Abstract

In this paper distribution of zeros of solutions to functional equations in the spaces of discontinuous functions is studied. It will be demonstrated that oscillation properties of functional equations are determined by the spectral radius of a corresponding operator acting in the space of essentially bounded functions. Distances between zeros of solutions are estimated.

Keywords: functional equations, zeros of solutions, Sturm separation theorem, oscillation, nonoscillation, spectral radius

1 Introduction

We consider the following functional equation

\[ y(t) = (Ty)(t), \quad t \in [0, +\infty), \]  \hspace{1cm} (1.0)

where \( T : L^\infty_{(0,\infty)} \to L^\infty_{(0,\infty)} \) is a linear continuous operator, \( L^\infty_{(0,\infty)} \) is the space of essentially bounded functions. The equation

\[ y(t) = \sum_{i=1}^{m} p_i(t)y(g_i(t)), \quad t \in [0, +\infty), \]  \hspace{1cm} (1.1)

\[ y(\xi) = \varphi(\xi), \quad \xi < 0, \]  \hspace{1cm} (1.2)

is a particular case of equation (1.0).

The theory of difference equations was intensively developed during the last two decades. Note in this connection the known monograph by R.P. Agarwal [1]. Let us stress now that the following difference equation

\[ y(n) = \sum_{k=-m}^{m} p_k y(n - k), \quad n \in [0, +\infty), \]  \hspace{1cm} (1.3)

\[ y(n) = \varphi_n, \quad n < 0. \]  \hspace{1cm} (1.4)
can be actually considered as a particular case of equation (1.1), (1.2). If we set in equation (1.1) $g_i(t) = t - i$, where $i$ is an integer number, and the coefficients $p_i(t)$ and the initial function $\varphi(t)$ are constants on each interval $[k, k + 1)$, we obtain that corresponding solutions $y(t)$ are constants ($y(t) = y_k$) on each interval $[k, k + 1)$. In this case we actually have a difference equation (1.3), (1.4). Obviously, each assertion obtained for functional equation (1.1) is also true for difference equation (1.3). One of the main aims of our approach is to derive conclusions about the behavior of solutions to functional equation (1.1) using corresponding properties of difference equation (1.3).

Note that questions of the action of the operator
\[
(Sy)(t) = \sum_{i=1}^{m} p_i(t) y(g_i(t)), \quad t \in [0, +\infty),
\]
where
\[
y(\xi) = 0, \ \xi < 0,
\]
in the space $L^\infty_{[0, +\infty)}$ and estimates of its spectral radius were considered by M.E.Drakhlin (see, for example, [12]) and part of his results were reflected in the known monograph by N.V.Azbelev, V.P.Maksimov and L.F.Rakhmatullina [3]. For the action of the operator $S : L^\infty_{[0, +\infty)} \rightarrow L^\infty_{[0, +\infty)}$ we assume that for each set $M$ from the equality $\text{mes}M = 0$ it follows the equality $\text{mes} g_i^{-1}(M) = 0$ for $i = 1, \ldots, m$. The necessary condition, which is practically very close to the sufficient condition, is the following: there are no intervals $[\nu, \mu]$ such that $g_i(t) = \text{const}$ for $t \in [\nu, \mu]$.

**Definition 1.1.** A function $y \in L^\infty_{[0, +\infty)}$, satisfying equation (1.1) for almost every $t \in [0, +\infty)$ is called a solution of (1.1).

Considering solutions in the space $L^\infty_{[0, +\infty)}$ we should define oscillation for non-continuous functions.

**Definition 1.2.** A point $\nu$ is a zero of a function $y \in L^\infty_{[0, +\infty)}$ if either product
\[
\lim_{t \to \nu^-} \text{ess sup}_{s \in [\nu, t]} y(s) \cdot \lim_{t \to \nu^+} \text{ess inf}_{s \in [\nu, t]} y(s),
\]
or
\[
\lim_{t \to \nu^-} \text{ess inf}_{s \in [\nu, t]} y(s) \cdot \lim_{t \to \nu^+} \text{ess sup}_{s \in [\nu, t]} y(s),
\]
is nonpositive.

