Identification of MIMO Hammerstein systems with non-linear feedback

DAPENG LUO†
Center of Excellence, Eaton Corporation, 13100 East Michigan Avenue, Galesburg, MI 49053, USA

AND

ALEXANDER LEONESSA‡
Mechanical Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0238, USA

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Subspace identification algorithms are proposed in this paper for a class of multi-input multi-output (MIMO) discrete-time closed-loop Hammerstein systems. In particular, instrumental variable (IV) techniques are applied to solve the correlations between the input signals and the signals that are fed back into the system. An augmented IV identification approach is introduced to address a problem related to the application of the numerical efficient RQ decomposition algorithm to a rank-decreased correlation matrix. The contribution of this paper is to provide a unified approach to identify a class of non-linear closed-loop systems and solve the related numerical issues during the implementation. Analysis and numerical examples are presented to demonstrate the validity of the algorithms proposed in this paper.

Keywords: subspace identification; non-linear identification; instrumental variable.

1. Introduction

The identification of non-linear multi-input multi-output (MIMO) Hammerstein systems, which are commonly encountered in model analysis and control, is still considered to be a challenging problem. State-space models represent an excellent choice for obtaining a minimal realization of MIMO systems as well as providing a platform for modern control techniques. Consequently, subspace identification methods are widely applied in the identification of such systems. Although open-loop subspace identification algorithms are well studied and applied to linear systems with great success (Overschee & Moor, 1994, 1996a,b), few applications can be found to systems with general non-linearities and large operating ranges. Furthermore, it is often necessary to perform identification experiments on systems operating in closed-loop (Söderström & Stoica, 1989; Overschee & Moor, 1997). Lacy & Bernstein (2001) proposed an intersection identification algorithm and an order approximation criterion for a class of MIMO Hammerstein systems with non-linear feedback loop. This is primarily a one-step ahead prediction algorithm where, at each step, the updated estimation of the state variables from current measurements is used to compensate for any identification error. Although the results obtained in Lacy & Bernstein (2001) are very promising, the required computational effort is considerable and difficult to achieve, especially in real-time applications. Finally, for control design purposes, it is desirable to obtain a system model which is independent from the identification process (Doherty et al., 1997).

†Email: dapengluo@eaton.com
‡Corresponding author. Email: leonessa@vt.edu
In this paper, we will present a subspace identification scheme for a class of MIMO Hammerstein system with a static non-linear feedback loop. The novel approach that we are going to consider allows consideration of non-linearities at both the input and the output, which represents an appealing model in many control problems. In particular, non-linear approximations are introduced in the form of finite sums of known basis functions with unknown coefficients, such as a truncated Fourier series. However, a problem arises in the closed-loop identification caused by the correlation between noise sequences and input signals, leading to biased identification results when directly using open-loop approaches (Ljung & McKelvey, 1996). In this paper, a subspace instrumental variable identification (SIVID) algorithm is presented to recover the non-linear closed-loop system model. The original aim of instrumental variable (IV) methods, in contrast to classical prediction error methods, is to allow the estimation of the system dynamics without specifying an accurate model for system or measurement noise signals (Viberg et al., 1995) and, in this paper, that idea is extended to solve the closed-loop identification problem. Specifically, a sequence of IV vectors uncorrelated with the system noise are constructed, and correlations are made between the IV vectors and the input/output signals of block Hankel matrices. Then, the system dynamics can be recovered from a canonical correlation matrix obtained from the correlated signals’ block Hankel matrices.

In general, the IV vectors consist of past or filtered input/output signals. Therefore, the corresponding IV-correlated input–output block Hankel matrices are rank decreased. As a result, the numerical stable and efficient RQ method described in Chapter 6 of Overschee & Moor (1996b) is not applicable since only a lower triangular matrix is used in calculation, and a correct partition of submatrices needs to be applied. To overcome this drawback, a novel-augmented IV matrix is introduced to replace the conventional ones. The analytical solution shows that, in the space of interest, the augmented subspace instrumental variable identification (ASIVID) algorithm has the identical least square solution as the previous one. Furthermore, the numerical implementation of this novel approach is shown to reduce the computational effort for identification problems large enough. The contribution of this paper is to present a systematic solution to address the identification problem for systems containing both non-linearities and closed-loop feedback and to extend the current IV approaches to solve numerical issues during their implementation.

The paper is organized as follows: notation and definitions are presented in Section 2. The problem formulation is presented in Section 3. In Section 4, we propose the identification algorithm for MIMO Hammerstein systems with non-linear feedback loops. The approximation techniques used in the simulation and two non-linear examples used to validate the methods presented in this paper are presented in Section 5 followed by our conclusions in Section 6.

2. Preliminaries

The following notation is used in this paper. Let the superscript $(\cdot)\top$ denote the transposition operator, $E(\cdot)$ the expectation operator, $\|\cdot\|_2$ the Euclidean vector norm, $\|\cdot\|_1$ the $l_1$ vector norm, $\|\cdot\|_F$ the Frobenius matrix norm, $I_m$ the identity matrix with dimension $m$ and $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices.

Suppose $Z \in \mathbb{R}^{m\times n}$, then $\mathcal{R}(Z)$ denotes the column space of $Z$, $\mathcal{R}(Z)\perp$ the orthogonal complement of $\mathcal{R}(Z)$ (Stewart & Sun, 1990), $\mathcal{R}(Z\top)$ the row space of $Z$. Denote by $Z^R \triangleq Z\top(ZZ\top)^\dagger$ the right inverse of $Z$, where $(\cdot)^\dagger$ denotes the Moore–Penrose pseudoinverse of a matrix. Furthermore, the operator $\Pi_Z \triangleq Z^R Z = Z\top(ZZ\top)^\dagger Z$ projects the row space of a matrix onto $\mathcal{R}(Z\top)$ and the operator $\Pi_Z\perp \triangleq I - \Pi_Z$ projects the row space of a matrix onto the orthogonal complement of $\mathcal{R}(Z\top)$. Denote
with $X\gamma Z \triangleq [X\Pi Y^\top][Z\Pi Y^\top]^\top Z$ the oblique projection of the row space of $X \in \mathbb{R}^{i \times n}$ along the row space $Y \in \mathbb{R}^{j \times n}$ on the row space of $Z \in \mathbb{R}^{m \times n}$.

Given a sequence $\{z_k\}$, $k = 1, 2, \ldots$, the averaging operator $\bar{E}$ is defined as (Viberg et al., 1997)

$$
\bar{E}[z_k] \triangleq \lim_{n_0 \to \infty} \frac{1}{n_0} \sum_{k=1}^{n_0} E[z_k].
$$

(2.1)

In order to simplify our notation in the following sections, we introduce the following definitions.

**Definition 2.1** Consider a generic signal $s(k) \in \mathbb{R}^n$, $k = 1, 2, \ldots$. For a given index $\alpha \in \mathbb{N}^+$, $\alpha > n$, define the ‘past’ and ‘future stacked signal vectors’ as follows:

$$
\begin{align*}
  s_p(k) & \triangleq [s^\top(k-\alpha), \ldots, s^\top(k-1)]^\top, \quad k = \alpha+1, \ldots, \\
  s_f(k) & \triangleq [s^\top(k), \ldots, s^\top(k+\alpha-1)]^\top, \quad k = \alpha+1, \ldots,
\end{align*}
$$

(2.2) (2.3)

The ‘block Hankel matrices’ associated with $s_p(k)$ and $s_f(k)$ are defined as

$$
\begin{align*}
  S_p(k) & \triangleq [s_p(k), \ldots, s_p(k+\beta-1)], \quad k = \alpha+1, \ldots, \\
  S_f(k) & \triangleq [s_f(k), \ldots, s_f(k+\beta-1)], \quad k = \alpha+1, \ldots,
\end{align*}
$$

(2.4) (2.5)

where $\beta \in \mathbb{N}^+$ represents the number of columns of block Hankel matrices.

### 3. Problem formulation

Consider the system shown in Fig. 1 and assume that its discrete-time state-space model is in the form of (Lacy & Bernstein, 2001)

$$
\begin{align*}
  x(k+1) &= Ax(k) + B_1f(u(k)) + B_2g(y(k)) + w(k), \quad k = 1, 2, \ldots, \\
  y(k) &=Cx(k) + D_1f(u(k)) + D_2g(y(k)) + v(k),
\end{align*}
$$

(3.1) (3.2)

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^l$, $k = 1, 2, \ldots$, are the state, input and output vectors, respectively. The matrices $A \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times r}$, $B_2 \in \mathbb{R}^{n \times s}$, $C \in \mathbb{R}^{l \times n}$, $D_1 \in \mathbb{R}^{l \times r}$ and $D_2 \in \mathbb{R}^{l \times s}$ are the linear subsystem dynamic matrices. The signals $w(k) \in \mathbb{R}^n$ and $v(k) \in \mathbb{R}^l$, $k = 0, 1, \ldots$, represent the process noise and the additive measurement noise, respectively. Note that the noise signals

![Fig. 1. MIMO Hammerstein model with a non-linear feedback loop.](http://imamci.oxfordjournals.org/Downloaded from http://finance.oxfordjournals.org)
can be coloured. The mappings \( f(\cdot) : \mathbb{R}^m \to \mathbb{R}^r \) and \( g(\cdot) : \mathbb{R}^l \to \mathbb{R}^s \) represent the input and feedback non-linearities, respectively.

