_t \int_{\Omega} \rho y^{(N)}(\varphi) \, dx \in C(\mathbb{R}^+) \quad \text{for any } \varphi \in S_N,
\]
so identity (17) and the initial condition (18) are well defined.

Moreover, the approximate Galerkin time-dependent problem (17) and (18) is equivalent to an IBVP for the time-dependent Schrödinger-like system of 1D (in space) equations
\[
t D_t (Mc) = \mathcal{H}_N c \quad \text{in } C\left( \mathbb{R}^+; \left[ H^{-1}(I_1) \right]^N \right), \quad \text{and } \quad c|_{t=0} = e^0 \in \left[ H^1_{0}(I_1) \right]^N, \tag{19}
\]
where \( \mathcal{H}_N c := -D_t \left( AD_t c + A^{(0)} c \right) + A^{(0)} D_1 c + B c \). Here the \( N \times N \)-matrices \( A, A^{(0)} \) and \( B \) have the entries, for \( 1 \leq k \leq N \) and \( 1 \leq \ell \leq N \),
\[
a_{k\ell} = \int_{I_2} \kappa_{11} \chi_{k} \chi_{\ell} \, dx_2, \quad a_{k\ell}^{(0)} = \int_{I_2} \chi_{k} (\kappa_{11} D_1 \chi_{\ell} + \kappa_{12} D_2 \chi_{\ell}) \, dx_2,
\]
\[
b_{k\ell} = \int_{I_2} \left( \sum_{j=1}^{2} \kappa_{ij} \chi_{k} \chi_{\ell} + C_{i2} \chi_{k} \chi_{\ell} \right) \, dx_2,
\]
satisfying the bound \( \|a_{k\ell}\|_{L^2(I_2)} + \|a_{k\ell}^{(0)}\|_{L^2(I_2)} + \|b_{k\ell}\|_{L^2(I_2)} \leq K \|\chi_{k}\|_{L^2(I_2)} \|\chi_{\ell}\|_{L^2(I_2)} \). The matrices \( A \) and \( B \) are self-adjoint and positive definite uniformly on \( I_1 \).
Remark 1. Equation in (19) can also be rewritten in the integral identity form
\[ t \left( D_t (\textbf{M}c)(\cdot,t), \textbf{d}(\cdot) \right)_{L^2} = \mathcal{L}_1^N (\textbf{c}(\cdot,t), \textbf{d}(\cdot)) \quad \text{for any} \quad \textbf{d} \in \left[ H^1_0(I_1) \right]^N \quad \text{and} \quad t \geq 0, \]
with the bounded and positive definite on $H^1_0(I_2) \times H^1_0(I_2)$ sesquilinear form
\[ \mathcal{L}_1^N (\textbf{c}, \textbf{d}) := \int_{I_1} \left[ (AD_1 \textbf{c} + A^{(0)} \textbf{c}, D_1 \textbf{d})_{C^{0}} + (A^{(0)*} D_1 \textbf{c} + B \textbf{c}, \textbf{d})_{C^{0}} \right] \, dx_1. \]

Proposition 3. The Galerkin IBVP (19) has a unique solution in the class (16).
Moreover, the conservation laws hold (compare with (5)):
\[ \| \sqrt{\alpha} y^{(N)} (\cdot, t) \|_{L^2(\Omega)} = \| \sqrt{\alpha} y^{(N),0} \|_{L^2(\Omega)}, \quad \| y^{(N)} (\cdot, t) \|_{\Omega} = \| y^{(N),0} \|_{\Omega} \quad \text{for any} \quad t \geq 0. \]

The existence and the conservation laws can be proved, for example, once more by the Fourier method based on expansions with respect to the system of eigenfunctions of the corresponding eigenvalue problem
\[ \mathcal{H}_0 \textbf{c} = \lambda \textbf{M} \textbf{c}, \quad \textbf{c}_{x_1=0, x_1} = 0. \]
The uniqueness can be proved by the energy method. Finally we turn to error bounds.

Proposition 4. Let conditions (9) and (11) be valid. Let also
\[ \psi^0 \in \textbf{H}^{0,0}(\Omega), \quad D_1 \psi^0 \in \textbf{H}^{0,0}(\Omega), \quad D_1 \psi \in L^1(0,T; \textbf{H}^{0,0}(\Omega)), \quad D_1 D_1 \psi \in L^1(0,T; \textbf{H}^{0,0}(\Omega)), \]
for some $\theta > 1$ and $T > 0$, as well as, in the case $\theta > 5/2$, condition (10) for $\beta = \theta - 5/2$ be valid too.
1. Let condition (13) be also valid. Then the following $C([0,T]; L^2(\Omega))$-error bound holds:
\[ \| y - y^{(N)} \|_{L^2(I_1; \Omega)} \leq K \left[ \| y^{(N),0} - \mathcal{P}^N_1 \|_{L^2(\Omega)} + N^{-\theta} \log N \left[ \| y^{(N),0} \|_{L^2(\Omega)} + \| D_1 \psi^{(N)} \|_{L^2(\Omega)} + \| D_1 D_1 \psi \|_{L^2(\Omega)} \right] \right]. \]
2. Let $\rho_0 = \rho$. Then the following $C([0,T]; H^1(\Omega))$-error bound holds:
\[ \| y - y^{(N)} \|_{L^2(I_1; H^1(\Omega))} \leq K \left[ \| y^{(N),0} - \mathcal{P}^N_1 \|_{H^1(\Omega)} + N^{-(\theta-1)} \log^{1/2} N \left[ \| y^{(N),0} \|_{H^1(\Omega)} + \| D_1 \psi^{(N)} \|_{H^1(\Omega)} + \| D_1 D_1 \psi \|_{H^1(\Omega)} \right] \right]. \]