Obviously, if $y$ is a continuous function, then a point $\nu$ is its zero if and only if $y(\nu) = 0$. Zeros determined by this definition include "traditional" zeros as well as points of sign change. Note, for example, that the function $y(t) = \sin \frac{1}{t}$, $t \in [-1, 1]$ has a zero at the point $t = 0$ according to Definition 1.2.

**Definition 1.3.** We say that a function $y \in L^\infty_{[0, +\infty)}$ nonoscillates if there exists $\nu$ such that $y(t)$ does not have zeros for $t \in (\nu, +\infty)$. If there exists a sequence of zeros of a function $y$ tending to infinity, we say that a function $y \in L^\infty_{[0, +\infty)}$ oscillates.

In the following papers [8, 10, 11, 18, 19, 20, 21, 22, 23] oscillation/nonoscillation properties of solutions of difference equations with continuous time were considered. Our main aim in this paper is the study of zeros’ distribution of equation (1.1).
2 Lower Estimate of the Spectral Radius and Oscillation Properties of Solutions

Consider the equation
\[ b(t)y(t) = a(t)y(t + m) + c(t)y(t - k), \quad t \in [0, +\infty), \] (2.1)
\[ y(\xi) = \varphi(\xi), \quad \xi < 0, \] (2.2)
and its ”interval version”
\[ b(t)y(t) = a(t)y(t + m) + c(t)y(t - k), \quad t \in [0, \mu], \] (2.3)
\[ y(\xi) = \varphi(\xi), \quad \xi < 0 \text{ or } \xi > \mu, \] (2.4)
where \( a, b, c \) are continuous positive functions, \( k \) and \( m \) are natural numbers. A corresponding operator \( T_{\nu \mu} : L^\infty_{[\nu, \mu]} \to L^\infty_{[\nu, \mu]} \) define as follows
\[ (T_{\nu \mu} y)(t) = \frac{a(t)}{b(t)} y(t + m) + \frac{c(t)}{b(t)} y(t - k), \] (2.5)
\[ y(\xi) = 0, \quad \xi < \nu \text{ or } \xi > \mu. \] (2.6)

**Theorem 2.1.**[5] Let the spectral radius \( r_{\nu \mu} \) of the operator \( T_{\nu \mu} \) satisfy the inequality \( r_{\nu \mu} > 1 \). Then there are no solutions of equation (2.1) without zero in at least one of the intervals \([\nu - k, \mu]\) or \([\nu, \mu + m]\).

**Theorem 2.2.**[5] There are no nonoscillatory solutions of equation (1.2) if for each \( \nu \) there exists \( \mu \) (\( \nu < \mu < +\infty \)) such that the spectral radius \( r_{\nu \mu} \) of the operator \( T_{\nu \mu} \) satisfies the inequality \( r_{\nu \mu} > 1 \).

**Remark 2.1.** All the known proofs of assertions about oscillation of solutions start with an assumption that there exists a nonoscillatory solution. Then authors try to obtain a contradiction to this assumption that proves oscillation of all solutions. It should be noted that the assumption of a nonoscillatory solution leads to the conclusion (see the proof of Theorem 2.1[5]) that \( r_{\nu \mu} \leq 1 \). It means that all known coefficient tests for oscillation represent various estimates \( r_{\nu \mu} > 1 \) of the spectral radius of a corresponding operator \( T_{\nu \mu} : L^\infty_{[\nu, \mu]} \to L^\infty_{[\nu, \mu]} \).

3 Upper Estimate of the Spectral Radius and Positivity of Solutions

In this paragraph conditions of the positivity of solutions of equation (1.1) on semi-axis are presented.
**Theorem 3.1.**[5] Let there exist a real positive number $\beta$ such that

$$b(t)\beta^k > a(t)\beta^{m+k} + c(t), \quad t \in [0, +\infty),$$

(3.1)

then the spectral radius of the operator $T_{\nu \mu}$ is less than one for every $0 \leq \nu < \mu < +\infty$, there exists nonnegative solution of equation (2.1), (2.2) for nonnegative $\varphi$, and the solution of equation (2.3)(2.4) is positive for each positive function $\varphi$.