Without loss of generality, the \( t \)-th component of \( f(\cdot), f_t(\cdot) : \mathbb{R}^m \to \mathbb{R}, t = 1, \ldots, r \), and the \( h \)-th component of \( g(\cdot), g_h(\cdot) : \mathbb{R}^l \to \mathbb{R}, h = 1, \ldots, s \), can be approximated by \( \hat{f}_t(\cdot) \) and \( \hat{g}_h(\cdot) \) of truncated series with \( p + 1 \) and \( q + 1 \) terms, respectively (Leonessa & Luo, 2001; Ljung, 1999),

\[
\hat{f}_t(u(k)) \triangleq \sum_{j=0}^{p} \mu^T_{tj} \eta_j(u(k)) = \mu^T_t \eta(k), \quad u(k) \in [u_{\min}, u_{\max}], \quad k = 1, 2, \ldots, \tag{3.3}
\]

\[
\hat{g}_h(y(k)) \triangleq \sum_{j=0}^{q} v^T_{hj} \xi_j(y(k)) = v^T_h \xi(k), \quad y(k) \in [y_{\min}, y_{\max}], \tag{3.4}
\]

where \( \mu_t \triangleq [\mu^T_0, \mu^T_1, \ldots, \mu^T_p] \), with \( \mu_{tj} \in \mathbb{R}^m \), \( j = 0, \ldots, p \), and \( \nu_h \triangleq [v^T_0, v^T_1, \ldots, v^T_q] \), with \( v_{hj} \in \mathbb{R}^l \), \( j = 0, \ldots, q \), are the expansion coefficient vectors and

\[
\eta(k) \triangleq [\eta^T_0(u(k)), \eta^T_1(u(k)), \ldots, \eta^T_p(u(k))]^T, \quad k = 1, 2, \ldots,
\]

\[
\xi(k) \triangleq [\xi^T_0(y(k)), \xi^T_1(y(k)), \ldots, \xi^T_q(y(k))]^T,
\]

where \( \eta_j(\cdot) \in \mathbb{R}^m, j = 0, \ldots, p \), and \( \xi_j(\cdot) \in \mathbb{R}^l, j = 0, \ldots, q \), represent the chosen ‘basis functions’. Bounds for the approximation residues (approximation errors) are defined as

\[
\epsilon_f \triangleq \max_{u \in [u_{\min}, u_{\max}]} ||f(u) - \hat{f}(u)||_1,
\]

\[
\epsilon_g \triangleq \max_{y \in [y_{\min}, y_{\max}]} ||g(y) - \hat{g}(y)||_1,
\]

where \( \hat{f}(u) \triangleq [\hat{f}_1(u), \ldots, \hat{f}_r(u)]^T \) and \( \hat{g}(y) \triangleq [\hat{g}_1(y), \ldots, \hat{g}_s(y)]^T \).

By using \( \hat{f}(u), \hat{g}(y) \) instead of \( f(u), g(y) \) and the decompositions (3.3), (3.4), the state-space model (3.1), (3.2) can be approximated with (Lacy & Bernstein, 2001)

\[
x(k + 1) = Ax(k) + Mz(k) + w(k), \quad k = 1, 2, \ldots, \tag{3.5}
\]

\[
y(k) = Cx(k) + Nz(k) + v(k), \tag{3.6}
\]

where \( M \triangleq [B_1 \mu^T, B_2 v^T] \in \mathbb{R}^{n \times ((p+1)m+(q+1)l)}, N \triangleq [D_1 \mu^T, D_2 v^T] \in \mathbb{R}^{l \times ((p+1)m+(q+1)l)} \) with \( \mu \triangleq [\mu_1, \ldots, \mu_r] \in \mathbb{R}^{(p+1)m} \) and \( v \triangleq [v_1, \ldots, v_s] \in \mathbb{R}^{(q+1)l} \) and \( \xi(k) \triangleq [\eta^T(k), \xi^T(k)]^T \in \mathbb{R}^{(p+1)m+(q+1)l} \). The following assumptions are introduced to allow the estimation problem to be solved and to facilitate the analysis.

(A1) The non-linear approximation residues are negligibly small, i.e. \( \epsilon_f \cong 0 \) and \( \epsilon_g \cong 0 \), or \( f(u) \cong \hat{f}(u), u \in [u_{\min}, u_{\max}] \), and \( g(y) \cong \hat{g}(y), y \in [y_{\min}, y_{\max}] \), if the basis functions are well chosen and the series orders are high enough.

(A2) The system (3.5–3.6) is asymptotically stable. Note that, although that is often the case in a closed-loop measurement situation, the state matrix \( A \) does not need to be Hurwitz.

(A3) The noise sequences \( \{w(k), v(k)\}, k = 1, 2, \ldots, \) are ergodic with the following form:

\[
E\left(\begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \begin{bmatrix} w(j) \\ v(j) \end{bmatrix}^T\right) = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \delta_{k,j} \geq 0, \quad k, j = 1, 2, \ldots,
\]
where, \( E(\cdot) \) denotes the statistical expectation, \( \delta_{k,l} \) denotes the Kronecker delta function, \( Q \in \mathbb{R}^{n \times n} \) and \( R \in \mathbb{R}^{l \times l} \) denote the variances of the random signals \( w(k) \) and \( v(k) \), respectively, and \( S \in \mathbb{R}^{n \times l} \) denotes the covariance between \( w(k) \) and \( v(k) \).

The following block matrix equations can be obtained directly from (3.5) and (3.6) (Lacy & Bernstein, 2001; Overschee & Moor, 1996b):

\[
Y_p(k) = \Gamma_a X_p(k) + H_a Z_p(k) + \Psi_a \mathcal{E}_p(k),
\]

\[
Y_l(k) = \Gamma_a X_l(k) + H_a Z_l(k) + \Psi_a \mathcal{E}_l(k),
\]

\[
X_l(k) = A^x X_p(k) + \Delta_a Z_p(k) + \Omega_a \mathcal{W}_p(k),
\]

where the block Hankel matrices \( X_p(k), X_l(k), Y_p(k), Y_l(k), Z_p(k), Z_l(k), \mathcal{E}_p(k), \mathcal{E}_l(k) \) and \( \mathcal{W}_p(k) \) are defined using the signals \( x(k), y(k), z(k), \varepsilon(k) \triangleq [w(k); v(k)] \) and \( w(k) \), respectively. In particular, state matrices \( X_p(k) \) and \( X_l(k) \) are defined as

\[
X_p(k) \triangleq [x(k - a), \ldots, x(k - a + \beta - 1)], \quad k = a + 1, \ldots,
\]

\[
X_l(k) \triangleq [x(k), \ldots, x(k + \beta - 1)], \quad k = a + 1, \ldots,
\]

where \( a, \beta \in \mathbb{N}^+ \), defined in Section 2, are numbers of block rows and columns, respectively.

Next, we introduce the extended observability matrix \( \Gamma_a \) and the extended controllability matrix \( \Delta_a \) defined as follows (Lacy & Bernstein, 2001):

\[
\Gamma_a \triangleq \left[ C^\top A^\top C^\top \ldots (A^{a-1})^\top C^\top \right] \top \in \mathbb{R}^{l \times n},
\]

\[
\Delta_a \triangleq \left[ A^{a-1} M \ldots A M M \right] \in \mathbb{R}^{n \times a((p+1)m+q+1)l}.
\]

Furthermore, the lower triangular block Toeplitz matrix \( H_a \) is defined as

\[
H_a \triangleq \begin{bmatrix}
N & 0 & \ldots & 0 \\
CM & N & \ldots & 0 \\
CAM & CM & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
CA^{a-2} M & CA^{a-3} M & \ldots & N
\end{bmatrix} \in \mathbb{R}^{l \times a((p+1)m+q+1)l}.
\]

Finally, two matrices \( \Omega_a \) and \( \Psi_a \), needed for later derivations, are defined as

\[
\Omega_a \triangleq \begin{bmatrix}
A^{a-1} \ldots A I
\end{bmatrix} \in \mathbb{R}^{n \times an},
\]

\[
\Psi_a \triangleq \begin{bmatrix}
0 & I & 0 & \ldots & 0 & 0 \\
C & 0 & 0 & I & \ldots & 0 \\
CA & 0 & C & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
CA^{a-2} & 0 & CA^{a-3} & \ldots & 0 & I
\end{bmatrix} \in \mathbb{R}^{l \times (a+1)n}.
\]
4. Identification scheme design

In this section, we will first analyse a deterministic algorithm which solves the identification problem and then show its drawbacks if directly applied to the system (3.5–3.6) with \( w(k), v(k) \not\equiv 0 \). Then, we will propose SIVID method to overcome these drawbacks.

