Large 2-transitive arcs

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Abstract

The projective planes of order $n$ with a collineation group acting 2-transitively on an arc of length $v$, with $n > v \geq n/2$, are investigated and several new examples are provided.
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1. Introduction and main results

A classical subject in studying finite geometries is the investigation of a finite projective plane $\Pi$ of order $n$ admitting a collineation group $G$ which acts 2-transitively on a $v$-arc $O$ of $\Pi$. The first remarkable result related to this problem dates back to 1967 and it is due to Cofman [10]. In that paper the author investigates the case $v = n + 1$, that is when $O$ is an oval. Cofman proves that $\Pi$ is Desarguesian and $O$ is a conic when $n$ is odd, under the additional assumption that all involutions in $G$ are homologies of $\Pi$. Few years later, Kantor [35] shows that Cofman’s result can be reached when $O$ is an oval and $\Pi$ has odd order $n$, only requiring that $G$ contains involutory homologies. Finally, in 1978 Korchmaros [39] shows that the assumption on involutions can be totally dropped. Also the case when $O$ is an oval in a finite projective plane of even order $n$ has inspired much work. This case has been essentially solved by Bonisoli and Korchmaros [7] in 1995, except when $G$ is either a Suzuki group or a semilinear 1-dimensional affine group. Maschietti [43] has recently solved the first case. The second one is essentially still open, despite several refinements due to Bonisoli [8] and to Biliotti and Francot [5].
In 1986 Abatangelo [1] dealt with the case \( v = n + 2 \), that is when \( \mathcal{O} \) is a hyperoval, proving that either \( \Pi \cong PG(2, 2) \) or \( \Pi \cong PG(2, 4) \). More recently, Biliotti and Francot [5] investigated the case \( v = n \), showing that the arc \( \mathcal{O} \) can be extended to an oval and \( G \leq A\Gamma L(1, v) \). As a result of all these investigations an almost complete classification of the doubly transitive arcs of length \( v \geq n \) has been achieved. When the length \( v \) of \( \mathcal{O} \) is smaller than \( n \), but close to \( n \), doubly transitive arcs seem to be rare. Nevertheless it seems possible to obtain a complete, or almost complete, classification even though sporadic examples in small planes are not easy to be determined. A first result in this direction is the paper of Biliotti and Montinaro [6] where the authors provide a characterization of the case \( v = n - 3 \). Aside from the trivial cases, the plane \( \Pi \) must have order 9 or 16 and \( G \) is isomorphic to \( PSL(2, 5) \) or \( AGL(1, 13) \), respectively.

The aim of this paper is to investigate the finite projective planes \( \Pi \) of order \( n \) admitting a collineation group \( G \) which acts 2-transitively on a \( v \)-arc \( \mathcal{O} \) of \( \Pi \), under the assumption \( v \geq n/2 \). Throughout the paper, the cases when \( G \) induces an almost simple group or an affine group on \( \mathcal{O} \) are investigated separately and for each of them new examples are given. In particular, the following result is obtained:

**Theorem 1.** Let \( \Pi \) be a projective plane of order \( n \) and let \( G \) be a collineation group acting 2-transitively on a \( v \)-arc \( \mathcal{O} \) of \( \Pi \). If \( v \geq n/2 \), then one of the following occurs:

1. \( v = n + 2 \), \( \mathcal{O} \) is a hyperoval and either \( n = 2 \) and \( A_4 \leq G \leq S_4 \), or \( n = 4 \) and \( A_5 \leq G \leq S_6 \).
2. \( v = n + 1 \) and either
   - (a) \( n \) is odd, \( \Pi \) is Desarguesian, \( \mathcal{O} \) is a conic and \( PSL(2, n) \leq G \), or
   - (b) \( n \) is even and one of the following holds:
     - (i) \( \Pi \) is Desarguesian, \( \mathcal{O} \) is a conic and \( PSL(2, n) \leq G \);
     - (ii) \( \Pi \) is a dual of a Lüneburg plane of order \( 2^{2r} \), \( r \) odd, \( r \geq 3 \), and \( Sz(2^r) \leq G \);
     - (iii) \( n \in \{2, 4\} \) or \( v \) is a prime power, \( v \equiv 3 \) mod 4, and \( G \leq A\Gamma L(1, v) \), or \( v = 7^2, 11^2, 23^2, 29^2 \) or \( 59^2 \).
3. \( v = n \), \( v \) is a prime power, \( \mathcal{O} \) is extendable to an oval and either \( G \leq A\Gamma L(1, v) \) or \( v \in \{5^2, 7^2, 11^2, 23^2, 29^2, 59^2, 3^4, 3^6\} \).
4. \( v < n \) and \( G \leq A\Gamma L(1, v) \). Moreover, one of the following occurs:
   - (a) \( v \) is even and \( n = 2v \);
   - (b) \( v \) is odd, \( v = n - 1 \), \( n \equiv 0 \) mod 4 and the involutions in \( G \) are elations;
   - (c) \( v \) is odd, \( n \) is a square and the involutions in \( G \) are Baer collineations.
5. \( v = n - 3 \), \( \Pi \cong PG(2, 9) \), \( \mathcal{O} \) is a complete 6-arc and \( PSL(2, 5) \leq G \).

The assumptions of Theorem 1 with the additional assumption \( v \geq n \) lead to the cases (1)–(3) (see [5, Theorem 4.4] and [43, Theorem 1.2]). So, our task is to complete the proof of Theorem 1 under the assumption \( v < n \). Nevertheless the case \( v = n - 3 \) has already been determined in [6]. So, we use sometimes the result in [6] in our proof. The cases when the socle of \( G \) is non-abelian simple and the case where it is solvable are investigated separately in Sections 3 and 4, respectively.

We remark that examples of type (4)(a) and (4)(c) occur in the Desarguesian planes, and examples of type (4)(b) occur in the Lorimer–Rahilly translation plane of order 16 and in the Johnson–Walker translation plane of order 16. A complete description of these examples is given in Section 4.
2. Preliminaries

We shall use standard notation, and Hering’s notation [22] to denote the different types of substructures of the projective planes. For what concerns finite groups the reader is referred to [33]. The necessary background about finite projective planes may be found in [31].

Let \( \Pi = (P, \mathcal{L}) \) be a finite projective plane of order \( n \). If \( G \) is a collineation group and \( P \in P \) \((l \in \mathcal{L})\), we denote by \( G(P) \) (by \( G(l) \)) the subgroup of \( G \) consisting of perspectivities with the center \( P \) (the axis \( l \)). Also, \( G(P, l) = G(P) \cap G(l) \). Furthermore, we denote by \( G(P, P) \) (by \( G(l, l) \)) the subgroup of \( G \) consisting of elations with the center \( P \) (the axis \( l \)). A collineation group \( G \) of \( \Pi \) is said strongly irreducible on \( \Pi \), if \( G \) does not fix any point, line, triangle and any proper subplane of \( \Pi \). Moreover, a collineation group \( G \) of \( \Pi \) is said totally irregular on \( \Pi \), if \( G_X \neq \{1\} \) for each point \( X \) of \( \Pi \).

A \( v \)-arc is a point-subset \( O \) of \( \Pi \) of size \( v \) such that any three points of \( O \) are not collinear. A line \( l \) of \( \Pi \) is called an external line, a tangent or a secant to \( O \), according to whether \( |O \cap l| = 0, 1 \) or 2, respectively. A point \( P \in \Pi - O \) is said a kernel for \( O \), if \( O \cup \{P\} \) is a \((v + 1)\)-arc. Finally, a \( v \)-arc \( O \) is said a complete \( v \)-arc, if it is not contained in any \((v + 1)\)-arc.

If a collineation group \( G \) acts 2-transitively on the points of a \( v \)-arc \( O \), then we call \( O \) a 2-transitive \( G \)-arc. Note that \( v \geq 3 \), since \( G \) is 2-transitive. If \( v = 3 \), there are exactly two examples: the stabilizer in \( PGL(3, h) \) of a triangle in \( PG(2, h) \) with \( h = 4 \) or 5. If \( v = 4 \), there are exactly three examples: the stabilizer in \( PGL(3, s) \) of a quadrangle in \( PG(2, s) \) with \( s = 5 \) or 7 or 8. Hence, we assume that \( v \geq 5 \).

Here some preliminary reductions for the group \( G \) are presented:

**Lemma 2.** The group \( G \) acts faithfully on \( O \).

**Proof.** Let \( N \) be the kernel of \( G \) on \( O \). Assume that there exists \( \alpha \in N \), \( \alpha \neq 1 \). Then \( \alpha \) is planar and \( o(Fix(\alpha)) \geq v - 2 \), since \( O \) is a \( v \)-arc. Hence \((v - 2)^2 \leq n \leq 2v\), by [31, Theorem 3.7]. Thus \( v = 5 \) and \( n = 9 \), since \( v \geq 5 \) by our assumption. Then \( o(Fix(\alpha)) \geq 4 \), since \( O \subseteq Fix(N) \) and \( O \) is 5-arc. A contradiction, since \( n = 9 \). □

**Lemma 3.** Each nontrivial \((C, a)\)-perspectivity in \( G \) has order 2. Furthermore, \( C \in \Pi - O \) and one of the following holds:

1. If \( v \) is odd, then \( a \cap O = \{D\} \) and the line \( CD \) is a tangent to \( O \). Moreover, each line of \([C] - \{CD\}\) which intersects \( O \) is a secant to \( O \).
2. If \( v \) is even, two cases arise:
   1. \( a \cap O = \emptyset \) and each line of \([C]\) which intersects \( O \) is a secant to \( O \);
   2. \( a \cap O = \{D_1, D_2\} \) and each line of \([C] - \{CD_1, CD_2\}\) which intersects \( O \) is a secant to \( O \). In particular, either \( CD_1 \neq CD_2 \) and \( a = D_1D_2 \) or \( a = CD_1 = CD_2 \).

**Proof.** Let \( \alpha \) be any nontrivial \((C, a)\)-perspectivity. Then \( C \in \Pi - O \) by Lemma 2. Clearly \( \alpha \) leaves \( l \cap O \) invariant for each line \( l \) through \( C \) and intersecting \( O \). Then \( o(\alpha) = 2 \) by Lemma 2, since \( |l \cap O| = 1 \) or 2. The remaining assertions are trivial. □

Now, we prove the following numerical lemma which will be useful in order to get through our task.
Lemma 4. Let \( q = p^r \) with \( p \) prime and let \( x \) and \( j \) be two positive integers, then the following hold:

(i) The ordered pair \((x, q^j) = (3, 4)\) is the positive integer solution of the Diophantine equation \( x^2 = 2q^j + 1 \).

(ii) The ordered pair \((x, q^j) = (5, 3)\) is the positive integer solution of the Diophantine equation \( x^2 = 2q^{2j} + 2q^j + 1 \).

(iii) The Diophantine equation \( x^2 = 2(q^j + 1) \) has no positive integer solutions for \( q \) even or for \( q \) a power of 3 and \( j = 3 \).

(iv) The Diophantine equation \( x^2 = 2(q^{2j} + q^j + 1) \) has no positive integer solutions.

Proof. Consider the Diophantine equation
\[
x^2 = 2q^j + 1.
\]
(1)

Then \((x - 1)(x + 1) = 2q^j\). Thus \( x > 2 \) and \( q \) must be even. Hence (1) becomes \( x^2 = 2rj + 1 \). It has no positive integer solutions by [52, Result A6.2], for \( rj > 2 \). It is a straightforward calculation to show that \((x, q^j) = (3, 4)\) is the positive integer solution of the Diophantine equation \( x^2 = 2rj + 1 \) for \( rj \leq 2 \). This completes the proof of (i).

Consider the Diophantine equation
\[
x^2 = 2q^{2j} + 2q^j + 1.
\]
(2)

By managing (2), we obtain the Pythagorean equation
\[
x^2 = q^{2j} + (q^j + 1)^2.
\]

Then \((x - q^j - 1)(x + q^j + 1) = q^{2j}\). Hence, we have \( x - p^{rj - 1} = p^{2rj - h} \) and \( x + p^{rj} + 1 = p^h \), where \( rj < h \leq 2rj \). Thus \( j > 0 \) and
\[
p^{2rj - h} + 2p^{rj} + 2 = p^h.
\]
(3)

Suppose that \( h \neq 2rj \). Note that \( 2rj - h < rj \) and \( 2rj - h < h \), since \( rj < h \leq 2rj \). Thus, dividing in (3) by \( p^{2rj - h} \), we obtain that \( p^{2rj - h} = 2 \). Hence \( h = 2rj - 1 \). Then (3) becomes \( 4 + 2rj - 1 = 2rj - 1 \). It is easily seen that this equation has no positive integer solutions. Hence \( h = 2rj \). Then (3) becomes \( 3 + 2p^{rj} = p^{2rj} \) and hence \((x, q^j) = (5, 3)\). This completes the proof of (ii).

Consider the Diophantine equation
\[
x^2 = 2(q^j + 1).
\]
(4)

Assume that \( q \) is even. A contradiction, since \( x^2 \equiv 2 \mod 4 \) being \( j > 0 \). Now, assume that \( j = 3 \) and \( q \) is a power of 3. Clearly (4) becomes \( x^2 = 2(q + 1)(q^2 - q + 1) \). In particular, \((q + 1, q^2 - q + 1) = 1\), since \( q \) is a power of 3. Thus there exists a divisor \( z \) of \( x \) such that \( z^2 = q^2 - q + 1 \), since \( q^2 - q + 1 \) is odd. A contradiction, since \((q - 1)^2 < (q^2 - q + 1) < q^2\) being \( q > 1 \). This completes the proof of (iii).

Finally, consider the Diophantine equation
\[
x^2 = 2(q^{2j} + q^j + 1).
\]
(5)

Then \( 2 \mid x^2 \) but \( 4 \nmid x^2 \), since \( q^{2j} + q^j + 1 \) is odd. A contradiction. This completes the proof of (iv). \( \square \)
3. The nonabelian simple case

In this section we analyze the case where $\Pi$ is a finite projective plane of order $n$ and $\mathcal{O}$ is a 2-transitive $G$-arc of length $v$ with $n > v \geq n/2$, under the assumption that the socle of $G$ is nonabelian simple. In particular, we prove the following theorem.

**Theorem 5.** Let $\Pi$ be a finite projective plane of order $n$ and let $\mathcal{O}$ be a 2-transitive $G$-arc of length $v$, with $n > v \geq n/2$. If $G$ is an almost simple group, then $\Pi \cong PG(2, 9)$, $\mathcal{O}$ is a complete 6-arc and $PSL(2, 5) \leq G$.

We start our investigation by giving some preliminary reductions for the structure of $G$.

**Lemma 6.** $G$ does not act on $\mathcal{O}$ as $PGL(2, 8)$ in its 2-transitive permutation representation of degree 28.

**Proof.** Suppose the contrary. Let $\sigma$ be any involution of $G$. Then $\sigma$ fixes exactly 4 points on $\mathcal{O}$, since $G \cong PGL(2, 8)$ acts on $\mathcal{O}$ in its 2-transitive permutation representation of degree 28. Then $\sigma$ is a Baer collineation of $\Pi$, since $\mathcal{O}$ is an arc. Thus $n \in \{36, 49\}$, since $28 < n \leq 56$ and $n$ is a square. Nevertheless the cases $n = 36$ and $n = 49$ cannot occur by [31, Theorem 3.6] and by [26, Theorem A], respectively. $\square$

Since $G$ does not act on $\mathcal{O}$ as $PGL(2, 8)$ in its 2-transitive permutation representation of degree 28, we may assume that $G$ coincides with its socle for the admissible cases.

Denote by $d_j(G)$, with $j \geq 0$, the primitive permutation representation degrees of $G$. In particular, $d_0(G)$ denotes the minimal one. Clearly $d_0(G) \leq d_j(G)$ for each $j > 0$. In particular, $d_0(G) \leq v$, where $v = |\mathcal{O}|$, since $G$ is 2-transitive on $\mathcal{O}$. In the following we treat the cases $d_0(G) = v$ and $d_0(G) < v$ separately.

**Assume that $d_0(G) = v$.**

**Lemma 7.** Let $L$ be a nonabelian simple group acting 2-transitively on a set of size $v$. If $d_0(L) = v$, then $d_j(L) > 2d_0(L) + 1$, except in the following cases:

1. $L \cong A_v$ and $v \in \{5, 6, 7, 8\}$;
2. $L \cong PSL(2, 7)$ and $v = 7$;
3. $L \cong PSL(2, 11)$ and $v = 11$;
4. $L \cong PSU(3, 3)$ and $v = 28$;
5. $L \cong Sp(2h, 2)$, $h \geq 3$, and $v = 2^{h-1}(2^h - 1)$;
6. $L \cong M_{11}$ and $v = 11$.