Let us write now the following particular cases of equation (2.3)

$$b(t)y(t) = a(t)y(t+1) + c(t)y(t-1), \quad t \in [0, +\infty),$$

(3.2)

and

$$b_0y(t) = a_0y(t+1) + c_0y(t-1), \quad t \in [0, +\infty),$$

(3.3)

in which $a_0, b_0$ and $c_0$ are positive constants.

Condition (3.1) for equation (3.3) can be written as follows

$$a_0\beta^2 - b_0\beta + c_0 < 0.$$  

(3.4)

**Theorem 3.2.**[5] Let the following inequality be fulfilled

$$b_0^2 > 4a_0c_0,$$  

(3.5)

then there exists a nonnegative solution of the equation

$$b_0(t)y(t) = a_0(t)y(t+1) + c_0(t)y(t-1), \quad t \in [0, +\infty),$$

(3.6)

$$y(\xi) = \varphi(\xi), \quad \xi < 0,$$  

(3.7)

for nonnegative $\varphi$ and the solution of the equation

$$b_0(t)y(t) = a_0(t)y(t+1) + c_0(t)y(t-1), \quad t \in [0, \mu),$$

(3.8)

$$y(\xi) = \varphi(\xi), \quad \xi < 0 \text{ or } \xi > \mu,$$  

(3.9)

is positive for every positive function $\varphi$.

The proof follows from Theorem 3.1 and the fact that inequality (3.4) has a real solution if and only if condition (3.5) is fulfilled.

**Remark 3.1.** It is known (see, for example, [5]) that the opposite inequality

$$b_0^2 < 4a_0c_0,$$  

(3.10)

implies oscillation of all solutions of equation (3.2). Thus inequality (3.5) cannot be improved.

**Theorem 3.3.**[5] Let inequalities (3.5) and

$$0 \leq a(t) \leq a_0, \quad b(t) \geq b_0, \quad 0 \leq c(t) \leq c_0$$

be fulfilled, then the spectral radius of the operator $T_{\nu \mu}$, where $m=k=1$, for every positive $\mu$ is
less than one, there exists nonnegative solution of the equation (3.2), (2.2) for nonnegative initial function \( \varphi \), and the solution of equation

\[
b(t)y(t) = a(t)y(t + 1) + c(t)y(t - 1), \quad t \in [0, \mu],
\]

\[
y(\xi) = \varphi(\xi), \quad \xi < 0 \text{ or } \xi > \mu,
\]
is positive for every positive function \( \varphi \).

**Remark 3.2.** Sufficient conditions of solutions’ nonnegativity for difference equations on a finite interval were obtained in Theorems 2.1-2.4 of the paper [15] in which a corresponding hypothesis on the size of this interval (see the condition (iv)) were assumed.

### 4 Eigenfunctions of the Operator \( T_{\nu \mu}^0 \)

Let us consider the equation

\[
b_0 y(t) = a_0 y(t + 1) + c_0 y(t - 1), \quad t \in [0, +\infty),
\]

where \( a_0, b_0, c_0 \) are positive constants,

\[
y(\xi) = 0, \quad \xi < 0.
\]

and construct eigenfunctions of the operator

\[
(T_{\nu \mu}^0 x)(t) = \frac{a_0}{b_0} x(t + 1) + \frac{c_0}{b_0} x(t - 1),
\]

\[
x(\xi) = 0, \quad \xi < \nu, \quad \xi > \mu.
\]

Let us find a solution of equation (4.1). Set \( y(t) = 1 \) for \( t \in [0, 1) \) and find from equation (4.1) that \( y(t) = \frac{b_0}{a_0} \) for \( t \in [1, 2) \). Then, from equation (4.1) we can find \( y(t) = \frac{k^2 - a_0 c_0}{a_0} \) for \( t \in [2, 3) \) and so on. We obtain the following solution \( y \) of equation (4.1) which is also an eigenfunction of the operator \( T_{\nu \mu}^0 \).

