Spaces with Congruence

Alexander Kreuzer
Fachbereich Mathematik
Universität Hamburg
Bundesstr. 55, 20146 Hamburg

1 Introduction

It is well known that every Euclidean plane $(E, \mathcal{L}, \alpha, \equiv)$ is isomorphic to an affine plane $\text{AG}(2,K)$ over a Pythagorean ordered commutative field $K$. We may consider $E$ as an quadratic separable field extension of $K$ with the corresponding involutory field automorphism $\tau: E \to E$ and we have $(a, b) \equiv (c, d)$ if and only if $(a - b)(a - b) = (c - d)(c - d)$ for any points $a, b, c, d \in E$.

$(E, \mathcal{L})$ denotes an affine plane, $(E, \mathcal{L}, \alpha)$ an ordered plane, and $\equiv$ denotes the congruence relation on $E \times E$. If we assume $(E, \mathcal{L}, \alpha)$ as an hyperbolic plane, there exists the corresponding theorem for hyperbolic planes (cf. [3]). For both proofs we consider first the group of motions, in particular the line reflections. For the definition of a motion and a line reflection we need only a congruence relation $\equiv$ (cf. [1, 7, 8]), but already for the proof that the line reflection is a motion it seems that additional assumptions on the geometry are necessary. For example K. Sørensen assumes here that for given lines $G_1, G_2$, there exist distinct lines $H_1, H_2$ through a common point $z$ which intersect $G_1, G_2$ (cf. [8]). Also for the proof that for any line $G$ and any point $x$ we have a unique perpendicular line through $x$, it seems that one needs additional assumptions.

Here we restrict ourselves not to planes. We consider a space with congruence not assuming any geometrical properties, and we deal with the group of motions. We assume that the planes satisfy the exchange property, which can be easily shown for ordered spaces. We also assume an additional property (W4), which is also valid in an ordered space with congruence.
2 Spaces with congruence

Let \((P, \mathcal{L})\) denote a linear space or incidence space.

A subspace is a subset \(U \subset P\) such that for all distinct points \(x, y \in U\) the unique line passing through \(x\) and \(y\), denoted by \(\overline{xy}\), is contained in \(U\). Let \(\mathcal{U}\) denote the set of all subspaces. For every subset \(X \subset P\) we define the following closure operation

\[
\overline{\cdot} : \mathcal{P}(P) \rightarrow \mathcal{U}, \quad X \mapsto \overline{X} \quad \text{by} \quad \overline{X} := \bigcap_{U \in \mathcal{U}} U \quad X \subset U
\]

For \(U \in \mathcal{U}\) we call \(\dim U := \inf \{[X]_1 : X \subset U \text{ and } \overline{X} = U\}\) the dimension of \(U\). A subspace of dimension two is a plane.

We introduce the concept of a space \((P, \mathcal{L}, \equiv)\) with congruence (cf. [8]).

We assume in this section a linear space \((P, \mathcal{L})\) which planes satisfy the following exchange condition.

(EC) Let \(S \subset P\) and let \(x, y \in P\) with \(x \in S \cup \{y\}\). Then \(y \in S \cup \{x\}\).

Let \(\equiv\) be a congruence relation on \(P \times P\), i.e.

\(\equiv\) is a equivalence relation with

\((a, b) \equiv (b, a)\) and

\((a, a) \equiv (b, c)\) if and only if \(b = c\).

We use the notation \((x_1, x_2, x_3) \equiv (y_1, y_2, y_3)\) if and only if \((x_i, x_j) \equiv (y_i, y_j)\) for \(i, j \in \{1, 2, 3\}\). \((P, \mathcal{L}, \equiv)\) is a space with congruence if the axioms (W1), (W2) and (W3) are satisfied.

(W1) Let \(a, b, c \in P\) be distinct and collinear, and let \(a', b' \in P\) with \((a, b) \equiv (a', b')\). Then there exists exactly one \(c' \in a'b'\) with \((a, b, c) \equiv (a', b', c')\).

(W2) Let \(a, b, x \in P\) be non-collinear and let \(a', b', x' \in P\) with \((a, b, x) \equiv (a', b', x')\). For any \(c \in \overline{ab}\) and \(c' \in \overline{a'b'}\) with \((a, b, c) \equiv (a', b', c')\) it holds \((x, c) \equiv (x', c')\).

(W3) For \(a, b, x \in P\) non-collinear there exists exactly one \(x' \in \{a, b, x\} \setminus \{x\}\) with \((a, b, x) \equiv (a, b, x')\).

Let \((P, \mathcal{L}, \equiv)\) be a space with congruence. Then \((P, \mathcal{L}, \equiv)\) is called a space with strong congruence, if in addition (W4) is satisfied:
(W4) Let $a, b, b'$ be distinct collinear points with $(a, b) \equiv (a, b')$ and let $a, c, c'$ be distinct collinear points with $(a, c) \equiv (a, c')$. Then $(b, c) \equiv (b', c')$.

Using that every plane satisfies the exchange condition, we can show the following important lemma for a space with congruence:

**Lemma 2.1** Let $a, b, x \in P$ be non-collinear points and let $x' \in \overline{a, b, x \setminus \{x\}}$ with $(a, b, x) \equiv (a, b, x')$. Then for any $c \in \overline{a, b, x}$ with $(x, c) \equiv (x', c)$ it holds $c \in a, b$.