**Proof.** Assume that $L \cong A_v$. Then the assertion follows by [33, Satz IV.4.6] and by [15, Appendix B], for $v \geq 9$. Assume that $L \cong PSL(d, q)$, $d \geq 2$, $q$ prime power and $(d, q) \neq (2, 2), (2, 3)$. Then $d_0(L) = \frac{q^{d-1} - 1}{q-1}$ for $(d, q) \neq (2, 5), (2, 7), (2, 9), (2, 11)$ by [12]. If $d = 2$, the assertion follows by [33, Hauptsatz II.8.27], unless $(d, q) = (2, 5), (2, 7), (2, 9), (2, 11)$. If $d = 3$, then the assertion follows by [18] and [46], unless $(d, q) = (3, 2)$. The group $SL(3, 2) \cong PSL(2, 7)$ is an exception when we refer to its 2-transitive permutation representation of degree 7. Now, assume that $d \geq 4$. Denote by $M_p(L)$ the index of the largest parabolic subgroup of $L$. If $|L| \geq 10^{12}$, then $M_p(L) \geq qd_0(L)$ by [40, Lemma 4(2)]. Actually, $M_p(L) > qd_0(L)$
by [12]. Thus $d_j(L) > qd_0(L)$ for $d_j(L) \geq M_p(L)$, and we have the assertion in this case. If $d_j(L) < M_p(L)$, then $(d_j, L, p) = 1$ by [40, Table II], and again $d_j(L) \geq qd_0(L)$ by [40, Lemma 4(2)]. Actually $d_j(L) > qd_0(L)$ since $(d_j, L, p) = 1$. Thus the assertion. If $|L| \leq 10^{12}$, then $(d, q) = (4, 3), (4, 4), (4, 5), (4, 2), (5, 2), (6, 2)$. Actually the case $(d, q) = (4, 2)$ is ruled out, since it does not satisfy the condition $d_0(L) = v$. For the remaining cases the assertion follows by a direct inspection of the list given in [15, Appendix B]. For the groups $L \cong PSU(3, q)$ and $q > 3$, $L \cong Sz(q)$ and $q = 2^{2k+1}$, $k \geq 1$, $L \cong ^2G_2(q)$ and $q = 3^{2k+1}$, $h \geq 1$, the assertion follows by [18] and [46], by [55] and by [38], respectively, since $d_0(L) = v$. Finally, for the sporadic simple groups the assertion follows by a direct inspection of [11]. □

Lemma 8. Let $\Pi$ be a finite projective plane of order $n$ and let $O$ be a 2-transitive $G$-arc of length $v$, with $n > v \geq n/2$. Then $d_j(G) > 2v + 1$ for each $j > 0$.

Proof. Deny. Then $G$ is one of the groups listed in Lemma 7.

Assume that $G \cong A_v$, $v \in \{5, 6, 7, 8\}$. Assume that $v \geq 7$. Let $\xi$ be any 3-cycle. Then $\xi$ fixes $v - 3$ points on $O$. Let $P$ be a point on $O$ fixed by $\xi$. Then $\xi$ fixes at least $v - 4$ lines through $P$. In particular, $\xi \in G_P$ and $G_P \cong A_{v-1}$. Assume that $n + 2 - v < v - 1$. Then the number of tangents to $O$ through $P$ is less than the minimal permutation representation degree of $A_{v-1}$. Thus $G_P$ fixes all the $n + 2 - v$ tangents to $O$ through $P$. So, $\xi$ fixes exactly $(n + 2 - v) + (v - 4)$ lines to $O$ through $P$. Hence $\xi$ fixes a subplane of $\Pi$ of order $n - 3$. Then $(n - 3)^2 \leq n$ by [31, Theorem 3.7]. A contradiction, since $v \geq 7$. Hence $n + 2 - v \geq v - 1$. Actually, $n + 2 - v = v - 1 + i$ with $i \in \{0, 1, 2, 3\}$, since $n \leq 2v$. Note that $G_P \cong A_{v-1}$ must have a nontrivial orbit on the set of tangents to $O$ through $P$, otherwise this case is ruled out by the above argument. Then $G_P$ fixes exactly $i$ tangents to $O$ through $P$ and acts 2-transitively on the remaining $v - 1$ tangents, since $d_0(G_P) = v - 1$ and $d_j(G_P) > v + 3$ for $j > 0$ by [15, Appendix B]. Thus $\xi$ is a planar element of $G$ fixing exactly $2(v - 4) + i - 1$ lines through $P$. Hence $(2v - 9)^2 \leq 2v$ by [31, Theorem 3.7]. A contradiction, since $v \geq 7$. Hence $v \in \{5, 6\}$. Let $v = 6$, then $6 < n \leq 12$. Actually $n \neq 9, 10$ and 12 by [6], by [31, Theorem 13.18], and by [34], respectively. Clearly $\Pi \cong PG(2, n)$ for $n = 7$ or 8. By [44], $\Pi \cong PG(2, 11)$ also for $n = 11$. Nevertheless these cases are ruled out by [54, Proposition 15]. Let $v = 5$. Then $5 < n \leq 10$. Each case, but $n = 6$ or 9, is ruled out by the same argument as above. Actually, also the case $n = 6$ cannot occur by [31, Theorem 13.18]. Hence $n = 9$. Then $\Pi \cong PG(2, 9)$ by [26, Theorem C]. A contradiction by [54, Proposition 11].

Assume that $G \cong PSL(2, 7)$. There exists an involution $\alpha$ in $G$ fixing 3 points on $O$, since $v = 7$ and the stabilizer in $G$ of any point of $O$ is isomorphic to $S_4$. Hence, $\alpha$ is a Baer collineation of $\Pi$ by Lemma 3. Then $n = 9$, since $7 < n \leq 14$ and $n$ is a square. Furthermore, $\Pi \cong PG(2, 9)$ by [26, Theorem C]. A contradiction by [54, Proposition 6].

Assume that $G \cong PSU(3, 3)$. Let $\sigma$ be any involution of $G$. Then $\sigma$ fixes exactly 4 points on $O$, since $G$ acts on $O$ in its 2-transitive permutation representation of degree 28. Then $\sigma$ is a Baer collineation of $\Pi$, since $\Pi$ is an arc. Thus $n \in \{36, 49\}$, since $28 < n \leq 56$ and $n$ is a square. Nevertheless the cases $n = 36$ and $n = 49$ cannot occur by [31, Theorem 3.6] and by [26, Theorem A], respectively.

Assume that $G \cong Sp(2h, 2)$ with $h \geq 3$. The group $G$ contains an involution fixing $2^{2h-2}$ points on $O$ by [15, Example 5.4.3]. Such an involution is a Baer collineation of $\Pi$ and $\sqrt{n} \geq 2^{2h-2} - 2$. Hence $(2^{2h-2} - 2)^2 \leq 2^h (2^h + 1)$ by [31, Theorem 3.7]. A contradiction, since $h \geq 3$.

Assume that $G \cong PSL(2, 11)$ or $M_{11}$, and $v = 11$. Clearly there exists an involution $\rho$ in $G$ fixing 3 points on $O$. Hence $\rho$ is a Baer collineation of $\Pi$ by Lemma 3. Then $n = 16$, since
$11 < n \leq 22$ and $n$ is a square. Let $Z_{11} \leq G$. Then $Z_{11}$ fixes a point $R \in \Pi - \mathcal{O}$, since $n = 16$. Furthermore, each line through $R$ intersecting $\mathcal{O}$ is a tangent to $\mathcal{O}$, since $Z_{11}$ is regular on $\mathcal{O}$. Then $Z_{11}$ fixes the 6 external lines to $\mathcal{O}$ through $R$, since $n + 1 = 17$. Moreover, $Z_{11}$ fixes 5 points on each of them, other than $R$. Thus, $Z_{11}$ fixes a subplane of order 5. Then $n \geq 5^2$ by [31, Theorem 3.7]. A contradiction, since $n = 16$. ∎

**Lemma 9.** $G$ is totally irregular on $\Pi$.

**Proof.** Suppose that there exists a point $X \in \Pi$ such that $G_X = \{1\}$. Clearly $X \in \Pi - \mathcal{O}$. Then $|G| \leq n^2 + n + 1 - v$. Hence $|G| \leq 4v^2 + v + 1$, since $n \leq 2v$. Furthermore $|G| = \theta v(v - 1)$, $\theta \geq 1$, since $G$ is 2-transitive on $\mathcal{O}$. So

$$\theta \leq \frac{4v^2 + v + 1}{v(v - 1)}.$$ 

Hence $\theta \leq 5$ for $v \geq 5$. Thus $G_{Y,Z}$ is abelian for any $Y, Z \in \mathcal{O}$, $Y \neq Z$, since $|G_{Y,Z}| = \theta$. Then $G \cong PSL(2, q)$ with $q \in \{5, 7, 9, 11\}$ by [2, Main Theorem], since $G$ is nonabelian simple. Nevertheless, these cases cannot occur by Lemma 8. Thus the assertion. ∎

**Proposition 10.** Let $\Pi$ be a finite projective plane of order $n$ and let $\mathcal{O}$ be a 2-transitive $G$-arc of length $v$, with $n > v \geq n/2$. Then one of the following occurs:

1. $G$ fixes a point on $\Pi - \mathcal{O}$ or an external line to $\mathcal{O}$. Furthermore, either $n = 2v$ or $n = 2v - 1$;
2. $G$ is strongly irreducible on $\Pi$.

**Proof.** Suppose that $G$ is not strongly irreducible on $\Pi$. Then $G$ fixes a point, or a line or a subplane of $\Pi$, since $G$ is nonabelian simple.

- Assume that $G$ fixes a point $P \in \Pi$.

Clearly $P \in \Pi - \mathcal{O}$. Furthermore, each line through $P$ intersecting $\mathcal{O}$ is a tangent to $\mathcal{O}$, since $G$ is primitive on $\mathcal{O}$. Let $\mathcal{E}$ be the set of external lines to $\mathcal{O}$ through $P$. Then $n + 1 = |\mathcal{E}| + v$. Furthermore $2 \leq |\mathcal{E}| \leq v + 1$, since $v + 1 \leq n \leq 2v$. Assume that $G$ fixes $\mathcal{E}$ elementwise. Assume also that $n > v + 1$. Hence $|\mathcal{E}| > 2$. If $G$ contains a perspectivity $\sigma$ then $C_{\sigma} = P$, since $G$ fixes $\mathcal{E}$ elementwise and $|\mathcal{E}| > 2$. Then $\sigma$ fixes $\mathcal{O}$ pointwise, since each line through $P$ intersecting $\mathcal{O}$ is a tangent to $\mathcal{O}$. A contradiction. Hence each involution in $G$ is a Baer collineation of $\Pi$. This yields $|\mathcal{E}| < v$ by [31, Theorem 3.7], since $G$ fixes $\mathcal{E}$ elementwise and $n + 1 = |\mathcal{E}| + v$. Thus $n < 2v - 1$. Let $b \in \mathcal{E}$, and let $\{Q\} \neq b \cap r$ where $r$ is any secant to $\mathcal{O}$. Then $Q^G \subset b$. Arguing as above with $Q$ in role of $P$, we find out that $|Q^G| > 1$, since $G$ is primitive on $\mathcal{O}$ and $r$ is a secant to $\mathcal{O}$ through $Q$. Then $|Q^G| = \lambda d_b(G)$, with $\lambda \geq 1$, where $d_b(G)$ denotes a primitive permutation representation degree of $G$. Then $h = 0$ by Lemma 8 and also $\lambda = 1$, since $Q^G \subset b$ and $n + 1 < 2v$. Moreover, $G$ fixes $b - |\{P\} \cup Q^G|$ pointwise, since $|b - (|\{P, X\} \cup Q^G|) < v$ and $d_b(G) = v$. So, each line $e \in \mathcal{E}$ contains a $G$-orbit of length $v$, say $E^G$, and $G$ fixes $e - (|\{P\} \cup E^G|$ pointwise. Then $G$ is planar, since $|\mathcal{E}| > 2$. Moreover, $o(\text{Fix}(G)) = n - v$, since $G$ fixes $\mathcal{E}$ elementwise, $|\mathcal{E}| = n + 1 - v$ and $G$ is 2-transitive on $\mathcal{O}$. Then $o(\text{Fix}(G_O)) = n - v + 1$, since $G_O$ fixes also the tangent $PO$ to $\mathcal{O}$ and $G$ is 2-transitive on $\mathcal{O}$. A contradiction by [31, Theorem 3.7], since $\text{Fix}(G) \subset \text{Fix}(G_O)$. Hence $n = v + 1$ and $|\mathcal{E}| = 2$. Clearly $G$ fixes a triangle $\triangle$ pointwise on $\Pi - \mathcal{O}$. Through each vertex of $\triangle$ there are $v$ tangents, since $G$ is primitive.
on \( \mathcal{O} \). Thus \( \triangle \cup \mathcal{O} \) is a hyperoval of \( \Pi \). Then \( n \equiv 2 \mod 4 \), since \( n = v + 1 \) and since \( v = q^i + 1 \) with \( i \in \{1, 2, 3\} \) by [4]. A contradiction by [31, Theorem 13.18]. As a consequence, \( \mathcal{E} \) contains a nontrivial \( G \)-orbit. Then \( |\mathcal{E}| \geq v \), since \( d_0(G) = v \). Then either \( |\mathcal{E}| = v \) and \( n = 2v - 1 \) or \( |\mathcal{E}| = v + 1 \) and \( n = 2v \), since \( n \leq 2v \). Thus the assertion (1).

- Assume that \( G \) fixes a line \( l \) on \( \Pi \).

We may also assume that \( G \) does not any point on \( l \), since we have already got through this case. Thus \( l \) is union of nontrivial \( G \)-orbits. Then \( n + 1 = 2v \) by Lemma 8, since \( n > v \). Thus the assertion (1).

- Assume that \( G \) is irreducible on \( \Pi \) but it leaves a proper subplane \( \Pi_0 \) of \( \Pi \) invariant.

Clearly \( G \) is irreducible on \( \Pi_0 \). Thus \( \Pi_0 \) is union of nontrivial \( G \)-orbits of points. Then

\[
\sum_{i=0}^{k} a_id_i(G) = m^2 + m + 1, \quad (6)
\]

where \( m = o(\Pi_0) \) and the coefficients \( a_i \) are nonnegative integers such that \( \sum_{i=0}^{k} a_i \geq 1 \). Clearly, either \( n \geq m^2 + m \) or \( n = m^2 \) by [31, Theorem 3.7]. Assume that \( n \geq m^2 + m \). Then

\[
\sum_{i=0}^{k} a_id_i(G) \leq n + 1 \leq 2v + 1. \quad (7)
\]

If there exists \( 1 \leq h \leq k \) such that \( a_h > 0 \), then \( 2v + 1 < a_h d_h(G) \) by Lemma 8. A contradiction. So \( a_0v = m^2 + m + 1 \) with \( a_0 \in \{1, 2\} \), since \( 2v \geq n \geq m^2 + m \). Actually \( a_0 = 2 \) is ruled out, since \( m^2 + m + 1 \) is odd. Hence \( v = m^2 + m + 1 \). Thus \( G \) is line 2-transitive on \( \Pi_0 \). Then \( \Pi_0 \cong PG(2, q) \) and \( PSL(3, q) \cong G \) by [31, Ostrom–Wagner’s Theorem]. Actually \( G \cong PSL(3, q) \), since \( G \) is simple. Let \( E \leq G \) such that \( E \) is an elementary abelian group of order \( q \) fixing \( q + 1 \) points on \( \mathcal{O} \). Then \( E \) fixes a subplane of \( \Pi \) of order at least \( q - 1 \). Let \( C \) by a cyclic subgroup of \( C \leq N_G(E) \) of order \( \frac{q-1}{d} \), with \( d = (3, q - 1) \), fixing a point on \( \mathcal{O} \) other than the \( q + 1 \) fixed by the group \( E \). Then \( Fix(E) \cap Fix(C) \) is a proper subplane of \( Fix(C) \). Hence

\[
Fix(E) \cap Fix(C) \subseteq Fix(C) \subsetneq \Pi.
\]

This yields \( (q - 1)^4 \leq n \) by [31, Theorem 3.7], since \( Fix(E) \cap Fix(C) \) is a plane of order at least \( q - 1 \). Then \( q = 3 \), since \( n \leq 2v \) with \( v = q^2 + q + 1 \) and since \( G \) is nonabelian simple. Then either \( n = 16 \) or \( n = 25 \), since \( 13 < n \leq 26 \) and \( C \) consists of a Baer involution for \( q = 3 \). Actually, the case \( n = 16 \) cannot occur by [6], since \( v = 13 \). Hence \( n = 25 \). Let \( X \in \mathcal{O} \) and let \( S \) be a Sylow 2-subgroup of \( G_X \). Then \( |S| = 16 \), since \( G \cong PSL(3, 3) \) and \( |\mathcal{O}| = 13 \). It is easily seen that there exists an involution \( \alpha \) in \( S \) fixing at least 4 tangents to \( \mathcal{O} \) through \( X \), since \( |S| = 16 \) and there are 14 tangents to \( \mathcal{O} \) through \( X \). On the other and, \( \alpha \) fixes exactly 3 secants to \( \mathcal{O} \) through \( X \), since \( \alpha \) fixes exactly 4 points on \( \mathcal{O} \). Hence, \( \alpha \) is a Baer collineation of \( \Pi \) fixing 7 lines through \( X \). A contradiction, since \( n = 25 \).