The quantities $K$ are independent of $T$. For $y^{(N),0} = \mathcal{P}^N_1 \psi^0$ or $\mathcal{P}^N_1 \psi^0$, the first summands on the right-hand sides of (22) and (23) can be omitted.

Proof. 1. The argument is rather standard in semidiscrete Galerkin methods for IBVP. For any $y$ with the properties like (15) and (16) of $y^{(N)}$, we have the following chain of identities following from the Galerkin and original integral identities (17) and (4):
\[ t \left( D_1 [\rho (y^{(N)} - y)], \phi \right)_{L^2} - \mathcal{L}_0 (y^{(N)} - y, \phi, \phi)_{L^2} = t \left( D_1 [\rho (y^{(N)} - y)], \phi \right)_{L^2} - \mathcal{L}_0 \left( (y^{(N)}, \phi)_{L^2} - t (D_1 (\rho y^{(N)}), \phi)_{L^2} - \mathcal{L}_0 (y, \phi) \right) \]
\[ = t (D_1 (\rho \phi), \phi)_{L^2} - t (D_1 (\rho y^{(N)}), \phi)_{L^2} - \mathcal{L}_0 (\psi - y, \phi) \quad \text{on} \quad (0,T), \]
for any $\phi \in S_N$. (24)

Here the term $t \left( D_1 [\rho (y^{(N)}), \phi \right)_{L^2}$ is understood actually as the left-hand side of identity (20), and the terms $t \left( D_1 [\rho (y^{(N)} - y)], \phi \right)_{L^2}$ and $t (D_1 (\rho y^{(N)}), \phi)_{L^2}$ are understood similarly.

For $D_1 \psi \in L^1(0,T; H^1(\Omega))$, setting $y := \mathcal{P}^N_1 \psi$ and $\epsilon^{(N)} := y^{(N)} - \mathcal{P}^N_1 \psi$ and using the definition (8) of $\mathcal{P}^N_1$, we obtain
\[ t \left( D_1 (\rho \epsilon^{(N)}), \phi \right)_{L^2} - \mathcal{L}_0 (\epsilon^{(N)}, \phi) = t \left( D_1 [\rho (\psi - \mathcal{P}^N_1 \psi)], \phi \right)_{L^2} \quad \text{on} \quad (0,T), \quad \text{for any} \quad \phi \in S_N. \]

Choosing $\phi = \epsilon^{(N)}$ and taking the imaginary part of the result, we get
\[ \frac{1}{2} \frac{d}{dt} \| \sqrt{\bar{\mathcal{P}}} \epsilon^{(N)} \|_{L^2}^2 = \text{Im} \left( t \int_{\Omega} (D_1 \psi - \mathcal{P}^N_1 D_1 \psi)(\epsilon^{(N)}, \rho \phi) \, dx \right) \]
\[ \leq \| \sqrt{\mathcal{P} (D_1 \psi - \mathcal{P}^N_1 D_1 \psi)} \|_{L^2} \| \sqrt{\mathcal{P} \epsilon^{(N)}} \|_{L^2} \quad \text{on} \quad (0,T), \]

where $\| \cdot \|_{L^2} := \| \cdot \|_{L^2(\Omega)}$ for brevity. Consequently
\[ \| \sqrt{\mathcal{P} \epsilon^{(N)}} \|_{L^2(\Omega)} \leq \| \sqrt{\mathcal{P} \epsilon^{(N)}} \|_{L^2(\Omega)} + 2 \left( \sqrt{\mathcal{P} (D_1 \psi - \mathcal{P}^N_1 D_1 \psi)} \right)_{L^2(\Omega)}. \]
This implies directly the corresponding $C([0, T]; L^2(\Omega))$-error bound
\[
\|\sqrt{\rho}(\psi - y^{(N)})\|_{C([0, T]; L^2(\Omega))} \leq \|\sqrt{\rho}\|_{L^2(\Omega)} + \|\sqrt{\rho}(\psi - P_N^{(1)} \psi)\|_{C([0, T]; L^2(\Omega))} \\
\leq \|\sqrt{\rho}(y^{(N,0)} - P_N^{(1)} \psi^{(0)})\|_{L^2} + \|\sqrt{\rho}(\psi^{0} - P_N^{(1)} \psi^{(0)})\|_{L^2} \\
+ 3 \|\sqrt{\rho}(D_t \psi - P_N^{(1)} D_t \psi)\|_{L^1(0, T; L^2(\Omega))}.
\]

Exploiting assumptions on $\psi^{0}$ and $D_t \psi$ as well as the $L^2(\Omega)$-approximation bound (14), we obtain (22).