4.1 Analysis of a deterministic identification scheme

In this section, we introduce the subspace identification scheme by presenting a deterministic identification algorithm. A remark showing the difference between the deterministic identification scheme and the deterministic–stochastic one will lead to the SIVID algorithm proposed in Section 4.2.

**Proposition 4.1 (SID algorithm)** Consider the system (3.5–3.6), and assume that the following conditions are satisfied:

(i) the pair \((A, C)\) is observable and the pair \((A, [M_1, M_2])\) is controllable,
(ii) \( w(k) \equiv 0 \) and \( v(k) \equiv 0 \),
(iii) the intersection of the row space of \( X_p \) and the row space of \( Z_f \) is empty, i.e. \( X_p \Pi Z_f = 0 \),
(iv) \( z(k) \) is a persistently exciting signal of order \( 2\alpha \), i.e. \( \text{rank} \left[ \begin{array}{c} Z_p \\ Z_f \end{array} \right] = 2\alpha \varrho \) with \( \varrho \triangleq (p + 1)m + (q + 1)l \) (Lacy & Bernstein, 2001).

Next, define

\[
\Xi_\alpha \triangleq Y_f/Z_f W_p,
\]

where \( W_p \triangleq [Z_p; Y_p] \in \mathbb{R}^{(q+l)\alpha \times \beta} \) and consider the Singular Value Decomposition (SVD)

\[
W_L \Xi_\alpha W_R = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 S_1 V_1^T,
\]

where \( W_L \) and \( W_R \) are weighting matrices, \( W_L \in \mathbb{R}^{\alpha \times \alpha} \) is of full rank and \( W_R \in \mathbb{R}^{\beta \times \beta} \) is such that \( \text{rank}(W_p) = \text{rank}(W_p W_R) \) (Overschee & Moor, 1996b). Then, we have \( \Gamma_a = W_L^{-1} U_1 S_1^{1/2} \) and \( X_f = S_1^{1/2} V_1^T W_R^{-1} \), and the order of the system given by (3.5), (3.6) is equal to the size of \( S_1 \).

**Proof.** The proof is analogous to the one for an open-loop deterministic dynamic system described in Overschee & Moor (1996b), except that the system matrices \( C \) and \( D \) are substituted by \( M \) and \( N \), and the input vector \( u \) by \( z \). \( \square \)

**Remark 4.1** Worth noting is that the block matrix equations (3.7–3.9) imply an interesting well-known result on system state construction. In particular, without loss of generality, when \( w(k) \equiv 0 \) and
\( v(k) \equiv 0 \), (3.7) can be written as

\[
X_p = \Gamma_a^\dagger Y_p - \Gamma_a^\dagger H_a Z_p. \tag{4.3}
\]

Substituting (4.3) into (3.9), we have

\[
X_f = A^\alpha \Gamma_a^\dagger Y_p - A^\alpha \Gamma_a^\dagger H_a Z_p + \Delta_a Z_p
\]
\[
= [\Delta_a - A^\alpha \Gamma_a^\dagger H_a A^\alpha \Gamma_a^\dagger] W_p = L_f W_p, \tag{4.4}
\]

where \( L_f \triangleq [\Delta_a - A^\alpha \Gamma_a^\dagger H_a A^\alpha \Gamma_a^\dagger] \). Equation (4.4) shows that the state can be written as a linear combination of past input–output data (Overschee & Moor, 1996b). In addition, (3.8) indicates that a prediction model for \( Y_f \) is

\[
\hat{Y}_f = \Gamma_a X_f + H_a Z_f. \tag{4.5}
\]

Equations (4.4) and (4.5) verify that ‘system states can be defined as the minimum amount of information about the past history of a system which is required to predict the future motion’ (Aström, 1970).

**Remark 4.2** Close connections among the intersection, the projection and the oblique projection algorithms are well studied in Overschee & Moor (1996b), Shi & MacGregor (2001) and Overschee & Moor (1994), the interested reader may refer to them and the numerous references therein for more details. Our focus is not about comparing those algorithms; however, oblique projection does provide a convenient tool for estimating the extended observability matrix \( \Gamma_a \) in one step instead of estimating its deterministic and stochastic components separately, as often done when applying Instrumental Variable techniques (Arun & Yung, 1990).

**Remark 4.3** Equations (3.8) and (4.4) also show the connection between the subspace methods and the least square method, which is seldom explicitly revealed in the literature. In particular, it follows from (3.8) and (4.4) that \( Y_f = \Gamma_a X_f + H_a Z_f = \Gamma_a L_f W_p + H_a Z_f \). The estimation of system matrices can be formulated as a least square problem \([\hat{\Gamma}_a, \hat{H}_a] = \text{arg} \min_{X \in \mathbb{R}^{(2n+1)\times (2n+1)}} \| Y_f - X \left( W_p^T Z_f^\dagger \right)^T \|^2_F \).

**Remark 4.4** The combined deterministic–stochastic identification problem for an open-loop system is already solved, under the assumption that the noise sequence is uncorrelated from the input signals (Overschee & Moor, 1996b). However, this is not the case when considering the closed-loop system (3.5) and (3.6) with \( w(k) \neq 0 \) and \( v(k) \neq 0 \) (Gustafsson, 2001; Ljung & McKelvey, 1996), where the noise signals are in general correlated with the input signals, which are functions of not only \( u(k) \) but also \( y(k) \). When applying the deterministic algorithm proposed in Proposition 4.1 directly on the deterministic–stochastic system (3.5) and (3.6) without considering the correlation, the identification result is biased (Ljung & McKelvey, 1996).

In Section 4.2, a SIVID algorithm is proposed to solve the issue discussed in Remark 4.4.

### 4.2 Subspace instrumental variable identification algorithm

Consider the case \( w(k) \neq 0 \) and \( v(k) \neq 0 \), thus \( z(k) \) is correlated with \( \varepsilon(k) \), which is the case for closed-loop systems. We remind that \( z(k) \triangleq [\eta(k)^T, \zeta(k)^T]^T \) and \( \varepsilon(k) \triangleq [w(k)^T, v(k)^T]^T \) are defined as concatenated vectors representing new input and noise vectors in the block matrix equations (3.7–3.9).

A SIVID algorithm is proposed in this section for the identification of the closed-loop non-linear system (3.5), (3.6), when the noise sequence is correlated to the input signals. The basic idea of IV approaches
is to construct an IV vector that is uncorrelated with the vector of noise sequence, while a certain rank condition must also hold to guarantee that the informative part of the input/output signals does not vanish in the IV correlation (Gustafsson, 1998). Then the noise vectors are significantly attenuated in the covariance matrix by correlating it to the IV vector. In the case that such an IV vector can be found, the noise vectors are allowed to have arbitrary spatial and temporal colour. Extensive discussions on how to find an IV vector that fulfills the above conditions can be found in Ljung (1999), Söderström & Stoica (1989) and Viberg et al. (1997).

Suppose that the following IV matrix \( \mathcal{Y}(k) \in \mathbb{R}^{n_0 \times \beta} \) is available

\[
\mathcal{Y}(k) \triangleq [\zeta(k), \ldots, \zeta(k + \beta - 1)], \quad k = \alpha + 1, \ldots,
\]

where \( \zeta(k), k = \alpha + 1, \ldots, \) are IV vectors. We will use \( \mathcal{Y} \) instead of \( \mathcal{Y}(k) \) in the following derivations. Then, by computing the correlation with \( \mathcal{Y} \) on both sides of (3.8), we obtain

\[
\tilde{Y}_f^0 = \Gamma_a \tilde{X}_f^0 + H_a \tilde{Z}_f^0 + \Psi_a \tilde{e}_f^0,
\]

where \( \tilde{Y}_f^0 \triangleq \tilde{E}(Y_f \mathcal{Y}^\top) \in \mathbb{R}^{l \times n_0}, \tilde{X}_f^0 \triangleq \tilde{E}(X_f \mathcal{Y}^\top) \in \mathbb{R}^{n \times n_0}, \tilde{Z}_f^0 \triangleq \tilde{E}(Z_f \mathcal{Y}^\top) \in \mathbb{R}^{\rho \times n_0} \) and \( \tilde{e}_f^0 \triangleq \tilde{E}(\mathcal{E}_f \mathcal{Y}^\top) \in \mathbb{R}^{(n+l) \times n_0} \).

As previously discussed, it is natural to construct \( \mathcal{Y} \) such that it is uncorrelated with the noise block Hankel matrix \( \mathcal{E}_f \) (Gustafsson, 2001), i.e.

\[
\tilde{E}_f^0 \equiv 0,
\]

which takes us to the following theorem.