\[
y(t) =
\begin{cases}
  1, & t \in [0, 1), \\
  \frac{b_0}{a_0}, & t \in [1, 2), \\
  \frac{k^2 - a_0 c_0}{a_0}, & t \in [2, 3), \\
  \frac{b_0}{a_0} (t^2 - 2a_0 c_0), & t \in [3, 4), \\
  \{b_0 (b_0^2 - 3a_0 c_0) + a_0^2 c_0^2 \} \frac{1}{a_0}, & t \in [4, 5), \\
  \vdots
\end{cases}
\]

5
Let us consider
\[ b(t)y(t) = a(t)y(t+1) + c(t)y(t-1), \quad t \in [0, +\infty), \]  
(4.6)
where \(a, b, c\) are positive functions and introduce the corresponding operator \(T_{\nu, \mu} : L^\infty_{[\nu, \mu]} \to L^\infty_{[\nu, \mu]}\) by equalities
\[ (T_{\nu, \mu} y)(t) = \frac{a(t)}{b(t)} x(t+1) + \frac{c(t)}{b(t)} x(t-1), \]
(4.7)
and (4.4).

**Theorem 4.1.** Let the function \(y\) determined by equality (4.5) be negative on the interval \((\mu, \mu+1)\) and at least one of the following two conditions either
\[ b_0 > \varepsilon + b(t), \quad a_0 \leq a(t), \quad c_0 \leq c(t), \quad t \in [0, +\infty), \]
(4.8)
or
\[ b_0 \geq b(t) > 0, \quad a_0 + \varepsilon < a(t), \quad c_0 + \varepsilon < c(t), \quad t \in [0, +\infty), \]
(4.9)
where \(\varepsilon > 0\), be satisfied. Then there are no positive (negative) solutions of equation (4.6) on at least one of the intervals \([\nu - 1, \nu + \mu]\) or \([\nu, \nu + \mu + 1]\).

**Theorem 4.2.** Let at least one of the following conditions (4.8) or (4.9) be fulfilled. Then there are no positive (negative) solutions of equation (4.6) on the following intervals \([\nu, \mu]\):
- If \(b_0^2 \leq a_0 c_0\), on intervals such that \(\mu - \nu \geq 3\).
- If \(b_0^2 \leq 2a_0 c_0\), on intervals such that \(\mu - \nu \geq 4\).
- If \(b_0^2 (3a_0 c_0 - b_0^2) > a_0^2 c_0^2\), on intervals such that \(\mu - \nu \geq 5\).

**Remark 4.1.** All solutions of equation (4.1) oscillate if \(b_0^2 < 4a_0 c_0\) [14], and the method allows us to estimate a zone of their positivity.

## 5 Applications to Oscillation of Partial Differential Equations

In this part oscillation properties of solutions of the following equation
\[ A(t) u_{xx}(\theta(t), x) + \sum_{i=1}^{m} p_i(t) u(g_i(t), x) = 0, \quad t \in [0, +\infty), x \in [0, \omega], \]
(5.1)
with the periodic
\[ u(t, 0) = u(t, \omega), \quad u_x'(t, 0) = u_x'(t, \omega), \quad t \in [0, +\infty), \]
(5.2)
Neumann
\[ u_x'(t, 0) = u_x'(t, \omega) = 0, \quad t \in [0, +\infty), \]
(5.3)
and Dirichlet
\[ u(t, 0) = u(t, \omega) = 0, \quad t \in [0, +\infty), \]  
boundary conditions will be studied.

**Definition 5.1.** We say that the solution \( u(t, x) \) of a PDE boundary value problem oscillates if for each \( t_0 \) there exists a zero of the solution \((t_1, x_1)\) such that \( t_1 > t_0, \ x_1 \in (0, \omega) \).

The following partial differential-difference equation
\[ u(p(t), x) + A(t)u''(t, x) + b(t, x)u(r(t), x) = 0, \ t > 0, x \in [0, \omega], \]
where \( p \) and \( r \) are monotone increasing functions such that \( p(t) \geq t, \ r(t) \leq t, \) was considered in [4], where estimates of zone of solutions’ positivity for the Dirichlet boundary value problem with equation (5.5) were obtained.