2. For $D_t \psi \in L^1(0, T; H_0^1(\Omega))$, now we set $y := P_N \psi$ and $q^{(N)} := y^{(N)} - P_N \psi$. In the case $\rho_0 = \rho$, using the definition (8) of $P_N$, from (24) we obtain
\[
I\{D_t(\rho q^{(N)}), \psi\}_{L^2} - \mathcal{L}_\Omega (q^{(N)}, \psi) = -\mathcal{L}_\Omega (\psi - P_N \psi, \psi) \quad \text{on} \quad (0, T), \quad \text{for any} \quad \psi \in S_N.
\]

Supposing that the property $D_t y^{(N)} \in L^1(0, T; H_0^1(\Omega))$ is valid, then choosing $\varphi = D_t q^{(N)}$ and separating the real part of the result, we get
\[
\frac{1}{2} \frac{d}{dt} (\|q^{(N)}\|_{L^2(\Omega)}^2) = \Re \mathcal{L}_\Omega (\psi - P_N \psi, D_t q^{(N)}) \quad \text{on} \quad (0, T).
\]

Integrating this equality and then integrating by parts, we have
\[
\|q^{(N)}(., t)\|_{L^2(\Omega)}^2 = \|q^{(N),0}\|_{L^2(\Omega)}^2 + 2\Re \left[ \mathcal{L}_\Omega \left( (\psi - P_N \psi)(., t), q^{(N)}(., t) \right) \right] \\
- \mathcal{L}_\Omega (\psi^{0} - P_N \psi^{0}, q^{(N),0}) - \int_0^t \mathcal{L}_\Omega \left( D_t (\psi - P_N \psi), q^{(N)} \right) \, dt \\
\leq \|q^{(N),0}\|_{L^2(\Omega)}^2 + 2\|\psi^{0} - P_N \psi^{0}\|_{L^2(\Omega)} \|q^{(N),0}\|_{L^2(\Omega)} + 2 (\|\psi - P_N \psi\|(., t))_{L^2(\Omega)} \\
+ \|D_t (\psi - P_N \psi)\|_{L^1(0, T; L^2(\Omega))} \|q^{(N)}\|_{C([0, T]; L^2(\Omega))} \quad \text{on} \quad (0, T),
\]

where $q^{(N),0} := q^{(N)}|_{t=0} = y^{(N),0} - P_N \psi^{0}$. Consequently
\[
\|q^{(N)}\|_{C([0, T]; L^2(\Omega))} \leq \|q^{(N),0}\|_{L^2(\Omega)} + 3\|\psi^{0} - P_N \psi^{0}\|_{L^2(\Omega)} + 4\|D_t (\psi - P_N \psi)\|_{L^1(0, T; L^2(\Omega))}.
\]

To remove the temporary assumption $D_t y^{(N)} \in L^1(0, T; H_0^1(\Omega))$, we can apply once more the Fourier method based on expansions with respect to the system of eigenfunctions of the eigenvalue problem (21): we rewrite identity (26) as an inhomogeneous equation like (19) for $q^{(N)}$, derive a bound like (27) for partial sums of the expansion for $q^{(N)}$ and then pass to the limit in the sums (see similar arguments in, for example, [11]).

Bound (27) implies the $C([0, T]; E(\Omega))$-error bound (compare with (25))
\[
\|y - y^{(N)}\|_{C([0, T]; E(\Omega))} \leq \|y^{(N),0} - P_N \psi^{0}\|_{L^2(\Omega)} + 4\|\psi^{0} - P_N \psi^{0}\|_{E(\Omega)} + 5\|D_t \psi - P_N D_t \psi\|_{L^1(0, T; E(\Omega))}.
\]

Exploiting assumptions on $\psi^{0}$ and $D_t \psi$ as well as the $H^1(\Omega)$-approximation bound (12), we get (23).

Since $P_N \psi^{0} - P_N^{(1)} \psi^{0} = P_N (\psi^{0} - P_N \psi^{0}) = -P_N^{(1)} (\psi^{0} - P_N \psi^{0})$, for $y^{(N),0} = P_N \psi^{0}$ or $P_N^{(1)} \psi^{0}$, the first terms on the right-hand sides in (25) and (28) either can be bounded by the second ones or are simply zero. This completes the proof. \qed

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