Assume that \( n = m^2 \). Let \( Y \in \mathcal{O} \). Then there exists a line \( r \) of \( \Pi_0 \) intersecting \( \mathcal{O} \) in \( Y \), since \( \Pi_0 \) is a Baer subplane of \( \Pi \). If \( r \) is secant to \( \mathcal{O} \) then \( r^G \subseteq \Pi_0 \), since \( G \) leaves \( \Pi_0 \) invariant. Note that \( |r^G| = \frac{v(v-1)}{2} \), since \( G \) is transitive on the secants to \( \mathcal{O} \). Then

\[
\frac{v(v-1)}{2} \leq m^2 + m + 1 \leq 2v + \sqrt{2v} + 1,
\]
since \( r^G \subseteq \Pi_0, n = m^2 \) and \( n \leq 2v \). Thus \( v = 5 \) or \( 6 \) and \( G \cong A_5 \). Then \( n = 9 \) and \( \Pi_0 \cong PG(2,3) \), since \( v < n < 2v \) and \( n \) is a square. So \( A_5 \) acts on \( \Pi_0 \cong PG(2,3) \) not trivially. A contradiction, since \( A_5 \not\cong PGL(3,3) \). Hence, each line of \( \Pi_0 \) intersecting \( O \) is tangent to \( O \). Thus \( \{Y\} = r \cap O \) and \( G_r \leq G_Y \). Set \( \theta = [G_Y : G_r] \). Then \( |r^G| = v\theta \), since \( v = [G : G_Y] \). Therefore \( v\theta \leq 2v + \sqrt{2v} + 1 \), since \( r^G \subseteq \Pi_0 \). It is easily seen that \( \theta \leq 2 \), since \( v \geq 5 \). Assume that \( \theta = 2 \). Then \( \Pi_0 = r^G \), since \( |\Pi_0 - r^G| \leq \sqrt{2v} + 1 \). Then \( 2v = m^2 + m + 1 \). A contradiction, since \( m^2 + m + 1 \) is odd. Hence \( \theta = 1 \). Then \( r^G \) is a 2-transitive line-orbit on \( \Pi_0 \). Note that \( r^G \subseteq \Pi_0 \), since \( v < n \) and \( n = m^2 \). Assume that \( v < m \). Then \( v^2 < n \), since \( n = m^2 \). A contradiction, since \( n \leq 2v \) and \( v > 2 \). Hence, we may assume that \( v \geq m \). Then \( v = m + \sqrt{m} + 1 \) by [5, Theorem 3.13, dual case]. Then \( m^2/2 \leq m + \sqrt{m} + 1 \), since \( n/2 \leq v \) and \( n = m^2 \). It is a straightforward calculation to show that no positive integer solutions arise from the previous inequality, since \( \sqrt{m} \) must be an integer. Thus \( G \) is strongly irreducible on \( \Pi \) and we have (2). \( \Box \)

**Theorem 11.** \( G \) is strongly irreducible on \( \Pi \).

**Proof.** Suppose that \( G \) fixes a point \( P \) of \( \Pi \). Then either \( n = 2v \) or \( n = 2v - 1 \) by Proposition 10. Assume that \( G \) contains involutory perspectivities. If \( n = 2v - 1 \), then \( G \cong PSU(3,2^t) \) by [28, Theorem 1], since \( G \) is totally irregular by Lemma 9. Let \( Z \) be the center of any Sylow 2-subgroup of \( G \). Then \( Z \) is an elementary abelian 2-group of order 2\(^t\) fixing a point \( Q \) on \( O \) and acting semiregularly on \( O \setminus \{Q\} \) by [18]. In particular, \( Z \) is a \((C,a)\)-homology group of \( \Pi \) for some point \( C \) of \( \Pi \setminus O \) and some tangent line \( a \) to \( O \) in \( Q \) by [28, Theorem 1], and by Lemma 3. Then \( Z \) is semiregular on \( b \cap O \) for any \( b \in [C \setminus \{C \cap Q\}] \) such that \( |b \cap O| > 0 \). Hence \( |Z| \mid |b \cap O| \). Thus \( t = 1 \) and \( G \cong PSU(3,2) \). A contradiction, since \( G \) must be nonabelian simple. If \( n = 2v \), then \( G \) is one of the groups listed in [28, Theorem 2]. Note that \( d_0(G) = v \) is odd for any of these groups. A contradiction by [31, Theorem 13.18], since \( n = 2v \). Hence, we may assume that each involution in \( G \) is a Baer collineation of \( \Pi \) and \( n \) is a square. By a direct inspection of the list given in [37] filtered with respect to the conditions \( d_0(G) = v \) and \( n \) square, where either \( n = 2v - 1 \) or \( n = 2v \), we have the following admissible groups:

1. \( G \cong A_\nu, \nu \geq 5 \);
2. \( G \cong PSL(d, q), d \geq 2, (d, q) \neq (2, 2), (2, 3) \);
3. \( G \cong PSU(3, q), q > 2 \);
4. \( G \cong Sp(h, 2), h \geq 3 \);
5. \( G \cong Sz(q), q = 2^{2k+1}, k > 0 \);
6. \( G \cong 2G_2(q), q = 3^{2k+1}, k > 0 \).

Assume that \( G \cong A_\nu, \nu \geq 5 \). The argument inside the proof of Lemma 8 can be used to show that actually \( v \geq 9 \). Let \( \zeta \) be any 3-cycle. Then \( \zeta \) fixes \( v - 3 \) points on \( O \). Then \( \zeta \) fixes a subplane of order \( v - 5 \), since \( O \) is an arc. Then \((v - 5)^2 \leq 2v \) by [31, Theorem 3.7], since \( n \leq 2v \). This yields \( v = 9 \), since \( v \geq 9 \). Then either \( n = 17 \) or \( n = 18 \). A contradiction in any case, since \( n \) must be a square.

Assume that \( G \cong PSL(d, q) \), with \((d, q) \neq (2, 2) \) and \((2, 3) \). It is easily seen that \( G \) contains a \( p \)-element \( \tau \) fixing \( q^{d-1} - 1 \) points on \( O \). Assume that \( d \geq 3 \). Then \( \tau \) fixes a subplane of \( \Pi \) of order at least \( q^{d-1} - 1 - 2 \). Therefore \((q^{d-1} - 1 - 2)^2 \leq 2(q^{d-1} - 1) \) by [31, Theorem 3.7], since \( n \leq 2v \) and \( v = q^{d-1}/q-1 \). This yields \( d = 3 \). Then \( n = 25 \) and \( G \cong PSL(3,3) \) by Lemma 4(ii) and (iv), since
Then the argument inside the proof of Proposition 10 can be used to rule out this case. Therefore \( d = 2 \). Then either \( n = 2(q + 1) \) and \( G \cong PSL(2, q) \) with \( q \) odd, or \( n = 9 \) and \( G \cong PSL(2, 5) \) by Lemma 4(i) and (iii). Actually the latter cannot occur by [26, Theorem A], since the involutions in \( G \) are Baer collineation of \( \Pi \). Hence, \( n = 2(q + 1) \) and \( G \cong PSL(2, q) \) with \( q \) odd. Then \( [P] \) consists of two 2-transitive orbits both of length \( q + 1 \) and a fixed external line to \( \mathcal{O} \) (it comes out from the proof of Proposition 10). Then an involution fixes exactly either 5 or 1 lines of \([P]\) according to whether \( q \equiv 1 \mod 4 \) or \( q \equiv 3 \mod 4 \), respectively. Each involution in \( G \) must fix \( \sqrt{n} + 1 \) lines of \([P]\), with \( \sqrt{n} \geq 2 \), since they are Baer collineations of \( \Pi \). Then the case \( q \equiv 3 \mod 4 \) is ruled out. Then \( q \equiv 1 \mod 4 \) and \( \sqrt{n} + 1 = 5 \). So \( n = 16 \) and hence \( q = 7 \), since \( n = 2(q + 1) \). A contradiction, since \( 7 \equiv 3 \mod 4 \).

Assume that \( G \cong PSU(3, q) \). Then \( n = 2(q^3 + 1) \) with \( q \) odd by Lemma 4(i) and (iii). Then there exists a subgroup \( D \cong \mathbb{Z}_{q^3 + 1} \), where \( d = (3, q + 1) \), which fixes exactly \( q + 1 \) points on \( \mathcal{O} \). Then \( D \) is planar on \( \Pi \) and \( o(\text{Fix}(D)) \geq q - 1 \). Let \( \sigma \) be the involution in \( D \). Clearly \( \text{Fix}(D) \subseteq \text{Fix}(\sigma) \). Assume that \( \text{Fix}(D) \varsubsetneq \text{Fix}(\sigma) \). Then \( q - 1 \leq \sqrt{n} \) by [31, Theorem 3.7], since \( \text{Fix}(\sigma) \) is a Baer subplane of \( \Pi \). A contradiction, since \( v < n \leq 2v \) with \( v = q^3 + 1 \) and \( q \) odd. Assume that \( \text{Fix}(D) = \text{Fix}(\sigma) \). Hence \( \text{Fix}(D) \) is a Baer subplane of \( \Pi \). Thus \( \text{Fix}(\rho) = \text{Fix}(D) \) for each \( \rho \in D, \rho \neq 1 \). Let \( P \in \mathcal{O} - \text{Fix}(D) \). Then there exists a line \( b \) of \( \text{Fix}(D) \) such that \( P \in b \) by the Baer property. Then \( D \) must be semiregular on \( (b \cap \mathcal{O}) - \text{Fix}(D) \). Then \( 2 + 1 \parallel 2 \) since \( |b \cap \mathcal{O}| = 2 \). This yields \( q = 5 \). Nevertheless this case cannot occur since \( d_0(G) < v \) for \( G \cong PSU(3, 5) \).

Finally, the group \( G \cong Sp(h, 2) \) is ruled out by the same argument inside the proof of Lemma 8, while the groups \( G \cong Sz(q) \), with \( q = 2k^2 + 1 \) and \( k > 0 \), and the group \( G \cong 2^G2(q) \), with \( q = 2^k \) and \( k > 0 \), are ruled out by Lemma 4(i) and (iii). Thus \( G \) does not fix any point of \( \Pi \).

Assume that \( G \) fixes a line \( l \) of \( \Pi \). Then \( n = 2v - 1 \), since \( G \) does not fix any point of \( \Pi \). The above arguments rule out this case. Hence, \( G \) is strongly irreducible on \( \Pi \) by Proposition 10 and we have the assertion. \( \Box \)

**Proposition 12.** Let \( \Pi \) be a finite projective plane of order \( n \) and let \( \mathcal{O} \) be a 2-transitive \( G \)-arc of length \( v \), with \( n > v \geq n/2 \). Then \( n \) is a square and the involutions in \( G \) are Baer collineations of \( \Pi \).

**Proof.** Suppose that \( G \) contains perspectivities. These are involutions by Lemma 3. Assume that \( n \) is even. Then \( G \cong PSL(3, 2^t) \), \( t > 1 \), or \( G \cong PSU(3, 2^t) \), \( t > 1 \), by [28, Theorem 2], since \( G \) is totally irregular and strongly irreducible on \( \Pi \). Actually, the group \( G \cong PSL(3, 2^t) \) cannot occur. Indeed, each involution in \( G \) fixes \( 2^t + 1 > 5 \) points on \( \mathcal{O} \), contrary to the fact that \( G \) contains involutory elations of \( \Pi \). Hence \( G \cong PSU(3, 2^t) \) with \( t > 1 \). Let \( S_i, i = 1, 2 \), be two distinct elementary abelian 2-subgroups of \( G \) of maximal order such that \( \langle S_1, S_2 \rangle = G \) (see [18]). Then \( S_i \) fixes a point \( C_i \) on \( \mathcal{O} \) and acts semiregularly on \( \mathcal{O} - \{C_i\} \), for each \( i = 1, 2 \). Furthermore, \( S_i \) does not contain perspectivities of center \( C_i \) by Lemma 3. Thus \( S_i = S_i(l_i, l_i) \) for some tangent \( l_i \) to \( \mathcal{O} \) in \( C_i \), since \( S_i \) is abelian. Then \( G \) fixes the point \( l_1 \cap l_2 \), since \( S_i \) fixes \( l_i \) pointwise for \( i = 1, 2 \). A contradiction, since \( G \) is strongly irreducible on \( \Pi \). Hence, we may assume that \( n \) is odd. Since \( G \) is a strongly irreducible and totally irregular group of \( \Pi \) containing involutory homologies and such that \( d_0(G) = v \), then \( G \) is one of the group listed below by [21], [23] and by [28, Theorem 1]:

1. \( G \cong PSL(2, q), q \) odd, \( q \notin \{5, 7, 9, 11\} \);
2. \( G \cong PSL(3, q), \Pi \cong PG(2, q), q \) odd;
(3) $G ≅ PSU(3, q)$, $q$ odd, $q > 4$;
(4) $G ≅ A_7$, $v = 15$.

The case $G ≅ PSL(3, q)$ and $Π ≅ PG(2, q)$ cannot occur, since it contradicts the assumption $n > v$. Also the case $G ≅ PSU(3, q)$ cannot occur. Indeed, in this case each involution in $G$ fixes $q + 1 > 4$ points on $O$, contrary to the fact that each involution in $G$ is a homology of $Π$. Now, assume that $G ≅ A_7$ and $v = 15$. Let $β$ be any involution in $G$. Then $β$ fixes $3$ points on $O$ by [48, Theorem 7.1 and Table I]. A contradiction by Lemma 3, since $G$ must contain involutory homologies. Hence $G ≅ PSL(2, q)$ with $q$ odd and $q \notin \{5, 7, 9, 11\}$. Let $P$ be any point of $O$. Denote by $T$ the subset of all tangents to $O$ through $P$. Then the Frobenius group $G_p ≅ E_q. Z_{q - 1}$ acts on $T$. Let $C$ be a Frobenius complement of $G_p$. Assume that $(1) < C_l < C$ for some $l \in T$. Then $C_l$ fixes at least $2$ tangents to $O$ through $P$, since $C_l < C$ being $C$ cyclic. Assume that $C_l$ contains an involution $α$. Clearly $α$ fixes one point $Q$ on $O - \{P\}$, since $v = q + 1$ and $q$ is odd. Thus $α$ fixes at least $3$ lines through $P$, namely $PQ$ and the tangents to $Q$ in $P$ fixed by $C_l$. Therefore $α$ is a homology with center $P$. A contradiction by Lemma 3. Thus $C_l$ has odd order. Let $σ$ be an involution in $G$ normalizing $C_l$. Then $C_l$ fixes at least $2$ tangents to $O$ through $Pσ$. Hence $C_l$ is planar on $Π$. Furthermore, $σ$ induces a perspective on $Fix(C_l)$. Thus $C_σ$ and $a_σ$ lie in $Fix(C_l)$. Let $γ$ be a generator of $C_l$, then $σ$ and $σγ$ are distinct involutory homologies of $Π$ both having center $C_σ$ and axis $a_σ$. A contradiction by [23, Proposition 5.2(d)].