Let us denote
\[ y(t) = \int_0^\omega x u(t, x) dx, \quad t \in [0, +\infty), \]
where \( z(t) \) is a positive eigenfunction of the following boundary value problems:
\[ z''(x) = 0, \quad z(0) = z(\omega), \quad z'(t, 0) = z'(\omega), \quad x \in [0, \omega], \quad t \in [0, +\infty). \]  
\[ z''(x) = 0, \quad z'(0) = z'(/\omega), \quad x \in [0, \omega], \quad t \in [0, +\infty). \]  
\[ z''(x) + \frac{n^2}{\omega^2} z(x) = 0, \quad z(0) = z(\omega) = 0, \quad x \in [0, \omega]. \]  

It is clear that we have \( z(x) = 1 \) for the problems (5.7), (5.8) and \( z(x) = \sin(\frac{n}{\omega} x) \) for (5.9). Note that if \( y(t) \) oscillates, then \( u(t, x) \) also oscillates.

Multiplying each term in equation (5.1) by \( z(x) = 1 \) or \( z(x) = \sin(\frac{n}{\omega} x) \) and integrating, we get the following equations for a function \( y \):
\[ \sum_{i=1}^m p_i(t) g_i(t) = 0, \quad t \in [0, +\infty), \]
for the problems (5.2) and (5.3), and
\[ \sum_{i=1}^m p_i(t) g_i(t) = A(t) \left( \frac{\pi}{\omega} \right)^2 y(\theta(t)), \quad t \in [0, +\infty), \]
for problem (5.4) respectively, where \( y(s) = 0 \) for \( s < 0 \).

Let us consider the following particular case of equation (5.1)
\[ b(t)u(t, x) - c(t)u(t - 1, x) = -A(t) u''_{xx}(t + 1, x) \quad t \in [0, +\infty), \ x \in [0, \omega], \]
Multiplying each term in equation (5.12) by \( z(x) = \sin(\frac{n}{\omega} x) \) and integrating, we get the following equations for a function \( y \):
\[ b(t)y(t) = A(t) \left( \frac{\pi}{\omega} \right)^2 y(t + 1) + c(t)y(t - 1), \quad t \in [0, +\infty), \]
Let us denote \( a(t) = A(t)(\frac{\pi}{\omega})^2 \).

**Theorem 5.1.** Let at least one of the conditions (4.8) or (4.9) be satisfied. Then there are no positive (negative) solutions of equation (5.12) with Dirichlet boundary conditions (5.4) in the following zones \([\nu, \mu] \times (0, \omega)\).

- If \( b_0^2 \leq a_0c_0 \), on zones such that \( \mu - \nu \geq 3 \).
- If \( b_0^2 \leq 2a_0c_0 \), on zones such that \( \mu - \nu \geq 4 \).
- If \( b_0^2(3a_0c_0 - b_0^2) > a_0^2c_0^2 \), on zones such that \( \mu - \nu \geq 5 \).

The proof follows by transforming the equation (5.12) into equation (5.13) and using Theorem 4.2.

Note the following assertion demonstrating the connection between the oscillatory character of solutions and the geometrical size of a zone \([\nu, \mu] \times (0, \omega)\).

**Corollary 5.1.** Let at least one of the conditions (4.8) or (4.9) be satisfied. If a zone \([\nu, \mu] \times (0, \omega)\) is narrow enough (i.e. \( \omega \) is small enough), then there are no positive (negative) solutions of problem (5.12), (5.4). In particular, in each zone \([\nu, +\infty) \times (0, \omega)\) all solutions oscillate.

Let us consider the following particular case of equation (5.1)

\[
a(t)(t + 1, x) + c(t)(t - 1, x) = -B(t)u_{xx}(t, x) \quad t \in [0, +\infty), x \in [0, \omega],
\]

Multiplying each term in this equation by \( z(x) = \sin(\frac{\pi}{\omega}x) \) and integrating, we get the following equation for a function \( y \) determined by equality (5.6):

\[
a(t)y(t + 1) + c(t)y(t - 1) = B(t)(\frac{\pi}{\omega})^2y(t), \quad t \in [0, +\infty),
\]

Let us denote \( b(t) = B(t)(\frac{\pi}{\omega})^2 \).