We call a bijective mapping $\phi: P \to P$ a **motion**, if $(x, y) \equiv (\phi(x), \phi(y))$ for all $x, y \in P$. Every motion is a collineation of $(P, \mathcal{L})$.

For a line $L \in \mathcal{L}$, $x \in P \setminus L$ and $a, b \in L$ with $a \neq b$ there exists by (W3) the unique point $x' \in \overline{L \cup \{x\} \setminus \{x\}}$ with $(a, b, x) \equiv (a, b, x')$. By (W2) $x'$ is independent of the choice of $a, b \in L$, hence we may denote $x' = L(x)$.

We call the following mapping **line reflection**

\[ \tilde{L}: P \to P; \quad x \to \begin{cases} 
  x & \text{if } x \in L, \\
  L(x) & \text{if } x \not\in L.
\end{cases} \]

**Theorem 2.2** Let $L$ be a line of a plane $E$. Then $\tilde{L} \upharpoonright E$ is an involutionary motion.

To show that $\tilde{L}$ is a motion at all, we need additional lemmas. We define for lines $A, B \in \mathcal{L}$:

\[ A \perp B \iff \tilde{\tilde{A}}(B) = B \text{ and } A \neq B. \]

**Lemma 2.3** If $A \perp B$, then $B \perp A$.

For a plane $E$ with congruence the following theorem is known by [8]:

**Theorem 2.4** Let $A, B, C$ be a lines of a plane $E$ through a point $a$. Then there exists a line $D \subset E$ through $a$ with $\tilde{A}\tilde{B}\tilde{C} \upharpoonright E = \tilde{D}$.

**Theorem 2.5** For a plane $E$ and distinct points $a, b, b' \in E$ with $(a, b) \equiv (a, b')$, there exists a line $L \subset E$ with $a \in L$ and $L \perp \tilde{\tilde{a}, \tilde{b}}$.

In general for points $a, b$ of a line $L$, there exists no point $b' \in L \setminus \{b\}$ with $(a, b) \equiv (a, b')$. But one can show:
Lemma 2.6 The following propositions are equivalent:
(a) There are points \(a, b, b'\) with \((a, b) \equiv (a, b')\).
(b) For any points \(x, y\) there is a point \(y' \in \overline{x, y} \setminus \{y\}\) with \((x, y) \equiv (x, y')\).
(c) There are two perpendicular lines \(A, B\) with \(A \cap B \neq \emptyset\).
(d) For a line \(L\) of any plane \(E\) and \(x \in L\), there exists a line \(G \subseteq E\) through \(x\) with \(L \perp G\).

For a space with congruence it is not known, if there may be more than one point \(y'\) on a line \(\overline{a, b} \setminus \{b\}\) with \((a, b) \equiv (a, b')\). Equivalent is the question, if in a plane \(E\) for \(x \in L \subseteq E\) there exists only one perpendicular line \(G \perp L\) in \(E\) through \(x\). Using the axiom (W4) we can show for a space with strong congruence:

Lemma 2.7 For distinct points \(a, b\), there exists at most one point \(y' \in \overline{a, b} \setminus \{b\}\) with \((a, b) \equiv (a, b')\).

In the following we assume, that there are two perpendicular lines with a non-empty intersection. Then by Lemmas 2.6 and 2.7, for any two points \(a, b\) there is exactly one point \(y' \in \overline{a, b} \setminus \{b\}\) with \((a, b) \equiv (a, b')\). Then one can show:

Theorem 2.8 Every line reflection \(\tilde{L}\) is an involutory motion.

If there are two perpendicular line with a non-empty intersection in a space with strong congruence, we can define point reflections. For distinct points \(a, x \in P\) we denote with \(a(x)\) the unique point \(a(x) \in \overline{a, x} \setminus \{x\}\) with \((a, x) \equiv (a, a(x))\). We call the following mapping point reflection:

\[
\tilde{a} : P \to P; \quad x \rightarrow \begin{cases} 
  x & \text{if } x = a, \\
  a(x) & \text{if } x \neq a.
\end{cases}
\]

Theorem 2.9 Every point reflection \(\tilde{a}\) is an involutory motion with \(x = \tilde{a}(x)\) if and only if \(x = a\).

3 Addition of points

In this section we assume that any two distinct points \(a, a'\) have a midpoint \(m \in \overline{a, a'}\) with \((a, m) \equiv (a', m)\). For a fixed point \(0 \in P\) we denote for any
point $a \in P \setminus \{0\}$ the unique midpoint of 0 and $a$ by $a/2$. We denote $0 = 0/2$. Then for the point reflection corresponding to $a/2$ we have
\[ \overline{a/2}(0) = a \] and \[ \overline{a/2}(a) = 0. \]

We define for points $a, b$ the addition $+$ on the point set $P$ by (cf. \cite{2, 5})
\[ a + b := \overline{a/2} \circ \overline{0(b)} = \overline{a/2} \circ \overline{0(b)}. \]

**Theorem 3.1** $(P, +)$ is a Bruck loop (cf. \cite{6}) with the neutral element 0. The point $-a := \overline{0(a)}$ is the inverse of $a \in P$. $(P, +)$ is associative if and only if for three points $a, b, c \in P$ the product $\overline{a \circ b \circ c}$ is a point reflection, too.

**References**