Note that the previous argument still works when $C_l = C$ and $C$ fixes at least $2$ tangents to $O$ through $P$. As a consequence, we may assume that $C$ fixes at most one tangent to $O$ through $P$ and acts semiregularly on the remaining ones. Thus either $\frac{q-1}{2} | n + 1 - q$ or $\frac{q-1}{2} | n - q$, since $|C| = \frac{q-1}{2}$ and $|T| = n + 1 - q$.

Assume that $\frac{q-1}{2} | n + 1 - q$. Then $n = 0 \frac{q-1}{2} + q - 1$ with $θ ≤ 2$, since $n ≤ 2(q + 1)$ and $q > 11$. Actually $θ = 1$ and $q ≡ 3 \mod 4$, since $n$ must be odd. Therefore $n = \frac{3}{2}(q - 1)$. Hence $|T| = \frac{q-1}{2}$ and $C$ is regular on $T$. Let $K$ be the Frobenius kernel of $G_p$. Hence $G_p = K.C$. Furthermore $K$ fixes $T$ elementwise, since $|K| = 1$, $C$ normalizes $K$ and $K$ is regular on $T$. Let $ξ$ be any nontrivial element of $K$. Then $ξ$ fixes at least $\frac{q-1}{2}$ points of $Π$, since $ξ$ fixes $T$ elementwise. If $ξ$ is planar then $o(Fix(ξ)) = \frac{q-1}{2} − 1$, since $ξ$ fixes exactly $\frac{q-1}{2}$ lines through $P$. Then $(\frac{q-1}{2} − 1)^2 ≤ \frac{3}{2}(q - 1)$ by [31, Theorem 3.7]. A contradiction, since $q > 11$. As consequence, the points fixed by $ξ$, possibly except one, must be collinear. Let $r$ be the line containing these points. Assume that $P \notin r$. Then the points fixed by $ξ$ are the intersections between $r$ and each line of $T$. Note that $K$ fixes $r$, since $K$ is abelian and since $ξ$ cannot be planar. Thus $K$ fixes the intersections between $r$ and each line of $T$, since $K$ fixes $T$ elementwise. Therefore $K$ must be semiregular on $PQ - \{P, Q\}$ for each $Q ∈ r \cap T$ where $t ∈ T$, since $K$ cannot contain planar elements. Then $q | n - 1$ and hence $q = 5$, since $n = \frac{3}{2}(q - 1)$. A contradiction. Hence $P ∈ r$ and $r ∈ T$. Assume that $ξ$ fixes a point $L$ not on $r$. Then $L$ is the unique point fixed by $ξ$ not on $r$, since $ξ$ cannot be planar. Since $\frac{q-1}{2} > 3$, there exists $w ∈ T - \{r, PL\}$. Then $ξ$ must be semiregular on $PL - \{P, L\}$ and on $w - \{P\}$. Hence $P | \frac{q-1}{2} - 2$ and $p | \frac{q-1}{2} - 1$, respectively.

A contradiction. Thus $Fix(ξ) \subset r$. Then $ξ$ fixes exactly $\frac{q-1}{2} − 1$ points on $r$ other than $P$, since $ξ$ fixes exactly $\frac{q-1}{2}$ lines through $P$ and since $Fix(ξ) \subset r$. Set $C = Fix(ξ) ∩ (r - \{P\})$. Then $K$ acts on $C$ with kernel $N$, since $K$ is abelian. Clearly $ξ ∈ N$. Arguing as above, with any element of $N$ in role of $ξ$, we show that $N$ must be semiregular on $u$ for any $u ∈ T - \{r\}$. Thus $|N| = \frac{q-1}{2} − 1$. Hence $N = \langle ξ \rangle$ and $|N| = 3$. Denote by $K = K/⟨ξ⟩$. Then $K$ is regular on each its orbit on $C$ by
[50, Proposition 3.2], since \( \bar{K} \) is abelian. Therefore \( \bar{K} \) is semiregular on \( \mathcal{C} \). Hence \( \frac{q}{3} \mid \frac{q-1}{2} - 1 \), since \( |\bar{K}| = q/3 \). A contradiction.

Assume that \( \frac{q-1}{2} \mid n-q \). Then \( n = \lambda \cdot \frac{q-1}{2} + q \) with \( \lambda \in \{1, 2\} \), since \( n \leq 2(q+1) \) and \( q \) is odd. Assume that \( \lambda = 1 \). Then \( n = \frac{3q-1}{2} \). Actually \( q \equiv 1 \mod 4 \), since \( n \) must be odd. It is easily seen that \( n \not\equiv 0, 1 \mod \frac{q+1}{2} \). Then \( n = q \) [29, Theorem B(II)], since \( q > 11 \). A contradiction, since \( n > v \) and \( v = q + 1 \). Hence \( \lambda = 2 \) and \( n = 2q - 1 \). Furthermore \( q \equiv 3 \mod 4 \), since the case \( q \equiv 1 \mod 4 \) is ruled out by the previous argument. Let \( \delta \) be an involutory \((\mathcal{C}, a_b)\)-homology. Clearly \( \delta \) fixes \( 2q - 1 \) points on \( \Pi - \mathcal{O} \), since \( n = 2q - 1 \). Since \( C_G(\delta) \cong D_{q+1} \) there are exactly \( \frac{q+1}{2} \) points of \( \text{Fix}(\delta) \) such that the stabilizer in \( G \) of each of them contains a Klein subgroup. Denote by \( \mathcal{K} \) the set of these points. Then \( \mathcal{K} \subseteq \mathcal{K} \), since commuting involutory homologies lie in a triangular configuration. Let \( Y \in \text{Fix}(\delta) - \mathcal{K} \). Then \( G_Y \) is a subgroup of \( PSL(2, q) \) of even order with no Klein subgroups. Then either \( G_Y \cong D_{2a} \) with \( a \mid \frac{q-1}{2} \), or \( G_Y \cong Z_b \) with \( b \mid \frac{q+1}{2} \) and \( b \) even by [33, Hauptsatz II.8.27]. Assume that \( G_Y \cong Z_b \) with \( b \mid \frac{q+1}{2} \) and \( b \) even. Then \( b > q/8 \), since \( |G : G_Y| \leq |\Pi - \mathcal{O}| \) and \( n = 2q - 1 \). Then \( G_Y \cong Z_{q+1} \) with \( f < 5 \), since \( b \mid \frac{q+1}{2} \) and \( b \) even. It is easily seen that in this case \( \delta \) fixes \( 2f \) points on \( Y^G(f) \). Denote by \( \lambda_f \) the number of this type of \( G \)-orbits on \( \Pi - \mathcal{O} \). Then \( \lambda_f \leq 4/f \), \( f < 5 \), since \( \lambda_f |Y^G(f)| \leq |\Pi - \mathcal{O}| \).

Assume that \( G_Y \cong D_{2a} \) with \( a \mid \frac{q-1}{2} \). Then \( a > q/16 \), since \( |G : G_Y| \leq |\Pi - \mathcal{O}| \) and \( n = 2q - 1 \). Then \( a = \frac{q-1}{2} \) with \( j < 5 \), since \( a \mid \frac{q+1}{2} \). In particular, \( \delta \) fixes \( \frac{q+1}{2} \) points on \( Y^G \). This yields that there are at most 2 orbits of this type, since \( |\text{Fix}(\delta) - \mathcal{K}| = \frac{3q+1}{2} \). Hence \( \mu \leq 2 \), where \( \mu \) denotes the number of this kind of \( G \)-orbits on \( \Pi - \mathcal{O} \).

Since \( |\text{Fix}(\delta) - \mathcal{K}| = \frac{3q+1}{2} \), we have the following Diophantine equation:

\[
2\lambda_1 + 4\lambda_2 + 6\lambda_3 + 8\lambda_4 + \frac{q+1}{2} \mu = \frac{3q+1}{2}.
\]

(8)

It is easily seen that (8) has no solutions. Indeed, \( 2\lambda_1 + 4\lambda_2 + 6\lambda_3 + 8\lambda_4 + \frac{q+1}{2} \mu \) is even while \( \frac{3q+1}{2} \) is odd, since \( q \equiv 3 \mod 4 \). \( \square \)

**Lemma 13.** Let \( \Pi \) be a finite projective plane of order \( n \) and let \( \mathcal{O} \) be a 2-transitive \( G \)-arc of length \( v \), with \( n > v \geq n/2 \). If \( G \) is a nonabelian simple group, then \( G \cong PSU(3, q) \), \( q = p^r \), \( p \) prime and \( r \geq 1 \).

**Proof.** Assume that \( G \cong PSU(3, q) \). We may also assume that \( q \) is even, since the argument of Theorem 11 rules out the case \( q \) odd. Let \( C \cong Z_{\frac{q+1}{2}} \) where \( d = (3, q+1) \). Then \( C \) is planar on \( \Pi \), since \( C \) fixes \( q+1 \) points on \( \mathcal{O} \) by [30] and \( q > 2 \). Again by [30], the group \( N_G(C)/C \cong PGL(2, q) \) acts on \( \text{Fix}(C) \cap \mathcal{O} \) in its 2-transitive permutation representation of degree \( q+1 \). Since \( q \) is even and \( C \cong Z_{\frac{q+1}{2}} \), it is easily seen that there exists a subgroup \( K \) of \( N_G(C) \) such that \( K \cong PSL(2, q) \). Clearly \( K \) acts not trivially on \( \text{Fix}(C) \). Set \( m = o(\text{Fix}(C)) \). Then \( m \geq q-1 \). Let \( U_0 \) be an elementary abelian 2-subgroup of \( K \) of order \( q \). Assume that each nontrivial element in \( U_0 \) is a Baer collineation of \( \text{Fix}(C) \). By [31, Result 1.4], we have that either \( q \mid \sqrt{m} \) or \( q \mid \sqrt{m} - 1 \). So \( q \leq \sqrt{m} \) in any case. Then \( q \leq \sqrt{4n} \), since \( m \leq \sqrt{4n} \) by [31, Theorem 3.7]. Then \( q^4 \leq 2(q^3 + 1) \), since \( n \leq 2v \) and \( v = q^3 + 1 \). A contradiction, since \( q > 2 \). Hence, we may assume that each nontrivial element in \( U_0 \) is a perspectivity of \( \text{Fix}(C) \), since the involutions in \( U_0 \) lie in a unique conjugate class. Furthermore, any two distinct commuting perspectivities cannot have the same center, since \( \text{Fix}(C) \cap \mathcal{O} \) is an arc of odd length of \( \text{Fix}(C) \) and each Sylow 2-subgroup of \( K \) fixes
exactly one point on $\text{Fix}(C) \cap \mathcal{O}$ and it is regular on the remaining ones. Assume that $q \neq 4$. Then $U_0$ must be an elation group of $\text{Fix}(C)$ by [36], since $U_0$ is an elementary abelian 2-group of order $q \geq 8$ and since any two distinct commuting perspectivities cannot have the same center. In particular, $U_0$ induces on $\text{Fix}(C)$ a group of elations with the same axis $s$, where $s$ is a tangent to $\text{Fix}(C) \cap \mathcal{O}$, since $U_0$ is abelian. Hence $U_0$ fixes exactly $q + 1$ collinear points, namely $\text{Fix}(C) \cap s$.

Let $U$ be a Sylow 2-subgroup of $G$ containing $U_0$ and let $\beta, \gamma \in U_0$ such that $\beta \neq \gamma$. Recall that each involution in $G$ is a Baer collineation of $\Pi$. Assume that $\text{Fix}(\beta) = \text{Fix}(\gamma)$. Let $Y \in \mathcal{O} - \text{Fix}(\beta)$. Then there exists a line $b \in \text{Fix}(\beta)$ such that $Y \in b$, since $\text{Fix}(\beta)$ is a Baer subplane of $\Pi$. Then $|b \cap \mathcal{O}| = 2$, since $Y \in (b \cap \mathcal{O}) - \text{Fix}(\beta)$. Nevertheless there exists an element in $\langle \beta, \gamma \rangle \cong E_4$ fixing $b \cap \mathcal{O}$ pointwise, since $|b \cap \mathcal{O}| = 2$ and $\text{Fix}(\beta) = \text{Fix}(\gamma)$. A contradiction. As a consequence, $\gamma$ induces on $\text{Fix}(\beta)$ either a Baer collineation or a perspectivity. Assume that $\gamma$ induces on $\text{Fix}(\beta)$ a Baer collineation. Then

$$\text{Fix}(\gamma) \cap \text{Fix}(\beta) \subseteq \text{Fix}(\beta) \subseteq \Pi.$$ 

A contradiction by [31, Theorem 3.7], since the order of $\text{Fix}(\gamma) \cap \text{Fix}(\beta)$ is at least $q$ while $n \leq 2(q^3 + 1)$ and $q > 2$. Hence, we may assume that each involution in $U_0 - \{\beta\}$ induces a perspectivity of axis $s \cap \text{Fix}(\beta)$. Since $U_0$ fixes $q + 1$ points on $s \cap \text{Fix}(\beta)$. Thus all involutions in $U_0$ fix $s \cap \text{Fix}(\beta)$ pointwise. Hence $U_0$ must be semiregular on $s - (s \cap \text{Fix}(\beta))$, since $|\text{Fix}(\beta) \cap s| = \sqrt{n} + 1$ and each involution in $U_0$ is a Baer collineation of $\Pi$. Then $U$ is semiregular on $s - (s \cap \text{Fix}(\beta))$, since each involution in $U$ lies in $U_0$. Thus $q^3 | n - \sqrt{n}$, since $|U| = q^3$ and $|s - (\text{Fix}(\beta) \cap s)| = n - \sqrt{n}$. Either $q^3 | \sqrt{n} - 1$ or $q^3 | \sqrt{n}$. So $q^3 \leq \sqrt{n}$ in any case. That is $q^6 \leq n$. A contradiction, since $n \leq 2(q^3 + 1)$. Assume that $q = 4$. Then $\sqrt{n} = 9$ or 10 or 11, since $65 < n \leq 128$. The case $\sqrt{n} = 10$ is ruled out by [31, Theorem 13.18]. If $\sqrt{n} = 11$, then $G$ must contain involutory homologies by [26], contrary to our assumption. Hence $\sqrt{n} = 9$. Assume that $U$ and $\beta$ are defined as above. Then group induced by $U$ on $\text{Fix}(\beta)$ has order 32. A contradiction, since $\sqrt{n} = 9$. Thus the assertion. 

**Lemma 14.** Let $\Pi$ be a finite projective plane of order $n$ and let $\mathcal{O}$ be a 2-transitive $G$-arc of length $v$, with $n > v \geq n/2$. If $G$ is a nonabelian simple group, then $G \cong 2G_2(q) = 3^{2k+1}$ and $k > 0$.

**Proof.** Assume that $G \cong 2G_2(q) = 3^{2k+1}$ and $k > 0$ (note that $k \neq 0$ by Lemma 6, since $2G_2(3) \cong P\Gamma L(2, 8)$). Let $\sigma$ be any involution in $G$. Then $\sigma$ is a Baer collineation of $\Pi$ by Proposition 12. Set $\mathcal{O}_\sigma = \mathcal{O} \cap \text{Fix}(\sigma)$. Then $|\mathcal{O}_\sigma| = q + 1$ by [41]. Furthermore $C_G(\sigma) = \langle \sigma \rangle \times H$, where $H \cong PSL(2, q)$. In particular, the group $H$ leaves $\text{Fix}(\sigma)$ invariant and $H$ acts in its 2-transitive permutation representation of degree $q + 1$ on $\mathcal{O}_\sigma$.

(A) $H$ is strongly irreducible on $\text{Fix}(\sigma)$.