**Theorem 5.2.** Theorem 5.1 is true for problem (5.14), (5.4).

Note the following assertion demonstrating the connection between the oscillatory character of solutions and the geometrical size of a zone \([\nu, \mu] \times (0, \omega)\).

**Corollary 5.2.** Let at least one of the conditions (4.8) or (4.9) be satisfied. If a zone \([\nu, \mu] \times (0, \omega)\) is wide enough (i.e. \( \omega \) is great enough), then there are no positive (negative) solutions of problem (5.14), (5.4). In particular, in each zone \([\nu, +\infty) \times (0, \omega)\) all solutions oscillate.

### 6 About Difference Equation

Consider the functional equation

\[
b_0y(t) = a_0y(t + 1) + c_0y(t - 1), \quad t \in (-\infty, +\infty),
\]
where \( a_0, b_0, c_0 \) are positive constants and its difference analog

\[
b_0 y(n) = a_0 y(n + 1) + c_0 y(n - 1), \quad n \in (-\infty, +\infty),
\]  

(6.2)

Let us search the solution of (6.2) in the form \( y(t) = \lambda^n \). The characteristic equation for (6.2) is the following

\[
a_0 \lambda^2 - b_0 \lambda + c_0 = 0,
\]  

(6.3)

and its roots are the following

\[
\lambda_{1,2} = \frac{b_0}{2a_0} \pm \frac{\sqrt{b_0^2 - 4a_0c_0}}{2a_0}.
\]  

(6.4)

If \( b_0^2 > 4a_0c_0 \), the roots are real and this case leads to nonoscillation and was considered above in the paragraph 3. The most interesting for us case is \( b_0^2 < 4a_0c_0 \) and the roots

\[
\lambda_{1,2} = \frac{b_0}{2a_0} \pm \frac{\sqrt{4a_0c_0 - b_0^2}}{2a_0}i,
\]  

(6.5)

are complex.

Let us denote for convenience the real and imaginary parts of these roots as follows

\[
\alpha = \frac{b_0}{2a_0}, \quad \beta = \frac{\sqrt{4a_0c_0 - b_0^2}}{2a_0},
\]  

(6.6)

and rewrite them in the form

\[
\lambda_{1,2} = \sqrt{\alpha^2 + \beta^2} \exp \{ \pm i\varphi \},
\]  

(6.7)

where

\[
\varphi = \arctg \frac{\beta}{\alpha} = \arctg \frac{\sqrt{4a_0c_0 - b_0^2}}{b_0}.
\]  

(6.8)

\[
\alpha^2 + \beta^2 = \frac{c_0}{a_0}.
\]  

(6.9)

We get the solutions

\[
Y_{1,2}(n) = \left( \frac{c_0}{a_0} \right)^{\frac{n}{2}} \cos n\varphi \pm i \sin n\varphi.
\]  

(6.10)

Let us write corresponding real solutions

\[
y_1(n) = \left( \frac{c_0}{a_0} \right)^{\frac{n}{2}} \cos n\varphi \sqrt{4a_0c_0 - b_0^2},
\]  

(6.11)

and

\[
y_2(n) = \left( \frac{c_0}{a_0} \right)^{\frac{n}{2}} \sin n\varphi \sqrt{4a_0c_0 - b_0^2},
\]  

(6.12)

**Remark 6.1.** In the case \( b_0^2 < 4a_0c_0 \), the inequality \( c_0 < a_0 \) implies that the solutions tend to zero exponentially when \( t \to +\infty \).
7 Sturm Principles of Zeros’ Separation