**Theorem 4.1 (SIVID algorithm)** Consider the system (3.5–3.6) and assume that

(i) there exists an IV matrix \( \mathcal{Y} \in \mathbb{R}^{n_0 \times \beta} \), where \( n_0 > n \), such that \( \tilde{E}_f^0 \equiv \tilde{E}(\mathcal{E}_f \mathcal{Y}^\top) \equiv 0 \),

(ii) the pair \((A, C)\) is observable and the pair \((A, [M_1, M_2, Q^{1/2}])\) is controllable,

(iii) \( \zeta(k) \) is a persistently exciting signal of order 2\( \alpha \), i.e. \( \text{rank} \left[ \begin{array}{c} Z_p \\ Z_f \end{array} \right] = 2 \alpha \varrho \) with \( \varrho = (p + 1)m + (q + 1)l \),

(iv) \( \text{rank}(\tilde{X}_f^0) = n \).

Next, define

\[
\mathcal{O}_a \triangleq \tilde{Y}_f^0 \Pi_{\tilde{Z}_f}^\perp,
\]

and consider the SVD

\[
W_L \mathcal{O}_a W_R = (U_1 U_2) \left( \begin{array}{cc} S_1 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} V_1^\top \\ V_2^\top \end{array} \right) = U_1 S_1 V_1^\top,
\]

where \( W_L \) and \( W_R \) are weighting matrices, \( W_L \in \mathbb{R}^{l \times l} \) is full rank and \( W_R \in \mathbb{R}^{l+\varrho \times (l+\varrho) \alpha} \) is such that \( \text{rank}(\Pi_{\tilde{Z}_f}^\perp) = \text{rank}(\Pi_{\tilde{Z}_f}^\perp W_R) \). Furthermore, \( S_1 \) contains the dominant singular values. Then, we have \( \Gamma_a = W_L^{-1} U_1 S_1^{1/2} \) and the order of the system (3.5–3.6) is equal to the size of \( S_1 \).
In the estimation of the extended observability matrix \( \Gamma_a \) obtained as in (Overschee & Moor, 1996b). Standard numerical tools applied to subspace into successful applications. Some numerically robust algorithms associated with open-loop identification are always essential when translating identification theories into practice, we substitute the non-increasing singular values of \( \tilde{\Gamma}_f \) the non-singular transformation matrix \( \tilde{\Gamma}_f \). From condition (iv) and the properties of \( \tilde{\Gamma}_f \), we obtain from (4.6) with \( \tilde{\Gamma}_f \) has full rank, thus \( \tilde{\Gamma}_f \) has full rank (Jansson & Wahlberg, 1998), which avoids a rank deficiency in the estimation of \( \Gamma_a \). From condition (iv) and the fact that \( \tilde{\Gamma}_f \) has full rank, it follows that rank \( (\tilde{\Gamma}_f) \) = \( n \). The estimation of the extended observability matrix \( \Gamma_a \) follows from (4.9) and (4.11)

\[
W_L O_a W_R = W_L \Gamma_a \tilde{\Gamma}_f \tilde{\Gamma}_f \Gamma_a = U_1 S_1 V_1^T,
\]

(4.12)

where the matrix \( U_1 \) consists of the principal singular vectors of \( \Gamma_a \), \( S_1 \) is a diagonal matrix with the non-increasing singular values of \( \tilde{\Gamma}_f \) on its diagonal. The matrix \( W_L O_a W_R \) is the product of the \( n \)-column matrix \( W_L \) and the \( n \)-row matrix \( \tilde{\Gamma}_f \). Since both matrices are of rank \( n \) (from condition (iv) and the properties of \( W_L \) and \( W_R \)), their product is also of rank \( n \). Hence, for any arbitrary transformation matrix \( T \in \mathbb{R}^{n \times n} \), (4.12) can be rewritten as

\[
W_L \Gamma_a = W_L U_1 S_1^{1/2} V_1^T,
\]

(4.13)

Without loss of generality, \( T \) is chosen as an identity matrix. Consequently, the ‘subspace estimate’ is obtained as \( \Gamma_a = W_L^{-1} U_1 S_1^{1/2} \), which completes the proof.

We have the following remarks on the implementation of SIVID algorithm.

**Remark 4.5** When \( g(t) \equiv 0 \), the system operates in open loop and an IV candidate is \( \gamma = [Z_p; Y_p; Z_t] \). In the closed-loop case, a natural choice is \( \gamma = [Z_p; Y_p] \) (Gustafsson, 2001; Ljung & McKelvey, 1996; Viberg et al., 1997).

**Remark 4.6** A finite-sample problem arises when implementing (2.1) in identification algorithms. In practice, we substitute \( \hat{E} \) with \( \hat{E}[z_k] \) \( \triangleq \frac{1}{n_0} \sum_{k=1}^{n_0} z_k \). It results in \( \lim_{n_0 \to \infty} \hat{E}[z_k] = \hat{E}[z_k] \), where the limit is taken with probability one (Viberg et al., 1997).

### 4.3 Consideration of numerical implementation

Stable and numerically efficient algorithms are always essential when translating identification theories into successful applications. Some numerically robust algorithms associated with open-loop identifications have been discussed in Overschee & Moor (1996b). Standard numerical tools applied to subspace
identification include Singular Value and RQ decompositions. The latter involves an essentially useful operation of decomposing a batch of concatenated matrices into a common factor $Q$ and a lower triangular matrix $R$. Therefore, the numerical implementation of subspace algorithms only involves the corresponding partitioned $R_{ij}$, $i, j = 1, 2, \ldots$, submatrices. Since $R$ is a lower triangular matrix, the computational effort is dramatically reduced in this way.

However, a problem arises when the RQ decomposition is directly applied to IV subspace algorithms. Note that the rank of $Y$, constructed by considering the filtered input–output signals, is often less than that of the concatenated input–output matrices, therefore the IV-correlated matrix, $[U_p^T, U_i^T, Y_p^T, Y_f^T]^T Y^T$, is rank deficient and fails to correctly partition the corresponding submatrices $R_{ij}$, $i, j = 1, 2, \ldots$. In this section, a modified IV subspace algorithm is introduced to solve this problem, at the expenses of some additional conditions required on the IV matrix.

Assume that an IV matrix $Y \in \mathbb{R}^{n_0 \times \beta}$ is available such that $\tilde{E}_f^0 \triangleq \tilde{E}(E_f Y^T) \equiv 0$ and $\tilde{E}_p^0 \triangleq \tilde{E}(E_p Y^T) \equiv 0$, where $n_0$ is the dimension of signals which can be used as IV. Since $\varepsilon(k) \triangleq [w(k); v(k)]$, it follows that also $\tilde{W}_p^0 \triangleq \tilde{E}(W_p Y^T) \equiv 0$. Then, by computing the correlation with $Y$ on both sides of (3.7–3.9), we obtain

\begin{align}
\tilde{Y}_p^0 & = \Gamma_a \tilde{X}_p^0 + H_a \tilde{Z}_p^0, \\
\tilde{Y}_f^0 & = \Gamma_a \tilde{X}_f^0 + H_a \tilde{Z}_f^0, \\
\tilde{X}_f^0 & = A^\alpha \tilde{X}_p^0 + \Delta_a \tilde{Z}_p^0.
\end{align}

(4.14) (4.15) (4.16)

where $\tilde{Y}_{p,f}^0 \triangleq \tilde{E}(Y_{p,f} Y^T) \in \mathbb{R}^{l \times n_0}$, $\tilde{X}_{p,f}^0 \triangleq \tilde{E}(X_{p,f} Y^T) \in \mathbb{R}^n \times n_0$ and $\tilde{Z}_{p,f}^0 \triangleq \tilde{E}(Z_{p,f} Y^T) \in \mathbb{R}^{\alpha \times n_0}$.

Consequently, solving for $\Gamma_a$ becomes a deterministic subspace identification problem where Proposition 4.1 can be applied after replacing (4.1) with $O_a \triangleq Y_f^0 / Z_f^0 W_p^0$. However, the rank-decreasing problem inherent to the classical instrumental variable subspace identification still exists. In the following, a close connection between (4.14–4.16) and an augmented system dynamics will be revealed, which leads to a novel algorithm where RQ decomposition can be applied.