Assume that $H$ fixes a line $z$ of $\text{Fix}(\sigma)$. Clearly $z \cap \mathcal{O}_\sigma = \emptyset$. Assume that the involutions in $H$ are perspectivities of $\text{Fix}(\sigma)$ (recall that the involutions in $H$ lie in a unique conjugate class). Then $z$ cannot be the axis of any perspectivity of $H$, since $H$ fixes $z$ and $H$ is nonabelian simple. Thus, if $\bar{\beta}$ is any involutory perspectivity lying in $H$ and $C$ is the center of $\bar{\beta}$, then $C \in z$. Assume that $H$ fixes $C$. Then each line through $C$ and intersecting $\mathcal{O}_\sigma$ is a tangent to $\mathcal{O}_\sigma$, since $H$ acts primitively on the points of $\mathcal{O}_\sigma$. So, $\bar{\beta}$ fixes $\mathcal{O}_\sigma$ pointwise. A contradiction, since $\bar{\beta}$ is perspectivity and $\mathcal{O}_\sigma$ is a $(q + 1)$-arc. Hence $H_C < H$. Then $[H : H_C] \geq \frac{1}{2}q(q - 1)$ by [33, Hauptsatz II.8.27], since $q \equiv 3 \mod 8$ and $H_C$ has even order. That is $|C_H| \geq \frac{1}{2}q(q - 1)$. Then
\[ \sqrt{n} + 1 \geq \frac{1}{2} q(q - 1), \] since \( C^H \subseteq [z] \). A contradiction, since \( n \leq 2(q^3 + 1) \) and \( q \geq 27 \). Hence, we may assume that the involutions in \( H \) are Baer collineations of \( \text{Fix}(\sigma) \). If there exists a point \( R \) on \( z \) such that \( H^R < H \) with \( H^R \) of even order, the previous argument can be used with \( H^R \) in role of \( H_C \) to rule out this case. Hence \( H_B = H \) for any point \( B \) on \( z \) such that \( H_B \) has even order. As a consequence, there are exactly \( \sqrt{n} + 1 \) points on \( z \), say \( B_e \), with \( 1 \leq e \leq \sqrt{n} + 1 \), such that \( H_B_e = H \), since the involutions in \( H \) are Baer collineations of \( \text{Fix}(\sigma) \). In particular, each line through any \( B_e \) and intersecting \( O_\sigma \) is a tangent to \( O_\sigma \), since \( H \) acts primitively on the points of \( O_\sigma \). Now, let \( \tilde{\rho} \) be any Baer involution in \( H \) and let \( J \) be any point of \( O_\sigma \). By the Baer property, there exists a line \( w \) of \( \text{Fix}(\tilde{\rho}) \) containing \( J \). Then \( w \) must be a secant to \( O_\sigma \), since \( H \) acts on \( O_\sigma \) in its 2-transitive permutation representation of degree \( q + 1 \) and \( q \equiv 3 \mod 8 \). On the other hand, \( w \cap z = \{ B_u \} \) for some \( 1 \leq u \leq \sqrt{n} + 1 \), since \( H_B u = H \) for each \( 1 \leq e \leq \sqrt{n} + 1 \). A contradiction, since each line through \( B_e \) intersecting \( O_\sigma \) is a tangent to \( O_\sigma \) for each \( 1 \leq e \leq \sqrt{n} + 1 \). Thus \( H \) does not fix any line of \( \Pi \).

Assume that \( H \) fixes a point \( P \) of \( \text{Fix}(\sigma) \). Then each line through \( P \) and intersecting \( O_\sigma \) is a tangent to \( O_\sigma \), since \( H \) acts primitively on the points of \( O_\sigma \). Thus, \( P \) cannot be the center of any perspectivity in \( H \), since \( H \) is faithful on \( O_\sigma \). Nevertheless each involution in \( H \) fixes \( P \). Then there exists a line \( r \) in \( [P] \) such that \( H_r \) has even order. In particular, \([H : H_r] > 1\), since \( H \) does not fix any line of \( \text{Fix}(\sigma) \). Arguing as above, we obtain \( |r^H| \geq \frac{1}{2} q(q - 1) \). Then \( \sqrt{n} + 1 \geq \frac{1}{2} q(q - 1) \), since \( r^H \subseteq [P] \). Again contradiction, since \( n \leq 2(q^3 + 1) \) and \( q \geq 27 \). Since \( H \) does not fix any point of \( \Pi \) and \( H \) is nonabelian simple, then \( H \) does not fix any triangle of \( \text{Fix}(\sigma) \). Thus \( H \) is irreducible on \( \text{Fix}(\sigma) \). Now, assume that \( H \) fixes a proper subplane \( \Pi_0 \) of \( \text{Fix}(\sigma) \). Then \( o(\Pi_0) < q \) by [31, Theorem 3.7], since \( \Pi_0 \subseteq \text{Fix}(\sigma) \subseteq \Pi \) and \( n \leq 2(q^3 + 1) \). A contradiction by [47, Theorem 1.1]. Therefore \( H \) is a strongly irreducible collineation group of \( \text{Fix}(\sigma) \).

(B) Commuting involutions of \( H \) are Baer collineations of \( \text{Fix}(\sigma) \) with distinct fixed Baer subplanes.

Assume that the involutions in \( H \) are perspectivities of \( \text{Fix}(\sigma) \). Since \( H \cong \text{PSL}(2, q) \), \( q \geq 27 \), is strongly irreducible on \( \text{Fix}(\sigma) \), then the involutions in \( H \) are homologies of \( \text{Fix}(\sigma) \) and no two of them share an axis or a center by [23, Proposition 5.2(d)]. So, \( H \) contains involutory homologies of \( \text{Fix}(\sigma) \) in a triangular configuration. Then \( H \) contains homologies of \( \Pi \) by [47, Proposition 2.6]. A contradiction by Proposition 12. Hence the involutions in \( H \) are Baer collineations of \( \text{Fix}(\sigma) \). Note that any line of \( \text{Fix}(\sigma) \) fixed by any Klein subgroup of \( H \) must be external to \( O_\sigma \), since \( H \) acts on \( O_\sigma \) in its 2-transitive permutation representation of degree \( q + 1 \) and \( q \equiv 3 \mod 8 \), being \( q = 3^{2k + 1} \) and \( k > 0 \). As a consequence, commuting involutions in \( H \) cannot fix the same Baer subplane in \( \text{Fix}(\sigma) \) and we have the assertion.

(C) The final contradiction.

Denote by \( \mathcal{E} \) the set of lines of \( \text{Fix}(\sigma) \) which are external to \( O_\sigma \). Clearly \( |\mathcal{E}| > 0 \) by (B). Actually, we have \( |\mathcal{E}| = n + \sqrt{n} + 1 - |T| - |S| \), where \( T \) and \( S \) denote the set of lines of \( \text{Fix}(\sigma) \) which are tangents and secants to \( O_\sigma \), respectively. As \( |T| = (q + 1)(\sqrt{n} - 1) \) and \( |S| = \binom{q + 1}{2} \), then

\[
|\mathcal{E}| = (\sqrt{n} + 1)(\sqrt{n} - q - 1) + \frac{q(q + 1)}{2} + 1.
\]

(9)
Denote by $A_j$, with $j \in J$, each $H$-orbit on $E$ with stabilizer of a point containing a subgroup of $H$ isomorphic to a Klein subgroup $E_4$. Note that $|A_j| > 1$ for each $j \in J$, since $H$ is strongly irreducible on $\Pi$. Furthermore $\bigcup_{j \in J} A_j \subseteq E$. Then

$$\sum_{j \in J} |A_j| \leq |E|. \tag{10}$$

We can associate in (10) the $H$-orbits $A_j$ having the same length. Hence, we obtain

$$\sum_{i \in I} \lambda_i |A_i| = \sum_{j \in J} |A_j|,
$$

where $I \subseteq J$ and $\lambda_i \geq 1$ for each $i \in I$. Then (10) becomes

$$\sum_{i \in I} \lambda_i [H : H_{l_i}] \leq |E|, \tag{11}$$

where $l_i \in A_i$ for each $i \in I$. Since $q \equiv 3 \mod 8$ and since $E_4 \leq H_{l_i}$ for each $i \in I$, then either $H_{l_i} \cong D_{2k}$, with $j | 2k + 1$ and $\theta$ even, or $H_{l_i} \cong PSL(2, 3^j)$, with $j | 2k + 1$ and $1 \leq j < 2k + 1$, or $H_{l_i} \cong S_4$ by [33, Hauptsatz II.8.27]. Thus $[H : H_{l_i}] \geq \frac{q(q-1)}{2}$ for each $i \in I$, since $q \geq 27$. Hence, by managing (11), we obtain

$$\lambda \frac{q(q-1)}{2} \leq |E|, \tag{12}$$

where $\lambda = \sum_{i \in I} \lambda_i$. Note that any subgroup of $H$ isomorphic to $E_4$ does not fix any secant to $O_\sigma$, since $H$ acts on $O_\sigma$ in its 2-transitive permutation representation of degree $q + 1$ and $q \equiv 3 \mod 8$. Hence, any subgroup of $H$ isomorphic to $E_4$ fixes at least $\sqrt{n} + 1$ external lines to $O_\sigma$. Moreover, since the subgroups of $H$ isomorphic to $E_4$ lie in a unique conjugate class and $NG(E_4) \cong A_4$ by [14, §246], we have that

$$\sqrt{n} + 1 \leq \sum_{i \in I} \lambda_i x_i, \tag{13}$$

where $x_i = \frac{|N_H(E_4)|}{|N_{H_{l_i}}(E_4)|}$ denotes the number of fixed elements in the $H$-orbit $A_i$ for each $i \in I$ by any subgroup of $H$ isomorphic to $E_4$. Clearly, $x_i \leq 3$, since $N_H(E_4) \cong A_4$ and $E_4 \leq N_{H_{l_i}}(E_4)$. By managing (13) and by bearing in mind the relation $\lambda = \sum_{i \in I} \lambda_i$, we obtain

$$\sqrt{n} + 1 \leq 3\lambda. \tag{14}$$

Now, composing (12) and (14), we have

$$\frac{q(q-1)(\sqrt{n}+1)}{6} < |E|. \tag{15}$$

Finally, composing (9) and (15), we obtain

$$\frac{q(q-1)(\sqrt{n}+1)}{6} < (\sqrt{n}+1)(\sqrt{n} - q) + \frac{q(q+1)}{2} + 1. \tag{16}$$

Now, observe the $\frac{q(q+1)}{2} + 1 \leq (q + 1)(\sqrt{n}+1)$, since $q^3 < n$. Hence, dividing in (16) by $\sqrt{n}+1$, we have

$$\frac{q(q-1)}{6} < \sqrt{n}.$$
Then $q = 27$, since $n \leq 2(q^3 + 1)$. Moreover, $\sqrt{n}$ must be an integer by (B). Then $\sqrt{n} \in \{12, 13, 14\}$, since $q^3 + 1 < n \leq 2(q^3 + 1)$ and $q = 27$. Let $\gamma$ be a Baer involution in $H$. Clearly $C_H(\gamma) \cong D_{28}$ and $C_H(\gamma)$ acts on $\text{Fix}(\gamma)$. Let $N$ be the kernel of $C_H(\gamma)$ on $\text{Fix}(\gamma)$. Then $\langle \gamma \rangle \leq N \leq C_H(\gamma)$ and $\text{Fix}(\gamma) = \text{Fix}(N)$. Actually $N = \langle \gamma \rangle$ and hence $C_H(\gamma)/N \cong D_{14}$, since $\text{Fix}(\gamma)$ is a Baer subgroup of $\text{Fix}(\sigma)$. Thus the cases $\sqrt{n} = 12$ and $\sqrt{n} = 14$ are ruled out by [34] and [31, Theorem 13.18], respectively. Hence $n = 13^q$. Let $E = \langle \sigma, \gamma \rangle$, where $\gamma$ is the involution in $H$ inducing $\gamma$ on $\text{Fix}(\sigma)$. Clearly, $E$ is a Klein subgroup of $G$ such that $\text{Fix}(E) = \text{Fix}(\gamma)$ by (B). Hence $\text{Fix}(E)$ is a subplane of $\text{Fix}(\sigma)$ of order 13 by (B). In particular, the group $N_G(E)$ acts on $\text{Fix}(E)$ with kernel, say $K$. Then $E \leq K \cong E \times D_{14}$, since $N_G(E) = (E \times D_{14}).Z_3$ by [38, Theorem C]. Actually, $K = E$ since $\text{Fix}(E) = \text{Fix}(\gamma)$ and $C_H(\gamma)/N \cong D_{14}$. Thus $\text{Fix}(E) \cong PG(2, 13)$ by [45]. Thus $Z_7.Z_3 \not\cong PSL(2, 13)$, since $Z_7 < N_G(E)/E$ and $Z_7$ fixes exactly a nonincident point-line pair in $PG(2, 13)$. A contradiction.

**Lemma 15.** Let $\Pi$ be a finite projective plane of order $n$ and let $\mathcal{O}$ be a 2-transitive $G$-arc of length $v$, with $n > v \geq n/2$. If $Z$ is an elementary abelian 2-subgroup of $G$ of maximal order consisting of Baer collineations of $\Pi$, then $\lvert Z \rvert \leq \sqrt{2v}$.

**Proof.** Let $Z$ be an elementary abelian subgroup of $G$ of maximal order consisting of Baer collineations of $\Pi$. Then $Z$ fixes a point $P$ of $\Pi$, since $n^2 + n + 1$ is odd. Furthermore each nontrivial element in $Z$ fixes exactly a point $n + 1$ lines of $[P]$, since $Z$ consists of Baer collineations. Then

$$\lvert Z \rvert = (\lvert Z \rvert - 1)(\sqrt{n} + 1) + n + 1 \quad (17)$$

by the Cauchy–Frobenius lemma. Thus $\lvert Z \rvert \leq \sqrt{n} + 1$. Then either $\lvert Z \rvert \leq \sqrt{n}$ or $\lvert Z \rvert \leq \sqrt{n} - 1$, since $Z$ is a 2-group. Then $\lvert Z \rvert \leq \sqrt{n}$ in any case. That is $\lvert Z \rvert \leq \sqrt{2v}$, since $n \leq 2v$. Thus the assertion.

**Lemma 16.** Let $\Pi$ be a finite projective plane of order $n$ and let $\mathcal{O}$ be a 2-transitive $G$-arc of length $v$, with $n > v \geq n/2$. If $G \cong PSL(d, q)$, $d \geq 2$, $q$ a prime power, then $d = 2$, $q$ is odd and $q \notin \{5, 7, 9, 11\}$. Furthermore, if $q \geq 307$ then the following occur:

(2) $q \equiv -1 \mod 8$ and $17 < \sqrt{n} \leq 32$;
(3) $q \equiv 1 \mod 8$, $q$ is a nonsquare and $17 < \sqrt{n} \leq 32$;
(4) $q \equiv 1 \mod 8$, $q$ is a square, $17 < \sqrt{n} \leq 225$ and there exists at least a $G$-orbit of external lines to $\mathcal{O}$ with stabilizer of a line isomorphic to $PSL(2, \sqrt{q})$.

**Proof.** Assume that $G \cong PSL(d, q)$, $d \geq 2$, $q$ a prime power. Clearly $G$ contains a $p$-element fixing a subplane of order at least $q^{-d-1}/q-1 - 2$. Then $d = 2$ by [31, Theorem 3.7]. Note that $q \notin \{5, 7, 9, 11\}$, since in these cases $d_0(G) < v$. Moreover, the case $G \cong PSL(2, q)$ with $q$ even is ruled out by Lemma 15, since $\lvert Z \rvert = q$, while $v = q + 1$ and $q \neq 4 (q \neq 4$ since $PSL(2, 4) \cong PSL(2, 5)$ and we proved that $q \neq 5$). Therefore $G \cong PSL(2, q)$ with $q$ odd and $q \notin \{5, 7, 9, 11\}$.

Let $\mathcal{E}$ be the set of the external lines to $\mathcal{O}$. Then $\lvert \mathcal{E} \rvert = n^2 + n + 1 - \lvert T \rvert - \lvert S \rvert$, where $T$ and $S$ denote the set of tangents and secants to $\mathcal{O}$, respectively. As $\lvert T \rvert = (q + 1)(n + 1 - q)$ and $\lvert S \rvert = (q + 1)$, then

$$\lvert \mathcal{E} \rvert = n(n - q) + \frac{q(q - 1)}{2}. \quad (18)$$
In particular, $|E| \leq \frac{1}{2}(5q^2 + 11q + 8)$, since $n \leq 2(q + 1)$. Let $(\alpha, \beta) \cong E_4$. Assume that $\text{Fix}(\alpha) = \text{Fix}(\beta)$. Then $\text{Fix}(\alpha) = \text{Fix}(\rho)$ for any nontrivial $\rho$ in $(\alpha, \beta)$. Let $P$ be any point of $O$. By the Baer property there exists a line $w$ of $\text{Fix}(\alpha)$ containing $P$. Actually, $\alpha$ fixes pointwise $w \cap O$, since $\text{Fix}(\alpha) = \text{Fix}(\rho)$ for any nontrivial $\rho$ in $(\alpha, \beta)$ and since $|w \cap O| \leq 2$. So, $\alpha$ fixes $O$ pointwise. A contradiction. Hence $\text{Fix}((\alpha, \beta)) \not\subseteq \text{Fix}(\alpha)$. Thus, $\alpha$ and $\beta$ share at least $\sqrt{n} + 1$ lines of $\mathcal{P}$. On the other hand, $\alpha$ and $\beta$ share either 0 or 3 secants to $O$ according to whether $q \equiv 3 \mod 4$ or $q \equiv 1 \mod 4$, respectively. Hence, any Klein subgroup of $E$ fixes at least $\sqrt{n} - 2$ lines on $\mathcal{E}$ in any case.