For second order ordinary differential equation
\[ x''(t) + q(t)x'(t) + p(t)x(t) = 0, \quad t \in (-\infty, +\infty), \]  
(7.1)
with continuous coefficients \( q \) and \( p \), the following classical principle of zeros separation were obtained by Sturm: between every two adjacent zeros of each nontrivial solution there is one and only one zero of every other nontrivial solutions. For the delay equation
\[ x''(t) + \sum_{j=1}^{m} p_j(t)x(t - \tau_j(t)) = 0, \quad t \in (-\infty, +\infty), \]  
(7.2)
it is not true without corresponding additional assumptions [2]. Analogs of Sturm separation theorem for equation
\[ x''(t) + \sum_{j=1}^{m} p_j(t)x(t - \tau_i(t)) = 0, \quad t \in [0 + \infty), \]  
(7.3)
\[ x(\xi) = 0, \quad \xi < 0, \]  
(7.4)
(7.2) were obtained in the papers [2, 6]. Importance of the Sturm separation theorem for equation (7.3) can be seen in the paper [7], where asymptotic properties of delay equation (7.3) were studied on this basis. In the works [8, 9, 10, 11] analogs of the Sturm separation theorem were obtained for equation (7.2) in a different form: every solution of equation (7.2) has zero on a interval which is larger than distance between two adjacent zeros \( \nu \) and \( \mu \), namely on the interval \([\nu - \tau, \mu]\), where \( \tau = \max_{j=1,...,m:-\infty<t<+\infty} \tau_j(t) \).

For difference equation (6.2) the Sturm separation theorem in its classical formulation is true. Moreover, it is clear from formulas (6.11) and (6.12) that every solution has zero on the interval of the length \( N \), if
\[ Narctg \sqrt{\frac{4aq0c0 - b0^2}{b0}} \leq \pi, \quad (N + 1)arctg \sqrt{\frac{4aq0c0 - b0^2}{b0}} \geq \pi. \]  
(7.5)
What about functional equation (6.1) in the space of discontinuous functions? On each interval of the length 1 we can get every number \( m \) of zeros. Actually let us take intervals \( \left( \frac{k}{m}, \frac{k+1}{m} \right) \), where \( k = 0, ..., m - 1 \), and define the function
\[ z_k(t) = (-1)^k \left( \frac{c0}{a0} \right)^{\frac{\pi}{2}} \sin arctg \sqrt{\frac{4aq0c0 - b0^2}{b0}}, \quad t \in \left( \frac{k}{m}, \frac{k+1}{m} \right), \quad k = 0, ..., m - 1 \]  
(7.6)
this function satisfies equation (6.1) and has \( m \) zeros on \([0,1)\). Sturm separation theorem can be formulated in the form: every nontrivial solution has at least one zero between two adjacent zeros of a basic solution, one of which in our case is the following
\[ u(t) = \left( \frac{c0}{a0} \right)^{\frac{\pi}{2}} \sin narctg \sqrt{\frac{4aq0c0 - b0^2}{b0}}, \quad t \in [n, n + 1). \]  
(7.7)
Another approach to formulation of the Sturm separation theorem is to consider separation of zeros only for basic solutions. In this case there is a chance to get the Sturm separation theorem in its classical form.
8 Approximate Formula for Estimates of Intervals between Zeros

Consider the functional equation

\[ by(t) = ay(t + 1) + cy(t - 1), \quad t \in (-\infty, +\infty). \quad (8.1) \]

**Lemma 8.1.** Let inequality \((4.8)\) or \((4.9)\) be fulfilled, \(N\) is defined by \((7.5)\), then the spectral radius \(r_{\nu\mu} > 1\) if \(\mu - \nu > N + 1\).

**Proof.** We set \(v(t) = u(t - v + 1)\). The inequality \(v(t) \leq (T_{\nu\mu}^0 v)(t)\) is fulfilled. Inequalities \((4.8)\) or \((4.9)\) imply that for a corresponding \(\lambda > 1\) the inequality \(\lambda v(t) \leq (T_{\nu\mu}^0 v)(t)\) is fulfilled. According to Theorem 5.4 [17] (see p.81) we get \(r_{\nu\mu} > 1\).

The use of Lemma 8.1 and Theorem 2.1 leads us to the following assertion.

**Theorem 8.1.** Let inequality \((4.8)\) or \((4.9)\) be fulfilled, \(N\) is defined by \((7.5)\), then every solution of equation \((8.1)\) has at least one zero on the interval of the length \(N + 2\).

9 Acknowledgment

This research was supported by The Israel Science Foundation (grant No. 828/07).

References