First, let $n_1 \triangleq 2(\rho + l)\alpha - n_0$ and define an augmented IV matrix

$$\tilde{Y} \triangleq [Y^T, I_{(n_1, \beta)}^T] \in \mathbb{R}^{2\alpha (\rho + l) \times \beta},$$

where $I_{(n_1, \beta)} = [I_{n_1 \times n_1}, 0_{n_1 \times (\beta - n_1)}]$. Accordingly, the IV-correlated matrices are defined as

\begin{align}
\tilde{Z}_x & \triangleq \tilde{E}(Z_x Y^T) \triangleq \tilde{E}_x \tilde{Z}_x \tilde{Z}_x^{-1} \in \mathbb{R}^{\alpha \times 2\alpha (\rho + l)}, \\
\tilde{Y}_x & \triangleq \tilde{E}(Y_x Y^T) \triangleq \tilde{E}_x \tilde{Y}_x \tilde{Y}_x^{-1} \in \mathbb{R}^{l \times 2\alpha (\rho + l)}, \\
\tilde{E}_x & \triangleq \tilde{E}(E_x Y^T) \in \mathbb{R}^{(n + l) \times 2\alpha (\rho + l)}, \\
\tilde{W}_p & \triangleq \tilde{E}(W_p Y^T) \in \mathbb{R}^{n \times 2\alpha (\rho + l)}, \\
\tilde{W}_f & \triangleq \tilde{E}(W_f Y^T) \in \mathbb{R}^{(\rho + l) \times 2\alpha (\rho + l)},
\end{align}

where the subscript ‘$x$’ can be ‘$p$’ for past and ‘$f$’ for future.
Next, by correlating (3.7–3.9) with \( \bar{Y} \), we obtain

\[
\bar{Y}_p = \Gamma_a \bar{X}_p + H_a \bar{Z}_p + \Psi_a \bar{\epsilon}_p, \tag{4.17}
\]

\[
\bar{Y}_t = \Gamma_a \bar{X}_t + H_a \bar{Z}_t + \Psi_a \bar{\epsilon}_t, \tag{4.18}
\]

\[
\bar{X}_t = A^a \bar{X}_p + \Delta_a \bar{Z}_p + \Omega_a \bar{\nu}_p. \tag{4.19}
\]

Next, the problem of finding \( \Gamma_a \) by using the augmented equations (4.17–4.19) can be solved by applying the RQ decomposition. However, since (4.17–4.19) have noise terms which do not appear in (4.14–4.16), one should wonder if (4.14–4.16) and (4.17–4.19) will provide the same solution for \( \Gamma_a \).

Strictly speaking, they do not; however, we are going to show that both equations have the same solution in the space of interest spanned by the rows of \( \bar{X}_0^0 \Pi_{\bar{Z}_0^1}^\perp \), which is the result presented in the SIVID algorithm (Theorem 4.1). Then, this result will be extended to formulate a new algorithm which accounts for the additional noise terms.

**Theorem 4.2** The least square solutions of \( \Gamma_a \) from (4.17–4.19) and (4.14–4.16) are identical in the row space of \( \bar{X}_0^0 \Pi_{\bar{Z}_0^1}^\perp \).

**Proof.** From (4.17) and (4.19), we have

\[
\bar{X}_t = A^a \Gamma_a^\dagger (\bar{Y}_p - H_a \bar{Z}_p - \Psi_a \bar{\epsilon}_p) + \Delta_a \bar{Z}_p + \Omega_a \bar{\nu}_p,
\]

\[
= [\Delta_a - A^a \Gamma_a^\dagger H_a A^a \Gamma_a^\dagger] [\bar{Z}_p^\top \bar{Y}_p^\top]^\top + \Omega_a \bar{\nu}_p - A^a \Gamma_a^\dagger \Psi_a \bar{\epsilon}_p
\]

\[
= L_t \bar{W}_p + \Omega_a \bar{\nu}_p - A^a \Gamma_a^\dagger \Psi_a \bar{\epsilon}_p. \tag{4.20}
\]

By substituting (4.20) into (4.18), we have

\[
\bar{Y}_t = \Gamma_a L_t \bar{W}_p + H_a \bar{Z}_t + \Gamma_a (\Omega_a \bar{\nu}_p - A^a \Gamma_a^\dagger \Psi_a \bar{\epsilon}_p) + \Psi_a \bar{\epsilon}_t. \tag{4.21}
\]

Equation (4.21) can be decomposed into two equations

\[
\bar{Y}_t^0 = \Gamma_a L_t \bar{W}_p^0 + H_a \bar{Z}_t^0 \quad \text{and}
\]

\[
\bar{Y}_t^1 = \Gamma_a L_t \bar{W}_p^1 + H_a \bar{Z}_t^1 + [\Gamma_a (\Omega_a \bar{\nu}_p^1 - A^a \Gamma_a^\dagger \Psi_a \bar{\epsilon}_p^1) + \Psi_a \bar{\epsilon}_t^1], \tag{4.23}
\]

where (4.22) can also be derived from (4.14–4.16) (refer to (4.4) and (4.5)), and in (4.23) the term within the bracket is caused by noise signals. Following similar steps to those presented in Remark 4.1, we obtain

\[
X_t^0 = L_t \bar{W}_p^0. \tag{4.24}
\]

Projecting both sides of (4.22) and (4.23) on the orthogonal complements of the row space of \( \bar{Z}_t^0 \) and \( \bar{Z}_t^1 \), respectively, and applying transpose on both sides of the equations, we can rewrite (4.22–4.23) as

\[
\bar{A}_0 \Gamma_a^\top = \bar{B}_0, \tag{4.25}
\]

\[
\bar{A}_1 \Gamma_a^\top = \bar{B}_1, \tag{4.26}
\]
where \( \tilde{B}_0^\top \triangleq Y_0^0 \Pi_0^\top \in \mathbb{R}^{l \times n_0} \), \( \tilde{B}_1^\top \triangleq Y_1^1 \Pi_1^\top - \{ \Gamma_a (Q_a \mathcal{W}_p^1 - A^a \Gamma_a^\top \mathcal{F}_p^1) + \mathcal{F}_a \mathcal{E}_p^1 \} \Pi_1^\top \in \mathbb{R}^{l \times n_1} \),
\( \tilde{A}_0^\top \triangleq L_t \tilde{W}_p^0 \Pi_0^\top \in \mathbb{R}^{n \times n_0} \), using (4.24), and \( \tilde{A}_1^\top \triangleq L_t \tilde{W}_p^1 \Pi_1^\top \in \mathbb{R}^{n \times n_1} \). Next, we will prove the proposition based on the row space of \( \tilde{A}_0^\top \), which is identical to the row space of \( X_0^0 \Pi_0^\top \).

Equations (4.25–4.26) can be rewritten as

\[
\begin{bmatrix}
\tilde{A}_0 \\
\tilde{A}_1
\end{bmatrix}
\Gamma_a^\top =
\begin{bmatrix}
\tilde{B}_0 \\
\tilde{B}_1
\end{bmatrix}.
\]  

(4.27)

Since \([ \tilde{A}_0^\top \tilde{A}_1^\top ] \in \mathbb{R}^{n \times 2a(q+l)} \) and \( n < 2a(q+l) \), there always exists a non-zero solution \( x \) to the system of equations \( \Gamma_a^\top = [ \tilde{A}_0^\top \tilde{A}_1^\top ] x \). Let us denote this solution \( x \) with \([ \tilde{\omega}_0^\top \tilde{\omega}_1^\top ]^\top \). Substituting this result in (4.27), we obtain

\[
\begin{bmatrix}
\tilde{\omega}_0 \\
\tilde{\omega}_1
\end{bmatrix}
= \begin{bmatrix}
\tilde{A}_0 \tilde{A}_0^\top & \tilde{A}_0 \tilde{A}_1^\top \\
\tilde{A}_1 \tilde{A}_0^\top & \tilde{A}_1 \tilde{A}_1^\top
\end{bmatrix}^{-1}
\begin{bmatrix}
\tilde{B}_0 \\
\tilde{B}_1
\end{bmatrix},
\]  

(4.28)

which, by applying the generalized Schur complement (Lei et al., 2003), is equivalent to

\[
[ \tilde{A}_1 \tilde{A}_1^\top - \tilde{A}_1 \tilde{A}_0^\top (\tilde{A}_0 \tilde{A}_0^\top)^\dagger \tilde{A}_0 \tilde{A}_1^\top ] \tilde{\omega}_1 = [ \tilde{A}_1 (I - \Pi_{\tilde{A}_0}) \tilde{A}_1^\top ] \tilde{\omega}_1
= \tilde{B}_1 - \tilde{A}_1 \tilde{A}_0^\top (\tilde{A}_0 \tilde{A}_0^\top)^\dagger \tilde{B}_0,
\]  

(4.29)

where \( \Pi_{\tilde{A}_0} \triangleq \tilde{A}_0^\top (\tilde{A}_0 \tilde{A}_0^\top)^\dagger \tilde{A}_0 \) is the orthogonal projection operator onto the row space of \( \tilde{A}_0 \). From (4.28), we have \( \tilde{\omega}_0 = (\tilde{A}_0 \tilde{A}_0^\top)^\dagger [\tilde{B}_0 - \tilde{A}_0 \tilde{A}_1^\top \tilde{\omega}_1] \). Therefore,

\[
\Gamma_a^\top = \tilde{A}_0^\top \tilde{\omega}_0 + \tilde{A}_1^\top \tilde{\omega}_1
= \tilde{A}_0^\top (\tilde{A}_0 \tilde{A}_0^\top)^\dagger [\tilde{B}_0 - \tilde{A}_0 \tilde{A}_1^\top \tilde{\omega}_1] + \tilde{A}_1^\top \tilde{\omega}_1
= \tilde{A}_0^R \tilde{B}_0 + (I - \Pi_{\tilde{A}_0}) \tilde{A}_1^\top \tilde{\omega}_1,
\]  