Denote by $B_j$, with $j \in J$, each $G$-orbit on $\mathcal{E}$ with stabilizer containing a Klein subgroup $E_4$. Note that $|B_j| > 1$ for each $j \in J$, since $G$ is strongly irreducible on $\mathcal{P}$. Furthermore $\bigcup_{j \in J} B_j \subseteq \mathcal{E}$. Then

$$
\sum_{j \in J} |B_j| \leq |\mathcal{E}|. \tag{19}
$$

We can associate in (19) the $G$-orbits $B_j$ having the same length. Then we obtain $\sum_{i \in I} \lambda_i |O_i| = \sum_{j \in J} |O_j|$, where $I \subseteq J$ and $\lambda_i \geq 1$ for each $i \in I$. Then

$$
\sum_{i \in I} \lambda_i [G : G_{i}] \leq |\mathcal{E}|, \tag{20}
$$

where $I \subseteq O_i$ for each $i \in I$.

Assume that $q \geq 307$. It is easily seen that $G_{i}$ cannot be isomorphic to $A_4$ or $S_4$ or $A_5$, since in these cases $|G : G_{i}| > \frac{1}{2}(5q^2 + 11q + 8)$ while $|\mathcal{E}| \leq \frac{1}{2}(5q^2 + 11q + 8)$. Assume also that $\text{PSL}(2, \sqrt{q}) \not\cong G_{i}$ for each $i \in I$. Then $[G : G_{i}] \geq \frac{q(q - 1)}{2}$ for each $i \in I$ by [33, Haupt- satz II.8.27]. Hence (20) becomes

$$
\lambda \frac{q(q - 1)}{2} \leq |\mathcal{E}|, \tag{21}
$$

where $\lambda = \sum_{i \in I} \lambda_i$. Therefore $\lambda \leq 5$, since $|\mathcal{E}| \leq \frac{1}{2}(5q^2 + 11q + 8)$ and $q \geq 307$.

Assume that $q \equiv \pm 3 \mod 8$. Then the Klein subgroups of $G$ lie in a unique conjugate class and $\text{NG}(E_4) \cong A_4$ by [14, §246]. Denote by $x_i$ the number of fixed lines in $B_i$ by any $E_4$ for each $i \in I$. Actually, $x_i = \frac{|\text{NG}(E_4)|}{|\text{NG}_{G_{i}}(E_4)|}$ by [47, relation (9)]. Then

$$
\sqrt{n} - 2 \leq \sum_{i \in I} \lambda_i x_i, \tag{22}
$$

since any Klein subgroup of $G$ fixes at least $\sqrt{n} - 2$ lines on $\mathcal{E}$ in any case. Then $x_i \leq 3$ for each $i \in I$, since $\text{NG}(E_4) \cong A_4$ and $E_4 \leq \text{NG}_{G_{i}}(A)$. As a consequence, $\sum_{i \in I} \lambda_i x_i \leq 3\lambda$. By managing (22), we obtain

$$
\frac{\sqrt{n} - 2}{3} \leq \lambda. \tag{23}
$$

Then $\sqrt{n} \leq 17$, since $\lambda \leq 5$. A contradiction, since $q \geq 307$ and $n > q + 1$.

Assume that $q \equiv \pm 1 \mod 8$. Then $\text{NG}(E_4) \cong S_4$ for any $E_4$ in $G$ and there are two conjugate classes of Klein subgroups in $G$ by [14, §246]. For each $i \in I$ denote by $x_i^{(1)}$ and $x_i^{(2)}$ the number of fixed lines in $B_i$ by any Klein subgroup of $G$ which lies either in one or in the other conjugate class. In particular, by [47, relation (9)], we have that

$$
x_i^{(h)} = \frac{|\text{NG}(E_4^{(h)})|}{|G_{i}|} \left| \{ U \leq G_{i} : U \text{ is a conjugates of } E_4^{(h)} \text{ in } G \} \right|, \tag{24}
$$
where \( h \in \{1, 2\} \). Moreover,
\[
\sqrt{n} - 2 \leq \sum_{i \in I} \lambda_i x_i^{(h)},
\]
with \( h = 1 \) or \( 2 \), since any Klein subgroup of \( G \) fixes at least \( \sqrt{n} - 2 \) lines on \( E \) in any case. It is easily seen that \( x_i^{(h)} \leq 6 \) for \( h \in \{1, 2\} \). Indeed, \( G_{l_i} \) cannot be isomorphic to \( A_4 \) or \( S_4 \) or \( A_5 \) since \( q \geq 307 \) and it might be \( G_{l_i} \cong D_{20} \) with \( \theta \equiv 2 \) mod 4 (see [33, Hauptsatz II.8.27]). As consequence, we obtain \( \sum_{i \in I} \lambda_i x_i^{(h)} \leq 6 \lambda \) and hence \( \sqrt{n} - 2 \leq 6 \lambda \) by (25). Since \( \lambda \leq 5 \), we have
\[
\sqrt{n} - 2 \leq 30.
\]
This yields \( \sqrt{n} \leq 32 \). On the other hand, \( \sqrt{n} > 17 \), since \( q > 307 \) and \( n > q + 1 \). Thus the assertions (1) and (2).

Assume that \( PSL(2, \sqrt{q}) \leq G_{l_i} \) for some \( r \in I \). Clearly \( q \) must be a square. Then \( [G : G_{l_i}] \geq \frac{q(q-1)}{2} \) if \( PSL(2, \sqrt{q}) \notin G_{l_i} \) and \( [G : G_{l_i}] \geq \frac{\sqrt{q}(q-1)}{2} \) if \( PSL(2, \sqrt{q}) \leq G_{l_i} \). Then, by managing (20), we obtain
\[
\mu_1 \frac{\sqrt{q}(q-1)}{2} + \mu_2 \frac{q(q-1)}{2} \leq |E|,
\]
where \( \lambda = \mu_1 + \mu_2 \). Recall that there are two conjugate classes of \( PSL(2, \sqrt{q}) \) and \( PGL(2, \sqrt{q}) \) in \( G \) by [14, §255]. Denote by \( C_1 \) and \( C_2 \) these classes. Denote by \( \mu_1^{(j)} \) the number of \( G \)-orbits on \( E \) such that the stabilizer of a line contains a subgroup isomorphic to \( PSL(2, \sqrt{q}) \) lying in \( C_j \), where \( j = 1 \) or \( 2 \). Then (26) becomes
\[
\mu_1^{(1)} \frac{\sqrt{q}(q-1)}{2} + \mu_1^{(2)} \frac{\sqrt{q}(q-1)}{2} + \mu_2 \frac{q(q-1)}{2} \leq |E|,
\]
where \( \mu = \mu_1^{(1)} + \mu_1^{(2)} \). Furthermore, by [47, Table III], it is easily seen that \( x_i \leq 6 \) for each \( i \in I \), since \( G_{l_i} \) cannot be isomorphic either to \( A_4 \), or to \( S_4 \), or to \( A_5 \), being \( q \geq 307 \). Thus \( \frac{\sqrt{n} - 2}{6} \leq \lambda \) by (25). Moreover, \( \mu_2 \leq 5 \) arguing as above with \( \mu_2 \) in the role of \( \lambda \). Hence
\[
\frac{\sqrt{n} - 2}{6} - 5 \leq \mu_1^{(1)} + \mu_1^{(2)}.
\]

**Step 1.** For \( q > 19^2 \) there are at most two of \( G \)-orbits on \( E \) having the stabilizer of a line isomorphic to the group \( PGL(2, \sqrt{q}) \).

Denote by \( \phi \) the number \( G \)-orbits on \( E \) having the stabilizer of a line isomorphic to \( PGL(2, \sqrt{q}) \). Let \( \gamma \) an element in \( G \) of order 4. Then \( \gamma \) fixes at least \( \frac{\sqrt{q} - 1}{2} \) lines in any of these \( \phi \) orbits by [47, Table IV]. Clearly \( Fix(\gamma) \subseteq Fix(\gamma^2) \), since \( Fix(\gamma^2) \) is a Baer subplane of \( \Pi \). Hence, \( \gamma \) fixes at most \( \sqrt{n} + \sqrt{n} \) external lines to \( \mathcal{O} \). Thus
\[
\phi \frac{\sqrt{q} - 1}{2} \leq \sqrt{n} + \sqrt{n}.
\]
At this point it is a straightforward calculation to show that \( \phi \leq 2 \) for \( q > 19^2 \), since \( n \leq 2(q + 1) \).

**Step 2.** For \( q > 19^2 \) the group \( PSL(2, \sqrt{q}) \) does not fix any subplane of \( \Pi \) pointwise.
Suppose that the group $PSL(2, \sqrt{q})$ fixes a subplane $\Pi_0$ of $\Pi$ of order $m$ pointwise for $q > 19^2$. Clearly $\Pi_0 \cap \mathcal{O} = \emptyset$ since $PSL(2, q)$ acts in its natural permutation representation of degree $q + 1$ on $\mathcal{O}$. Let $\sigma$ be an involution in $G$ normalizing the selected $PSL(2, \sqrt{q})$. Then $\sigma$ induces on $\Pi_0$ either the identity or a nontrivial quasisimilarity. Then there are at least $m + 1$ lines of $\Pi_0$ fixed by $\sigma$ and external to $\mathcal{O}$ in any case, since $m = o(\Pi_0)$ and $\Pi_0 \cap \mathcal{O} = \emptyset$. As a consequence, there are at least $m + 1$ lines of $\Pi_0$ fixed by $PGL(2, \sqrt{q})$ and external to $\mathcal{O}$ in any case. Since $G$ is strongly irreducible on $\Pi$, since there are at most two orbits $G$-orbits on $\mathcal{E}$ having the stabilizer of a line isomorphic to the group $PGL(2, \sqrt{q})$ for $q > 19^2$ and since $PGL(2, \sqrt{q})$ fixes exactly one lines in each of these $G$-orbits, being $PGL(2, \sqrt{q})$ is maximal in $G$, we have that $m + 1 \leq 2$ for $q > 19^2$. A contradiction, since $m \geq 2$.

**Step 3.** $n \leq 15^4$.

Let $H \cong PSL(2, \sqrt{q})$. Assume that $\mu_1^{(1)} > 3$. By Step 2, all the lines of $\mathcal{E}$ which are fixed by a $H$, possibly except one, are concurrent to a point $X$. Hence, there are at least $\mu_1^{(1)} - 1$ lines of $\mathcal{E}$ through $X$ which are fixed by $H$. In particular, $H \leq G_X$. Pick a Klein subgroup $A$ of $H$. Then $A$ fixes at least $\mu_1^{(1)} - 1$ lines of $\mathcal{E}$ through $X$, since $Fix(H) \subseteq Fix(A)$. Furthermore $A$ fixes a triangle $\triangle$ whose sides are secants to $\mathcal{O}$. If $X \notin \triangle$, then $\triangle \cup \{X\}$ is a quadrangle and hence $A$ is planar. Therefore $\mu_1^{(1)} - 2 \leq \sqrt[n]{4}$. If $X \in \triangle$, then $A$ fixes at least $\mu_1^{(1)} - 1$ lines through each vertex of $\triangle$, since there exists an an element of order $3$ normalizing $A$ and acting transitively on $\triangle$. Hence $A$ is planar and $\mu_1^{(1)} - 2 \leq \sqrt[n]{4}$ in any case. Arguing as above with $\mu_2^{(1)}$ in role of $\mu_1^{(1)}$, we obtain

$$\mu_1^{(1)} + \mu_2^{(2)} \leq 2 \sqrt[n]{4} + 4.$$  \hspace{1cm} (30)

Now composing (28) with (30), we obtain

$$\frac{\sqrt[n]{4} - 2}{6} - 5 \leq 2 \sqrt[n]{4} + 4,$$

which produces $\sqrt[n]{4} \leq 15$. Then $\sqrt[n]{4} \leq 15^2$ in any case and we have the assertion (3).  \hfill $\square$

**Proposition 17.** Let $\Pi$ be a finite projective plane of order $n$ and let $\mathcal{O}$ be a 2-transitive $G$-arc of length $v$, with $n > v \geq n/2$. If $G$ is a nonabelian simple group such that $d_0(G) = v$, then $G \not\cong PSL(d, q)$, $d \geq 2$, $q$ a prime power.

**Proof.** Clearly $G \cong PSL(2, q)$, $q$ is a power of an odd prime and $q \notin \{5, 7, 9, 11\}$ by Lemma 16. Recall that each involution in $G$ must be a Baer collineation of $\Pi$. Hence $n$ is a square. With the aid of GAP [17], we solve the Diophantine equation $n = q + k$, where $n$ is a square and $1 < k < q + 1$ and with respect to the reductions for $n$ and $q$ provided by Lemma 16. For each solution $(n_0, q_0, k_0)$ found, we evaluate $|\mathcal{E}(n_0, q_0, k_0)|$, where $\mathcal{E}(n_0, q_0, k_0)$ denotes the set of external lines to $\mathcal{O}$ corresponding to the solution $(n_0, q_0, k_0)$. Clearly $\mathcal{E}$ must be union of nontrivial $G$-orbits, since $G$ is strongly irreducible on $\Pi$. Since the length of each nontrivial $G$-orbit is a multiple of a degree $d_j = d_j(n_0, q_0, k_0)$, $j \geq 0$, of a primitive permutation representation of $G$, then we evaluate the primitive representation degrees of $G$ less than $|\mathcal{E}|$. Hence each solution $(n_0, q_0, k_0)$ must be a solution of the Diophantine equation $|\mathcal{E}| = \sum_{j} \theta_j d_j$. We also filter the solutions of the Diophantine equation $n = q + k$ with respect to the conditions arising from
[26, Theorem A], since the involutions in $G$ must be Baer collineations of $\Pi$. Moreover, when $q \geq 307$ and $q$ is a nonsquare we select the solutions satisfying the inequality

$$\frac{q(q-1)(\sqrt{n}-2)}{6} \leq n(n-q) + \frac{q(q-1)}{2}$$

arising from the composition of the inequalities (18), (21) and (23) of Lemma 16. We stress that the same computer search is made for $q \leq 307$ and in this case we take in consideration that there might be $G$-orbits on $E$ with stabilizer of a point isomorphic to $A_4$ or $S_4$ or $A_5$. As a result of this search, we obtain $(q,n) = (13, 2^4)$, $(13, 5^2)$, $(17, 5^2)$, $(31, 2^6)$, $(37, 2^6)$, $(181, 17^2)$, $(181, 2^8)$. Nevertheless, it is easily seen that no one of these cases geometrically occurs. Indeed, in these cases, for each involution $\sigma$ in $G$ the group induced by $C_G(\sigma)$ on the Baer subplane $Fix(\sigma)$ is isomorphic to $D_{q-1}$ and this contains planar elements of $Fix(\sigma)$. A contradiction by [31, Theorem 3.7].

**Theorem 18.** In a finite projective plane $\Pi$ of order $n$ there are no 2-transitive $G$-arcs $\mathcal{O}$ of length $v$, with $n > v \geq n/2$, such that $G$ is a nonabelian simple group with $d_0(G) = v$.

**Proof.** By the classification of the finite 2-transitive groups (e.g. see [37]) and by Lemmas 13, 14, Proposition 17 and by the argument of Theorem 11 for $G \cong A_v$, we have to investigate the following groups in order to prove the present theorem:

1. $G \cong S_2(2^r)$, $r$ odd, $r > 1$;
2. $G \cong M_v$, $v = 12, 22, 23, 24$;
3. $G \cong Co_3$.