(4.30)

where \( \tilde{A}_0^R \triangleq \tilde{A}_0^\top (\tilde{A}_0 \tilde{A}_0^\top)^\dagger \) is the right inverse of \( \tilde{A}_0 \). Then, by left multiplying \( \tilde{A}_0 \) on both sides of (4.30), we have \( \tilde{A}_0 (I - \Pi_{\tilde{A}_0}) \tilde{A}_1^\top \tilde{\omega}_1 = 0 \) and

\[
\tilde{A}_0 \Gamma_a^\top = \tilde{A}_0 \tilde{A}_0^R \tilde{B}_0.
\]  

(4.31)

Furthermore, by left multiplying \( \tilde{A}_1 \) on both sides of (4.30) and using (4.29), we have

\[
\tilde{A}_1 \Gamma_a^\top = \tilde{A}_1 \tilde{A}_0^R \tilde{B}_0 + \tilde{A}_1 (I - \Pi_{\tilde{A}_0}) \tilde{A}_1^\top \tilde{\omega}_1,
= \tilde{A}_1 \tilde{A}_0^R \tilde{B}_0 + \tilde{B}_1 - \tilde{A}_1 \tilde{A}_0^\top (\tilde{A}_0 \tilde{A}_0^\top)^\dagger \tilde{B}_0,
= \tilde{B}_1.
\]  

(4.32)

Note that (4.31) is different from (4.25) as the right-hand side of (4.31) lies in the row space of \( \tilde{A}_0 \tilde{A}_0^R \) instead of \( \tilde{B}_0 \) itself. However, since \( \tilde{A}_0 \in \mathbb{R}^{n_0 \times n} \) with \( n_0 > n \), the least square solutions of \( \Gamma_a \) from (4.25) and (4.31) are identical. Furthermore, (4.30) implies that \( \Gamma_a \) can be represented by the projection
of $\tilde{B}_0^\top$ on the row space of $\tilde{A}_0$ with an additional term which lies in its complementary space. Therefore, we can conclude that $\Gamma_\alpha$ in the form of (4.30) is indeed a least square solution of (4.17–4.19).

On the other hand, $\Gamma_\alpha^\top = \tilde{A}_0^R \tilde{B}_0$ is a least square solution of (4.25), which is derived from (4.22) (or (4.14–4.16)). Therefore, the least square solutions of $\Gamma_\alpha$ from (4.21) and (4.22) are actually identical in the row space of $\tilde{A}_0^\top$ which is that of $\tilde{X}_f^0 \Pi_1^{\perp \tilde{Z}_i}$, so are those from (4.17–4.19) and (4.14–4.16), which completes the proof.

**Remark 4.7** The regressor in (4.27) is a $2\alpha(\varrho + l) \times n$ matrix with $2\alpha(\varrho + l) > n$ while that in (4.25) is a $n_0 \times n$ matrix with $n_0 > n$ which indicates the uniqueness of the least square solutions of $\Gamma_\alpha$ from (4.17–4.19) and (4.14–4.16). Furthermore, the fact that $A_0(I - \Pi_{\tilde{A}_0})\tilde{A}_0^\top \tilde{w}_1 = 0$ implies that any change of $\tilde{B}_1$, say $\tilde{B}_1^\top = \tilde{Y}_i^1 \Pi_1^{\perp \tilde{Z}_i}$, does not affect the solution in the row space of $\tilde{A}_0$ which is identical in the row space of $\tilde{X}_f^0 \Pi_1^{\perp \tilde{Z}_i}$.

We have the following corollary.

**Corollary 4.1** Consider the following equations:

\[
\tilde{Y}_p = \Gamma_\alpha \tilde{X}_p + H_a \tilde{Z}_p, \tag{4.33}
\]
\[
\tilde{Y}_i = \Gamma_\alpha \tilde{X}_i + H_a \tilde{Z}_i, \tag{4.34}
\]
\[
\tilde{X}_i = A^a \tilde{X}_p + \Delta_a \tilde{Z}_p. \tag{4.35}
\]

The least square solution of $\Gamma_\alpha$ from (4.33–4.35) and (4.14–4.16) are identical in the row space of $\tilde{X}_f^0 \Pi_1^{\perp \tilde{Z}_i}$.

**Proof.** Following Remark 4.7, it can be induced that the noise terms in the system (4.17–4.19), which are only contained in the matrix $\tilde{B}_1$, do not affect the solution of $\Gamma_\alpha$ in the row space of $\tilde{X}_f^0 \Pi_1^{\perp \tilde{Z}_i}$ which is the space of interest; however, they do affect the solution in its complementary space which we do not care about. In another word, the noise terms can be ignored in the augmented system functions (4.17–4.19), which leads to (4.33–4.35).

Note that, from simple steps applied to (4.33–4.35), we obtain

\[
[ \tilde{Y}_f^0 \tilde{Y}_i^1 ] = \Gamma_\alpha [ L_t \tilde{w}_0 L_t \tilde{w}_p^1 ] + H_a [ \tilde{Z}_f^0 \tilde{Z}_i^1 ],
\]

or,

\[
\tilde{Y}_i = \Gamma_\alpha L_t \tilde{w}_p + H_a \tilde{Z}_i, \tag{4.36}
\]

which leads to the following theorem.

**Theorem 4.3** (ASIVID algorithm) Consider the system (3.5–3.6) satisfying the conditions described in Theorem 4.1 and the augmented IV-correlated equations (4.33–4.35). Define

\[
\hat{\Omega}_a \triangleq \tilde{Y}_i / \tilde{Z}_i \tilde{w}_p \tag{4.37}
\]

and consider the SVD

\[
W_L \hat{\Omega}_a W_R = (U_1 U_2) \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T, \tag{4.38}
\]
where the weighting matrices $W_L \in \mathbb{R}^{l \times l}$ is full rank and $W_R \in \mathbb{R}^{2(l+\rho) \times 2l}$ is such that \( \text{rank}(\Pi_{\tilde{z}_t}^\perp) = \text{rank}(\Pi_{\tilde{z}_t}^\perp W_R) \). Furthermore, $S_1$ contains the dominant singular values. Then, we have $\Gamma_a = W_L^{-1} U_1 S_1^{1/2}$ and the order of the system (3.5–3.6) is equal to the size of $S_1$.

**Proof.** The proof is identical to that of Theorem 4.1 after replacing (4.10) with (4.36).

**Remark 4.8** Worth noting is that the choice of the lower block of the augmented IV matrix is not unique. Our choice was $I_{\beta \times n_1}$, which is the simplest way to ensure the full rank property when implementing the correlation with the input–output block Hankel matrix. It is obvious that by multiplying a constant factor $\gamma \geq 0$ with $I_{\beta \times n_1}$ this property is not changed. The special case when $\gamma = 0$ is equivalent to that of the non-augmented IV matrix (Theorem 4.1), and the algorithm proposed in this section is equivalent to the case when $\gamma = 1$. The parameter $\gamma$ can be used to adjust the correction term of $\Gamma_a$ lying in the orthogonal complement space of $\tilde{X}_f^0 \Pi_{\tilde{z}_t}^\perp$. In this way, the numerical stability of RQ decomposition and SVD could be improved, which is a topic of further investigation.

Suppose that an IV matrix $Y \in \mathbb{R}^{n_0 \times \beta}$ is available such that $\tilde{X}_f^0 \triangleq \tilde{X} (\mathcal{E}_f Y) \equiv 0$ and $\tilde{E}_p^0 \triangleq \tilde{E} (\mathcal{E}_f Y) \equiv 0$. The Augmented IV subspace estimation algorithm for the linear system (3.1–3.2) can be described as follows.

**Algorithm 4.1 ASIVED algorithm**

(i) Transform $Y$ into an augmented matrix $\tilde{Y} \triangleq [Y^T, I_{\beta \times n_1}]^T \in \mathbb{R}^{2(m+l) \times \beta}$, where $n_1 \triangleq 2(m+l)\alpha - n_0$ and $I_{\beta \times n_1} = [I_{\beta \times n_1}, 0_{n_1 \times (\beta - n_1)}]$.

(ii) Compute IV correlated matrices $\tilde{Z}_{p,t} = \tilde{E}(\tilde{Z}_{p,t} \tilde{Y}^T)$ and $\tilde{Y}_{p,t} = \tilde{E}(\tilde{Y}_{p,t} \tilde{Y}^T)$.

(iii) The numerical implementation of oblique projection $\hat{O}_\alpha \triangleq \tilde{Y}_{l/\tilde{z}_t} \tilde{W}_p$ is described below.