Assume that $G \cong S_2(q)$, $q = 2^r$, $r$ odd and $r > 1$. Let $S$ be a Sylow 2-subgroup of $G$ and let $Z = Z(S)$. Then $Z$ is a group of order $q$ consisting of all involutions of $S$ by [42, Theorem 24.2]. Furthermore each element in $Z$ is a Baer collineation of $\Pi$ by Proposition 12. Then $q \mid \sqrt{n}$ or $q \mid \sqrt{n} - 1$ by relation (17) of Lemma 15, since $|Z| = q$. It is easily seen that $\sqrt{n} = q + 1$, since $v < n \leq 2v$ and $v = q^2 + 1$. Let $\alpha \in Z$, $\alpha \neq 1$. Clearly $\alpha$ is a Baer collineation of $\Pi$. Assume that $Fix(\delta) = Fix(\alpha)$ for some $\delta \in Z$, $\delta \neq \alpha$. Let $B \in \mathcal{O} - Fix(\alpha)$. Then there exists a line $y \in Fix(\alpha)$ such that $B \in y$, since $Fix(\alpha)$ is a Baer subplane of $\Pi$. Then $|b \cap \mathcal{O}| = 2$, since $y \in (b \cap \mathcal{O}) - Fix(\alpha)$. Nevertheless $\alpha$ fixes $b \cap \mathcal{O}$ pointwise, since $|b \cap \mathcal{O}| = 2$ and since $Fix((\alpha, \delta)) = Fix(\alpha)$ being $\langle \alpha, \delta \rangle \cong E_4$. A contradiction, since $Z$ fixes a point on $\mathcal{O}$ and it is semiregular on the remaining ones by [42]. As a consequence, $Z$ acts on $Fix(\alpha)$ with kernel $\langle \alpha \rangle$. Set $H = Z/\langle \alpha \rangle$. Assume that $q + 1$ is a nonsquare. Then $q = 8$ and hence $|H| > 8$, since $|H| = q/2$, being $q = 2^r$ with $r$ odd and $r > 1$. Thus $H = H(Q,l)$ with $Q \notin \mathcal{O}$ in $Fix(\alpha)$ by [36, Lemma 3.1(ii)], since $\sqrt{n} = q + 1$ is an odd nonsquare. Therefore $Z$ fixes the $\sqrt{n} + 1$ lines of $[Q] \cap Fix(\alpha)$. Then $Z$ is semiregular on $[Q] - [Q] \cap Fix(\alpha)$, since $Z$ consists of all the involutions of $S$ and since $Fix(S) \subseteq Fix(Z)$. Hence $q^2 \mid \sqrt{n}(\sqrt{n} - 1)$, since $|[Q] - [Q] \cap Fix(\alpha)| = n - \sqrt{n}$. A contradiction. Assume that $q + 1$ is a square. Then $q = 8$ and $\sqrt{n} = 9$ by [52, Result A5.1], since $q = 2^r$, $r$ is odd and $r > 1$. Hence $|H| = 4$. Furthermore, $H$ consists of homologies of $Fix(\alpha)$ by [26, Theorem A]. In particular, $H$ consists of homologies of $Fix(\alpha)$ lying in a triangular configuration, since $H$ is abelian and since the case $H = (Q,l)$ is ruled out by the above argument. Then there exists a nontrivial element in $Z$ which is a homology of $\Pi$ by [47, Proposition 2.6]. A contradiction, since each involution in $S$ must be a Baer collineation of $\Pi$. 
Assume that \( G \cong M_12 \) and \( v = 12 \). Then \( n \) is a square, since each involution in \( G \) is a Baer collineation of \( \Pi \) by Proposition 12. Then \( n = 16 \), since \( v < n \leq 2v \) and \( v = 12 \). Let \( O \in \mathcal{O} \). Then \( G_O \cong M_{11} \). Clearly \( M_{11} \) fixes the 6 tangents to \( \mathcal{O} \) through \( O \), since \( d_0(G_O) = 11 \). So, each involution in \( G_O \) fixes 6 lines through \( O \) and hence \( \sqrt{n} = 5 \). A contradiction by [31, Theorem 3.7], since \( n = 16 \).

Assume that \( G \cong M_n \) and \( v = 22 \), or 23, or 24. Each of these groups contains an elementary abelian 2-subgroup of order 16 by [53, Lemmas 1.3, 1.4 and 1.5]. Then these groups are ruled out by Lemma 15, since \( 16 > \sqrt{2v} \) for \( v = 22, 23, 24 \).

Finally, assume that \( G \cong Co_3 \). By [16], there exists a nontrivial element in \( G \) fixing a subplane of order at least 34. A contradiction by [31, Theorem 3.7], since \( n \leq 2v \) and \( v = 276 \). Thus the assertion. \( \square \)

Thus the case \( d_0(G) = v \) has been completed. Now, assume that \( d_0(G) < v \).

**Theorem 19.** Let \( \Pi \) be a finite projective plane of order \( n \) and let \( \mathcal{O} \) be a 2-transitive \( G \)-arc of length \( v \), with \( n > v \geq n/2 \). If \( G \) is a nonabelian simple group such that \( d_0(G) < v \), then \( \Pi \cong PG(2, 9) \), \( \mathcal{O} \) is a complete 6-arc and \( G \cong PSL(2, 5) \).

**Proof.** The nonabelian simple groups such that \( d_0(G) < v \) are listed in Table 1 (see [37]).

<table>
<thead>
<tr>
<th>Group</th>
<th>( v )</th>
<th>( d_0(G) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G \cong PSL(2, 5) )</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>( G \cong PSL(2, 7) )</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>( G \cong PSL(2, 9) )</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>( G \cong PSL(2, 11) )</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>( G \cong PSL(4, 2) )</td>
<td>15</td>
<td>8</td>
</tr>
<tr>
<td>( G \cong A_7 )</td>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>( G \cong PSU(3, 5) )</td>
<td>126</td>
<td>50</td>
</tr>
<tr>
<td>( G \cong M_{11} )</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>( G \cong HS )</td>
<td>176</td>
<td>100</td>
</tr>
<tr>
<td>( G \cong Sp(2h, 2), h \geq 3 )</td>
<td>( 2^{h-1}(2^h + 1) )</td>
<td>( 2^{h-1}(2^h - 1) )</td>
</tr>
</tbody>
</table>

Assume that \( G \cong PSL(2, 7) \). Then \( 6 < n \leq 12 \). The cases \( n = 10 \) and \( n = 12 \) are ruled out by [31, Theorem 13.18], and [34], respectively. If \( n = 7 \) or 8, then \( \Pi \cong PG(2, n) \). Also for \( n = 9 \) and \( n = 11 \), we have that \( \Pi \cong PG(2, n) \) by [6] and [44], respectively. Therefore \( \Pi \cong PG(2, n) \) in any case. Actually \( n = 9 \) by [54, Proposition 15]. Hence the assertion.

Assume that \( G \cong PSL(2, 9) \). Then \( 8 < n \leq 16 \). The cases \( n = 10, 12 \) or 14 cannot occur by the same argument as above. Furthermore, the case \( n = 15 \) cannot occur by [27]. In the remaining cases, but \( n = 16 \), we have that \( \Pi \cong PG(2, n) \) by [26, Theorem C], [44] and [45], respectively. Nevertheless these cases are ruled out by [54, Proposition 15]. Finally, the case \( n = 16 \) cannot occur by [13, Results 4.6(a) and (b), and 4.8(i)], and by [54, Proposition 15], since \( \mathcal{O} \) is a 8-arc.

Assume that \( G \cong PSL(2, 9) \). Then \( 10 < n \leq 20 \). Assume also that \( G \) contains involutory perspectivities. Let \( A \cong E_4 \). Then \( A \) fixes a triangle \( \Delta \) whose vertices lie in \( \Pi - \mathcal{O} \) and whose sides are secants to \( \mathcal{O} \). Thus each involution is a homology and hence \( n \) is odd. In particular, \( n \leq 19 \). Let \( X \in \Delta \). It is easily seen that \( G_X \cong D_8 \). Since \( n \leq 19 \), then \( |G| > |\Pi - (\mathcal{O} \cup X^G)| \). Hence \( G \) is totally irregular on \( \Pi \). Then \( n = 19 \) by [29, Theorem B(II)], since \( n \leq 19 \). Let \( C \cong Z_4 \). Clearly \( C \) fixes two points on \( \mathcal{O} \). Let \( Y \) be one of these points, then the involution in \( C \) fixes at least 3 of the 11 tangents to \( \mathcal{O} \) through \( Y \). Therefore the involution in \( C \) must be a homology of \( \Pi \) of center \( C \). A contradiction by Lemma 3. Thus, each involution in \( G \) is a Baer
collineation of \( \Pi \) and \( n \) is a square. Hence \( n = 16 \), since \( 10 < n \leq 24 \). Assume that \( G \) fixes a line \( l \) of \( \Pi \). Clearly \( l \) is external to \( \mathcal{O} \). Furthermore, each secant \( s \) to \( \mathcal{O} \) intersects \( l \). Then \( G_s \cong D_8 \) fixes \( l \cap s \). Since the secants to \( \mathcal{O} \) are 45 and \( n + 1 = 17 \), there are \( s_1 \) and \( s_2 \) which are secants to \( \mathcal{O} \) and such that \( s_1 \cap l = s_2 \cap l \). Set \( \{ P \} = s_1 \cap l \). Then \( \langle G_{s_1}, G_{s_2} \rangle \leq G_P \). In particular, either \( G_P = G \) or \( G_P \cong S_4 \), since \( G_s \cong D_8 \) for \( i = 1, 2 \) and \( G_{s_1} \neq G_{s_2} \). If \( G_P = G \) then each line through \( P \) intersecting \( \mathcal{O} \) would be a secant, since \( P \in s_1, s_1 \) is a secant to \( \mathcal{O} \) and \( G \) is transitive on \( \mathcal{O} \). A contradiction, since \( G \) is primitive on \( \mathcal{O} \). Hence \( G_P \cong S_4 \). Then \( l \) consists of a \( G \)-orbit of length 15, namely \( P^G \), and of two points fixed by \( G \). It easily seen that each involution fixes 5 points on \( P^G \). Hence each involution fixes 7 points on \( l \). A contradiction, since each involution in \( G \) is a Baer collineation of \( \Pi \) and \( n = 16 \). Hence \( G \) does not fix any line of \( \Pi \). Denote by \( \mathcal{E} \) the set of external lies to \( \mathcal{O} \). Clearly \( |\mathcal{E}| = 148 \). Then \( \mathcal{E} \) is union of nontrivial \( G \)-orbits of lines, since \( G \) does not fix any line of \( \Pi \), since each \( G \)-orbit is the index of proper subgroup of \( G \) and since \( |\mathcal{E}| = 148 \), then \( \sum_{j \in J} \lambda_j \|G : G_{z_j}\| = 148 \), where \( G_{z_j} \) is known by [33, Hauptsatz II.8.27]. Moreover, any Baer involution fixes 8 external lines to \( \mathcal{O} \), since it fixes 5 secants and 8 tangents to \( \mathcal{O} \). Then \( \sum_{j \in J} \lambda_j x_j = 8 \), where \( x_j \) denotes the number of lines fixed by any involutions in \( z_j^G \).

Note that \( x \) is the same for any involution in \( G \), since they lie in a unique conjugate class. In particular, if \( \sigma \) is any involution of \( G \), then it easily seen that \( x_j = |C_G(\sigma) : |C_{G_{z_j}}(\sigma)| \), where \( C_G(\sigma) \cong D_8 \) and \( C_{G_{z_j}}(\sigma) \) is known by [33, Hauptsatz II.8.27]. Finally, with the aid of GAP [17], we solve the system of the two Diophantine equations finding no solutions.

Assume that \( G \cong PSL(2, 11) \). Assume also that \( G \) contains Baer involutions of \( \Pi \). Then \( n = 16 \), since \( 12 < n \leq 24 \). Let \( Z \leq G \) such that \( Z \cong Z_{11} \). Then \( Z \) fixes exactly one point \( P \) on \( \mathcal{O} \). It is easily seen that \( Z_{11} \) fixes a subplane of \( \Pi \) of order 5. A contradiction by [31, Theorem 3.7], since \( n = 16 \). Hence each involution in \( G \) is a perspectivity of \( \Pi \). Furthermore, it is easily seen that \( G \) is totally irregular on \( \Pi \), since \( n \leq 24 \). Moreover, \( G \) is strongly irreducible on \( \Pi \) by [28, Theorems 1 and 2]. Then each involution in \( G \) is a homology of \( \Pi \) by [23, Proposition 5.2(d)]. Thus \( n \) is odd. As consequence \( n \) is a prime, since the cases \( n = 15 \) and \( n = 21 \) cannot occur by [27], and by [31, Theorem 3.6], respectively. Then \( \Pi \cong PG(2, n) \) by [25, Remark 6.1], since \( n \) is a prime and \( n < 24 \). A contradiction by [54, Proposition 15].

Assume that \( G \cong PSL(4, 2) \) or \( G \cong A_7 \). Note that \( A_7 \leq G \) in any case. Hence we may assume that \( G \cong A_7 \) in order to investigate both cases at the same time. Any involution in \( G \) fixes exactly 3 points on \( \mathcal{O} \) by [48, Theorem 7.1 and Table I]. Thus any involution in \( G \) is a Baer collineation of \( \Pi \) by Lemma 3. Hence either \( n = 16 \) or \( n = 25 \). Actually the case \( n = 16 \) cannot occur by [13, Results 4.6(a) and (b), and 4.8(i)], since \( G_X \cong PSL(2, 7) \) for any \( X \in \mathcal{O} \) and \( G_X \) is transitive on the 14-arc \( \mathcal{O} - \{X\} \). Hence \( n = 25 \). Note that \( G_X \) acts on the set of \( K \) of tangents to \( \mathcal{O} \) in \( X \). Clearly \( |K| = 12 \). Denote by \( k \) the number of elements in \( K \) fixed by \( G_X \). Since the primitive permutation representation degrees of \( G_X \) less than 12 are 7 and 8, we have that \( 12 = k + 7a_1 + 8a_2 \) with \( a_1, a_2 \geq 0 \). It is easily seen that \( (k, a_1, a_2) = (4, 0, 1) \), since each involution in \( G_X \) is a Baer collineation of \( \Pi \) and \( n = 25 \). Let \( \gamma \in G_X \) such that \( o(\gamma) = 4 \). Then \( \gamma \) does not fix any point on \( \mathcal{O} - \{X\} \), \( \gamma^2 \) fixes exactly two distinct points \( X_1 \) and \( X_2 \) on \( \mathcal{O} - \{X\} \) and \( X_1 \gamma = X_2 \) by [48, Theorem 7.1 and Table I]. Denote by \( \tilde{\gamma} \) the collineation induced by \( \gamma \) on the Baer subplane \( Fix(\gamma^2) \). Then \( \tilde{\gamma} \) must be an involutory perspectivity of \( Fix(\gamma^2) \), since \( Fix(\gamma^2) \) has order 5 and \( X_1 \tilde{\gamma} = X_2 \) with \( X_1, X_2 \in Fix(\gamma^2) \). Nevertheless, \( \tilde{\gamma} \) fixes exactly 4 lines in \( [X] \cap Fix(\gamma^2) \), namely those fixed by the whole group \( G_X \), since \( X_1X \tilde{\gamma} = X_2X \tilde{\gamma} \). A contradiction.

Assume that \( G \cong PSU(3, 5) \). Note that any involution \( \sigma \) is a Baer collineation of \( \Pi \), since \( \sigma \) fixes 6 points on \( \mathcal{O} \). Thus \( n \) is a square. Hence \( \sqrt{n} \in \{12, 13, 14, 15\} \), since \( 126 < n \leq 252 \). By [31, Theorem 13.18], the case \( \sqrt{n} = 14 \) cannot occur. Also the cases \( \sqrt{n} = 12 \) or 15 cannot occur. 
by [34] and [27], respectively, since \( C_G(\sigma)/\langle \sigma \rangle \cong PGL(2, 5) \). Hence \( \sqrt{n} = 13 \). Then \( \text{Fix}(\sigma) \cong PG(2, 13) \) by [45]. A contradiction by [54, Proposition 15], since \( |\text{Fix}(\sigma) \cap \mathcal{O}| = 6 \).

Assume that \( G \cong M_{11} \) and \( v = 12 \). It is easily seen that \( G \) contains Baer collineation of \( \Pi \) (see, for example, [15, Appendix B]). Thus \( n = 16 \), since \( 12 < n \leq 24 \). Let \( Z_{11} \leq G \). Then \( Z_{11} \) fixes a point \( O \in \mathcal{O} \), since \( v = 12 \). Then \( Z_{11} \) fixes the 6 tangents to \( \mathcal{O} \) through \( O \), since \( n + 1 = 17 \). Moreover, \( Z_{11} \) fixes 5 points on each of them, other than \( O \). Thus \( Z_{11} \) fixes a subplane of order 5. Then \( n \geq 5^2 \) by [31, Theorem 3.7]. A contradiction, since \( n = 16 \).