(a) The RQ decomposition of the input-output block Hankel matrix

\[
\begin{pmatrix}
\tilde{U}_p \\
\tilde{U}_f \\
\tilde{Y}_p \\
\tilde{Y}_f
\end{pmatrix} = \begin{pmatrix}
\alpha & \alpha & l & l & 2\alpha(\beta + l)
\end{pmatrix} \begin{pmatrix}
R_{11} & 0 & 0 & 0 & Q_1^T \\
R_{21} & R_{22} & 0 & 0 & Q_2^T \\
R_{31} & R_{32} & R_{33} & 0 & Q_3^T \\
R_{41} & R_{42} & R_{43} & R_{44} & Q_4^T
\end{pmatrix} \triangleq RQ^T.
\]

(b) Define

\[
R_{\tilde{f}_p} \triangleq R((2\alpha + l)\alpha + 1 \ldots 2(\beta + l)\alpha, \cdot) \in \mathbb{R}^{li \times 2(\beta + l)i},
\]

\[
R_{\tilde{w}_p} \triangleq [R(1 \ldots 2\alpha, \cdot); R(2\alpha + 1 \ldots (2\alpha + l)\alpha, \cdot)] \in \mathbb{R}^{(\beta + l)i \times 2(\beta + l)i},
\]

\[
R_{\tilde{z}_t} \triangleq R(\rho \alpha + 1 \ldots 2\rho \alpha, 1 \ldots 2\rho \alpha) \in \mathbb{R}^{\rho i \times 2\rho i},
\]
and let
\[
R_{\hat{y}_l,\hat{z}_l} \triangleq [R_{\hat{y}_l}(\cdot, 1 \ldots 2q\alpha) - R_{\hat{y}_l}(\cdot, 1 \ldots 2q\alpha)R_{\hat{z}_l}]^\top R_{\hat{z}_l},
\]
\[
R_{\hat{y}_l}(\cdot, 2q\alpha + 1 \ldots 2(q + l)\alpha),
\]
\[
R_{\hat{w}_p,\hat{z}_l} \triangleq [R_{\hat{w}_p}(\cdot, 1 \ldots 2q\alpha) - R_{\hat{w}_p}(\cdot, 1 \ldots 2q\alpha)R_{\hat{z}_l}]^\top R_{\hat{z}_l},
\]
\[
R_{\hat{w}_p}(\cdot, 2q\alpha + 1 \ldots 2(q + l)\alpha).
\]

(c) Obtain
\[
\hat{O}_\alpha = R_{\hat{y}_l,\hat{z}_l}(R_{\hat{w}_p,\hat{z}_l})^\top R_{\hat{w}_p}.
\]

(iv) Perform the SVD on the weighted matrix \( \hat{O}_\alpha \), if the system order \( n \) is known, we have
\[
W_L \hat{O}_\alpha W_R = \hat{U} \hat{S} \hat{V}^\top \approx \hat{U}_1 \hat{S}_1 \hat{V}_1^\top,
\]
where \( \hat{U}, \hat{V}, \hat{S}, \hat{U}_1, \hat{V}_1, \hat{S}_1, W_L \) and \( W_R \) are defined as before.

(v) Compute \( \hat{I}_\alpha = W_L^{-1} \hat{U}_1 \hat{S}_1^{1/2} \) and \( \hat{X}_f = S_1^{1/2} V_1^\top W_R^{-1} \).

(vi) Recover the system matrices \( \hat{A}, \hat{B}, \hat{C} \) and \( \hat{D} \) from \( \hat{I}_\alpha \) and \( \hat{X}_f \).

5. Numerical simulations and experiments

In this section, we first introduce the truncated series expansion used in the identification algorithms, then present some numerical examples for model recovery and system order determination.

5.1 Orthonormal series approximation and prediction quality measurements

The main motivations for utilizing orthonormal series in non-linear identification approaches are model simplification, fast convergence and smaller set of coefficients to be determined. Furthermore, such a choice reduces both bias and variance of estimated coefficients (Ljung, 1999; Van den Hof et al., 1995).

In particular, by choosing the \( \eta_j(\cdot), j = 0, \ldots, p/2, \) to be the \( j \)th harmonics of both \( \sin(\cdot) \) and \( \cos(\cdot) \) functions, (3.3) becomes a truncated Fourier series expansion. Thus, the non-linear function \( f_t(u(k)) \), defined in the interval \([u_{\min}, u_{\max}]\), is approximated by the following expansion (Leonessa & Luo, 2001):

\[
\hat{f}_t(u(k)) \equiv \mu_t^\top \eta(u(k)) \triangleq \mu_{t_0} + \sum_{i=1}^{p/2} \mu_{t_i} \eta_i(u(k)), \quad t = 1, 2, \ldots, r,
\tag{5.1}
\]

where \( \mu_t \triangleq [\mu_{t_0}, \mu_{t_1}^\top, \ldots, \mu_{t_{p/2}}^\top]^\top \), with \( \mu_{t_j} \triangleq [\mu_{t_{j_1}}, \mu_{t_{j_2}}]^\top, j = 1, \ldots, p/2, \) and \( \eta(u(k)) \triangleq [1, \eta_1^\top(u(k)), \eta_2^\top(u(k)), \ldots, \eta_{p/2}^\top(u(k))]^\top \), with \( \eta_j(u(k)) \triangleq [\eta_{c_j}(u(k)), \eta_{s_j}(u(k))]^\top \), where
\[
\eta_{c_j}(u(k)) \triangleq \cos(j \pi (u(k) - u_m)/L),
\]
\[
\eta_{s_j}(u(k)) \triangleq \sin(j \pi (u(k) - u_m)/L), \quad j = 1, 2, \ldots, p/2.
\tag{5.2}
\]
and $u_m \triangleq (u_{\text{min}} + u_{\text{max}})/2$, $L \triangleq (u_{\text{max}} - u_{\text{min}})/2$. The truncated Fourier expansion for $g_h(y(k))$ can be constructed following identical steps.

The quality of prediction is measured by the prediction errors (in percentage) (Overschee & Moor, 1996b)

$$
e_i = \begin{cases} 
100 \times \sqrt{\frac{\sum_{k=1}^{N}(y_i(k) - \hat{y}_i(k))^2}{\sum_{k=1}^{N}y_i^2(k)}}, & \text{if } e(i) \leq 100, \quad i = 1, \ldots, l, \\
100, & \text{if } e(i) > 100,
\end{cases}$$

where $i$ is the index of output channels and $N$ is the number of samples used for validation.

### 5.2 Example 1: System with knowledge of non-linearities

The purpose of this simulation is to study the case when a dynamic system has input/feedback non-linearities in the form of sine/cosine functions such that no approximation error is added. In this way, the performance of the proposed algorithm to noise-contaminated data can be examined.

#### 5.2.1 System description

In this simulation, the identification algorithms presented in Section 4.3 are tested by using data sampled from a system with linear dynamics and non-linear input $f(u) = \sin(u)$ and feedback $g(y) = \sin(y) + 0.5\sin(2y)$. Suppose the model of the system is written as

$$
\begin{align*}
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} &= \begin{bmatrix} 0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix} x_1(t) \\
 x_2(t)
\end{bmatrix} + \begin{bmatrix} 0 \\
 g(y_2(t)) + f(u(t))
\end{bmatrix} + \begin{bmatrix} \omega_1(t) \\
 \omega_2(t)
\end{bmatrix}, \\
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} &= \begin{bmatrix} 1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix} x_1(t) \\
 x_2(t)
\end{bmatrix} + \begin{bmatrix} v_1(t) \\
v_2(t)
\end{bmatrix},
\end{align*}
$$

where $\omega_1(\cdot)$, $\omega_2(\cdot)$, $v_1(\cdot)$ and $v_2(\cdot)$ are Gaussian normal white noise signals and the sampling period is $T = 0.05$ s. The input signal $u$ was a length of $N = 6000$, zero mean, white noise sequence uniformly distributed between $-1$ and $1$. The input–output is shown in Fig. 2.

#### 5.2.2 Identification results

The input and feedback non-linearities are parameterized using the truncated Fourier series with $p = 8$ and $q = 8$, respectively. The number of block rows defined in (2.2) and (2.3) is $a = 6$. In the following simulations, sampled data of length $N$ are used for identification, then an additional 1000 samples are used to validate the identification algorithm.

**Case 1.** Noise free

Consider a case when the system (5.3–5.4) is noise free, i.e. $\omega_1(t)$, $\omega_2(t)$, $v_1(t)$ and $v_2(t)$ are identical to zero. The identification results by using SIVID and SID algorithms are shown in Fig. 3.

We found that, when the system is noise free, both SIVID and SID algorithms provide good prediction results. We have repeated the experiment from $N = 4000$ to $N = 7000$, where the results are almost identical. The results from Fig. 3, where $N = 6000$, show that, without additive noise, the identification results by using the open-loop SIVID algorithm and the open-loop SID algorithm are very similar, since there is no correlation between the concatenated input vector $z$ and noise vector $\varepsilon$. In this case, the open-loop algorithm works well to predict this unknown system and the closed-loop one does not provide any additional benefit.
Case 2. System with additive noises

The process and measurement noises, \( \omega_1(\cdot), \omega_2(\cdot), v_1(\cdot) \) and \( v_2(\cdot) \) are considered to be Gaussian normal white noise signals, with covariance 0.090, 0.087, 0.0025 and 0.0025, respectively. The identification results by using SIVID and SID algorithms are shown in Fig. 4, where \( N = 6000 \). The results in Fig. 4 clearly show that, when the system has additive noises, the closed-loop SIVID algorithm has better prediction performance than the open-loop SID algorithm. Similar results are shown more clearly in Fig. 5, where \( N = 7000 \) is used.

5.3 Example 2: System without knowledge of non-linearities

5.3.1 System description Next, we consider the Van der Pol oscillator with an input non-linearity \( f(u) = u^2 \) and feedback non-linearity \( g(y_1, y_2) = (1 - y_1^2)y_2, \)

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
(1 - y_1^2(t))y_2(t)
\end{bmatrix} +
\begin{bmatrix}
u_1(t) \\
u_2(t)
\end{bmatrix} +
\begin{bmatrix}
\omega_1(t) \\
\omega_2(t)
\end{bmatrix},
\]

(5.5)

\[
\begin{bmatrix}
y_1(t) \\
y_2(t)
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
v_1(t) \\
v_2(t)
\end{bmatrix},
\]

(5.6)

where \( u(t), \omega_1(t), \omega_2(t), v_1(t) \) and \( v_2(t), t \geq 0 \) are defined as those in Section 5.2. The non-linearities are represented by truncated trigonometric series, and non-linear approximation residue will exist. The input–output map is shown in Fig. 6.
5.3.2 Identification results The number of block rows defined in (2.2) and (2.3) is $\alpha = 6$. 

Case 1. Noise free

Consider the case when the system (5.5–5.6) is noise free, i.e. $w_1(t)$, $w_2(t)$, $v_1(t)$ and $v_2(t)$ are identical to zero. The input and feedback non-linearities are parameterized using the truncated Fourier series with $p = 6$ and $q = 6$, respectively. The identification results by using SIVID and SID algorithms are shown in Fig. 7. The results show that the SIVID and the SID algorithms provide almost identical prediction.
qualities when the system is noise free, regardless of the existence of the non-linear approximation residues. It also implies that the approximation residues have little correlation with the concatenated input $z$. The prediction errors in percentage, as introduced in Section 5.1, are $e = [81.0, 95.7]$ for the SID algorithm and $e = [81.5, 94.6]$ for the SIVID algorithm. When the length of the Fourier series is increased, i.e. $p = 8$ and $q = 8$, the identification results are shown in Fig. 8. The prediction errors are $e = [50.9, 69.3]$ for the SID algorithm and $e = [49.5, 66.3]$ for the SIVID algorithm. The comparison
between Figs 7 and 8, or the corresponding prediction errors, indicates an improvement of the prediction quality as $p$ and $q$ increase, since the non-linear approximation residues will decrease accordingly. When $p$ and $q$ continue to increase, however, a numerical problem arises due to the computational burden which increases accordingly. On the other hand, when $p$ and $q$ approach certain limits, the improvement of the prediction quality is not obvious as $p$ and $q$ increase further, although still can be qualitatively shown by the comparisons between prediction errors. For example, when $p = 14$ and
When implementing the proposed algorithm, the use of an instrumental variable vector that does not perfectly satisfy the assumptions may introduce additional errors due to, e.g. the loss of useful information. This may sometimes deteriorate the results provided by the SIVID approach, as shown in the previous example. The next example will clearly show that when the system contains additional noise signals, the benefits of the SIVID approach outweigh the eventual deterioration caused by an imperfect choice of the IV vector.

**Case 2. System with additive noise**
In this example, the process and measurement noises, $\omega_1(\cdot), \omega_2(\cdot), v_1(\cdot)$ and $v_2(\cdot)$ are considered to be Gaussian normal white noise signals, with covariance 0.160, 0.163, 0.0025 and 0.0025, respectively. The identification results by using the SIVID and the SID algorithms are shown in Fig. 9 which clearly show that, when the system has additive noises, the closed-loop SIVID algorithm provides much better prediction performance than the open-loop SID algorithm.

### 5.4 Investigation of computational efficiency
Consider the system described in Case 2 of Section 5.3 with $p, q = 4$ and $\alpha = 6$. A comparison of the computing time (ms) when using the ASIVID and the SIVID algorithms is performed, and the average result from 91 simulations is considered. The average computing times at various data length from 2000 to 58000 are presented in Table 1 which shows that the computing efficiency of the ASIVID algorithm
is considerable when the length of data used in identification is large. When the data length is short, however, the ASIVID algorithm does not show computing efficiency over the SIVID algorithm. This is due to the fact that when the data length increases, the block Hankel matrix dimensions increase as well, hence some matrix operations like projection and inversion become more and more overwhelming, which is not the case when the data length is short. The comparison is also shown in Fig. 10, where the computing time of both algorithms increases linearly with respect to the data length.
6. Conclusion

The main contribution of this paper is the presentation of a unified approach to solve SIVID problems for MIMO Hammerstein systems with non-linear feedback, when a truncated series expansion is chosen to approximate the non-linearities appearing in the system. Furthermore, numerical issues are addressed by extending existing IV approaches during their implementation. The proposed approach overcomes the correlations between the system input signals and noise signals in the closed-loop system by choosing...
apparent VI signals. Simulation results have shown that the SVID approach can greatly improve the prediction performance, proving that this approach provides a powerful tool to model a non-linear grey-box closed-loop system. Moreover, an ASIVID algorithm is proposed to address the computational efficiency problem, aroused by the concern of heavy matrix computations. We have rigorously proved that, when using the proposed algorithm, the ASIVID approach has an identical solution in the space of interest as that of its unaugmented counterpart. Furthermore, the ASIVID approach allows the use of powerful tools such as RQ decomposition, which makes it much more efficient in its total computational effort, especially when the volume of processing data heavily increases.
TABLE 1 Van der Pol oscillator in Section 5.3: comparison of average computing efficiency between the ASIVID algorithm and the SIVID algorithm with $p, q = 4$ and $\alpha = 6$

<table>
<thead>
<tr>
<th>Data length</th>
<th>ASIVID algorithm</th>
<th>SIVID algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>$1.4799 \times 10^{+003}$</td>
<td>$1.0711 \times 10^{+003}$</td>
</tr>
<tr>
<td>6000</td>
<td>$2.3877 \times 10^{+003}$</td>
<td>$2.1975 \times 10^{+003}$</td>
</tr>
<tr>
<td>10000</td>
<td>$3.1969 \times 10^{+003}$</td>
<td>$3.3210 \times 10^{+003}$</td>
</tr>
<tr>
<td>14000</td>
<td>$4.0242 \times 10^{+003}$</td>
<td>$4.3306 \times 10^{+003}$</td>
</tr>
<tr>
<td>18000</td>
<td>$4.8365 \times 10^{+003}$</td>
<td>$5.2947 \times 10^{+003}$</td>
</tr>
<tr>
<td>22000</td>
<td>$5.6734 \times 10^{+003}$</td>
<td>$6.3048 \times 10^{+003}$</td>
</tr>
<tr>
<td>26000</td>
<td>$6.5022 \times 10^{+003}$</td>
<td>$7.3462 \times 10^{+003}$</td>
</tr>
<tr>
<td>30000</td>
<td>$7.3170 \times 10^{+003}$</td>
<td>$8.3754 \times 10^{+003}$</td>
</tr>
<tr>
<td>34000</td>
<td>$8.1331 \times 10^{+003}$</td>
<td>$9.3809 \times 10^{+003}$</td>
</tr>
<tr>
<td>38000</td>
<td>$8.9568 \times 10^{+003}$</td>
<td>$1.0399 \times 10^{+004}$</td>
</tr>
<tr>
<td>42000</td>
<td>$9.7425 \times 10^{+003}$</td>
<td>$1.1417 \times 10^{+004}$</td>
</tr>
<tr>
<td>46000</td>
<td>$1.0549 \times 10^{+004}$</td>
<td>$1.2420 \times 10^{+004}$</td>
</tr>
<tr>
<td>50000</td>
<td>$1.1358 \times 10^{+004}$</td>
<td>$1.3427 \times 10^{+004}$</td>
</tr>
<tr>
<td>54000</td>
<td>$1.2168 \times 10^{+004}$</td>
<td>$1.4452 \times 10^{+004}$</td>
</tr>
<tr>
<td>58000</td>
<td>$1.2973 \times 10^{+004}$</td>
<td>$1.5439 \times 10^{+004}$</td>
</tr>
</tbody>
</table>

FIG. 10. Van der Pol oscillator in Section 5.3: comparison of average computing efficiency between the ASIVID algorithm and the SIVID algorithm with $p, q = 4$ and $\alpha = 6$. 
REFERENCES