Assume that \( G \cong HS \) and \( v = 176 \). Note that \( G \) contains an involution \( \alpha \) fixing at least four points on \( \mathcal{O} \) by [15, Appendix B]. Thus \( \alpha \) is a Baer collineation of \( \Pi \) and hence \( n \) is a square. Then \( 13 \leq \sqrt{n} \leq 18 \), since \( 176 < n \leq 352 \). Actually, \( \sqrt{n} \neq 14, 15 \) and 18 by [27] and by [31, Theorem 13.18], since \( C_G(\alpha)/\langle \alpha \rangle \) is nonsolvable by [53, Lemma 1.9(b)]. Furthermore, there exists an element \( \gamma \) in \( G \) fixing a subplane of \( \Pi \) of order \( m \geq 14 \) by [16]. Thus, either \( n = 256 \) or 289. Let \( P \in \mathcal{O} \). Then \( G_P \cong M_{22} \) acts on the set \( T \) of tangents to \( \mathcal{O} \) through \( P \). Then \( M_{22} \) fixes at least 5 tangents to \( \mathcal{O} \) through \( P \), since \( |T| = 82 \) or 115 and since 22 and 77 are the primitive permutation representation degrees of \( M_{22} \) which are less than 115. So \( m \geq 19 \). A contradiction, since \( n \leq 289 \).

Finally, the argument inside the proof of Lemma 8 rules out the group \( G \cong Sp(2h, 2) \) with \( h \geq 3 \). Thus the assertion. \( \square \)

Theorem 5 easily follows by Theorems 18 and 19.

4. The solvable case

In this section we analyze the case where \( \Pi \) is a finite projective plane of order \( n \) and \( \mathcal{O} \) is a 2-transitive \( G \)-arc of length \( v \) with \( n > v \geq n/2 \), under the assumption that the socle of \( G \) is solvable. In particular, we prove the following theorem.

**Theorem 20.** Let \( \Pi \) be a finite projective plane of order \( n \) and let \( \mathcal{O} \) be a 2-transitive \( G \)-arc of length \( v \), with \( n > v \geq n/2 \). If the socle of \( G \) is solvable, then \( G \leq A\Gamma L(1, v) \). Moreover, one of the following occurs:

(1) \( v \) is even and \( n = 2v \);
(2) \( v \) is odd, \( v = n − 1 \), \( n \equiv 0 \) mod 4 and the involutions in \( G \) are elations;
(3) \( v \) is odd, \( n \) is a square and the involutions in \( G \) are Baer collineations.

Note that, in this context, the socle of \( G \) is an elementary abelian \( p \)-group for some prime \( p \), since \( G \) is faithful and 2-transitive on \( \mathcal{O} \). Hence, the arc \( \mathcal{O} \) is endowed with the structure of a \( d \)-dimensional vector space over \( GF(p) \) and the zero vector in \( \mathcal{O} \) is denoted by \( O \). Thus \( |\mathcal{O}| = v = p^d \), \( d \geq 1 \). Then \( G = TG_O \), where \( T \) is the full translation group of \( \mathcal{O} \) and \( G_O \leq GL(d, p) \).

We treat the case \( p = 2 \) and \( p \) odd separately.

Assume that \( p = 2 \).

**Lemma 21.** The following hold:

(i) \( T = T(l, l) \) for some external line \( l \) to \( \mathcal{O} \);
(ii) \( n = 2^{d+1} \);
(iii) $\Pi - l$ is partitioned in $2^{d+2}$ incomplete 2d-arcs;
(iv) There exists a hyperoval of $\Pi$ containing $O$.

**Proof.** Assume that $T$ contains a Baer collineation of $\Pi$. Then each nontrivial element in $T$ is a Baer collineation of $\Pi$ by [50, Proposition 4.2], since $G$ is 2-transitive on $O$. Note that $T$ fixes a point $P$ on $\Pi$, since $n^2 + n + 1$ is odd. Furthermore, each nontrivial element in $T$ fixes exactly $\sqrt{n} + 1$ lines through $P$. Thus $2^d | (2^d - 1)(\sqrt{n} + 1) + n + 1$ by [31, Result 1.4]. Hence $2^d | n - \sqrt{n}$. Either $2^d | \sqrt{n} - 1$ or $2^d | \sqrt{n}$ So $2^d \leq \sqrt{n}$ in any case. That is $2^{d+1} \leq n$.

A contradiction, since $n \leq 2v$, $v = 2^d$ and $v > 4$. Hence, we may assume that $T$ is a group of perspectivities of $\Pi$. Furthermore, any two of them cannot have the same center, since $T$ is semiregular on $O$. Then $T = T(l, l)$ for some line $l$ external to $O$ by [35], since $T$ is abelian and $|T| > 4$. That is the assertion (i). Let $C \in l$ such that $T(C, l) \neq (1)$. Then all the lines through $C$ intersecting $O$ are secants to $O$, since $T$ is semiregular on $O$. Let $z$ be one of these lines and let $R \in z \cap O$. Then $|C^G_R| = 2^d - 1$, since $G$ is 2-transitive on $O$. Furthermore $C^G_R \subset l$, since $G$ fixes $l$ being $T(l, l) < G$. Moreover, there are exactly $2^{d-1}$ secants to $O$ through each $X \in C^G_R$, since $T(X, l) \neq (1)$ and $T$ is regular on $O$. Thus each secant to $O$ contains exactly one point of $C^G_R$, since the secants to $O$ are $2^{d-1}(2^d - 1)$. As a consequence, we have that $|T(X, l)| = 2$ for each $X \in C^G_R$. Since $n > v$ and $v = 2^d$, there exists a point $D$ on $l$ such that $T(D, l) = (1)$. Then all the lines of $D$ which intersect $O$ are tangents to $O$, since $T$ is regular on $O$. Thus $O$ is incomplete. Furthermore, $T$ is semiregular on the subset $E$ of $D - \{l\}$ consisting of the external lines to $O$. Then $2^{2d} | n - 2^d$, since $|E| = n - 2^d$. Hence $n = 2v = 2^{d+1}$, since $v < n \leq 2v$ and $v = 2^d$. Hence the assertion (ii). Let $O'$ be any other $T$-orbit on $\Pi - l$. If $O' \subset a$ for some line $a$ of $\Pi$, then $T$ fixes $a$. So $T$ fixes the lines $a$ and $l$ and hence the point $a \cap l$. So $T = T(a \cap l, l)$. A contradiction. Thus $O'$ is not contained in any line of $\Pi$. Let $r$ be any secant to $O'$. Then $T_r \neq (1)$, since $T$ is an elementary abelian 2-group acting regularly on $O'$. Actually $|T_r| = 2$, since $T_r = T(r \cap l, l)$. Thus $|r \cap O'| = 2$, since $T_r$ is regular on $r \cap O'$. Hence $O'$ is a 2d-arc. Actually $O'$ is an incomplete 2d-arc by the above argument with $O'$ in role $O$. In particular, $\Pi - l$ is partitioned in $2^{d+2}$ incomplete 2d-arcs left invariant by $T$. Hence the assertion (iii). Let $Y \in l$ such that $T(Y, l) \neq (1)$. Let $s \in [Y] - \{l\}$, such that $s \cap O = \emptyset$. Let $Q \in s$ such that $Q \notin l$, and set $O'' = QT$. Clearly $O''$ is a 2d-arc of $\Pi$, since $O'' \subset \Pi - l$. In particular, $O$ and $O''$ do not share any line of $[Y]$ by the construction of $O''$. Let $b$ be any secant to $O$. Then $T(b \cap l, l) \neq (1)$ by the above argument. Assume that $|b \cap O''| > 0$. Then $|b \cap O''| = 2$, since $T(b \cap l, l) \neq (1)$ and $T$ is regular on $O''$. Clearly $b \notin [Y]$. Furthermore $b \cap l, Y \in C^G_R$, since $T(b \cap l, l) \neq (1)$ and $T(Y, l) \neq (1)$. Thus there exists $g \in G_R$ such that $(b \cap l)g = Y$. Then $bg \in [Y] - \{l\}$. Furthermore, both $|bg \cap O| = 2$ and $|bg \cap O''| = 2$. A contradiction, since $O$ and $O''$ do not share any line of $[Y]$. Therefore $O \cup O''$ is a $n$-arc of $\Pi$. Let $O \in O$. Then there are exactly $2^d + 2$ tangents to $O$ through $O$, since $n = 2^{d+1}$. Furthermore $2^d$ of these lines are also tangents to $O''$, since $O \cup O''$ is an arc. Hence, there exist two lines $a_1, a_2 \in [O]$ which are tangents to $O$ and external to $O''$. Let $\{U_i\} = a_i \cap l$, with $i = 1, 2$. Assume that there exists a line $c$ through $U_i$, with $i = 1$ or 2, which is tangent both to $O$ and $O''$. We may assume that $i = 1$ without loss of generality. Let $\{P\} = c \cap O$. Then $P \tau = O$ for some $\tau \in T$, since $T$ is transitive on $O$. Thus $U_1 P \tau = U_1 O$. That is $c \tau = a_1$. A contradiction, since $c$ intersects $O''$ while $a_1$ does not. Hence both $U_1$ and $U_2$ are kernels for $O \cup O''$. Thus

$$O \cup O'' \cup \{U_1, U_2\}$$

is a hyperoval of $\Pi$ containing $O$, and we have the assertion (iv). \(\Box\)
Theorem 22. Let $\Pi$ be a projective plane of order $n$ and let $O$ be a 2-transitive $G$-arc of even length $v$, with $n > v \geq n/2$. If $G$ is of affine type on $\Pi$, then $v = n/2$ and $G_O \leq \Gamma L(1, n/2)$.

Proof. Recall that $|O| = 2^d$, $d \geq 1$. By [20] a structure of $d^*$-dimensional vector space $V$ over a field $L \cong GF(2^h)$, $h | d$, $d = h d^*$, may be defined on $O$ in such a way that $G \leq A \Gamma L(V)$ and $O$ is identified with the zero-vector of $V$. Assume that $d$ is even. Then each involution in $G$ is an elation of $\Pi$, since $n = 2^{d+1}$ is a nonsquare. Then any involution fixes either 0 or 2 points on $O$ by Lemma 3(2). Then $G$ is solvable by [3] and [19]. Actually $G_O \leq \Gamma L(1, n/2)$ by [32], since $|O| = 2^d$, and we have the assertion. Assume that $d$ is odd. Then $d^*$ is odd, as $d^* \mid d$. By the classification of the finite 2-transitive groups of affine type, either $G_O \leq \Gamma L(1, n/2)$ or $SL(d^*, 2^h) \leq G_O$, with $d^* \geq 3$. Assume that $SL(d^*, 2^h) \leq G_O$, with $d^* \geq 3$. Let $\tau \in G_O$ inducing a transvection on $O$. Clearly $\tau$ fixes exactly $2^{d-h}$ points on $O$. Hence $\tau$ is a Baer collineation of $\Pi$ and $n$ is a square, since $d^* \geq 3$. Then $(2^{d-h} - 2)^2 \leq 2^d$ by [31, Theorem 3.7], since $o(\text{Fix}(\tau)) \geq 2^{d-h} - 2$. This yields $2(d - h) \leq d$. Hence $d \leq 2h$. A contradiction, since $d^* \geq 3$. Thus $G_O \leq \Gamma L(1, n/2)$ and we have the assertion. 

Example 23. In the Lorimer–Rahilly translation plane of order 16 and in the Johnson–Walker translation plane of order 16 there exists a 8-arc on which the group $G \cong AGL(1, 8)$ acts 2-transitively. In particular, this arc is contained in a hyperoval.

Let $\Pi$ be the Lorimer–Rahilly translation plane of order 16 or the Johnson–Walker translation plane of order 16. Denote by $l_\infty$ the line at infinity of $\Pi$ and let $T$ be the full translation group of $\Pi$. By [9, Tables 11 and 12], there exists a hyperoval $\mathcal{H}$ containing the origin $O$ of the plane which is left invariant by a group $T_0.Z_7$, where $T_0$ is a subgroup of $T$ of order 16 and $Z_7$ lies in the translation complement fixing $O$. Furthermore, $\mathcal{H} \cap l_\infty = \{A, B\}$, $T_0$ fixes $A$ and $B$ and $T$ is regular on $\mathcal{H}_0 = \mathcal{H} - (\mathcal{H} \cap l_\infty)$. Now, $Z_7$ leaves a subgroup $T_1$ of $T_0$ of order 8 invariant, since $T_0 - \{1\} \cong PG(3, 2)$ and $Z_7$ fixes a nonincident point-plane pair in $PG(3, 2)$ by [48, Table I]. Clearly $Z_7$ is transitive on $T_1 - \{1\}$. Set $O = O^{T_1}$. Clearly $O \subset \mathcal{H}_0$. Moreover, the group $T_1.Z_7$ is 2-transitive on 8-arc $O$, since $Z_7$ acts on $T_0 - \{1\}$ and on $\mathcal{H}_0 - \{O\}$ in the same way by [50, Proposition 4.2]. Actually, the group $T_1.Z_7$ splits $\mathcal{H}_0$ in two 8-arcs and $T_1.Z_7$ acts 2-transitively on each of them.

We remark that in the other known finite projective planes of order 16 there are no examples of 8-arc on which a group isomorphic to $AGL(1, 8)$ acts 2-transitively (see [51]).

Now, assume that $p$ is odd.

Lemma 24. Let $\Pi$ be a finite projective plane of order $n$ and let $O$ be a 2-transitive $G$-arc of odd length $v$, with $n > v \geq n/2$. Then one of the following occurs:

1. $n = v + 1$, $v \equiv 3 \mod 4$, $O$ is contained in a hyperoval, any involutory dilatation of $G$ is an elation of $\Pi$ and $G \leq A \Gamma L(1, v)$. Moreover, if $n = 2^t$, $t \geq 2$, then $v$ is a Mersenne prime.
2. $n$ is a square and any involutory dilatation of $G$ is a Baer collineation of $\Pi$.

Proof. Let $O \in \Pi$ and let $\alpha$ be the involutory $O$-dilatation in $G_O$. If $\alpha$ is a Baer collineation of $\Pi$, then $n$ is a square and we have the assertion (2). So, we may assume that $\alpha$ is a perspectivity of $\Pi$. Then $C_\alpha \neq O$ by Lemma 3. In particular, $a_\alpha \cap O = \{O\}$ and $C_\alpha O$ is a tangent to $O$. If $T$ contains a nontrivial planar element $\tau$, then $\alpha$ induces a perspectivity on $\text{Fix}(\tau)$, since $\alpha$
inverts \( \tau \) and \( \alpha \) is a perspectivity of \( \Pi \). Then \( a_\alpha \) is a secant to \( \text{Fix}(\tau) \) and hence \( \tau \) fixes \( a_\alpha \cap \mathcal{O} \). A contradiction, since \( T \) is semiregular on \( \mathcal{O} \) while \( a_\alpha \cap \mathcal{O} = \{0\} \). Hence, we may assume that \( T \) does not contain nontrivial planar elements. Let \( \beta \) be the involutory \( O' \)-dilatation, with \( O' \neq O \) then \( a_\beta \cap \mathcal{O} = \{O'\} \) and \( C_\beta O' \) is a tangent to \( O \) arguing as above with \( \beta \) in role of \( \alpha \). Clearly \( a_\alpha \neq a_\beta \). If \( C_\alpha = C_\beta \), then \( \alpha \) fixes \( C_\beta O' \cap \mathcal{O} \). A contradiction, since \( O' \neq O \) and \( C_\beta O' \cap \mathcal{O} = \{O'\} \). Hence, we may assume that \( C_\alpha \neq C_\beta \). Moreover, \( C_\alpha \neq a_\beta \), otherwise \( \alpha \) would fix \( a_\beta \cap \mathcal{O} = \{O'\} \) which is clearly impossible. Arguing as above with \( \beta \) in role of \( \alpha \) we have \( C_\beta \neq a_\alpha \). Thus there exists a nontrivial element \( \gamma \) in \( \langle \alpha, \beta \rangle \cap T \) which is a generalized homology of \( \Pi \) by [22, Lemma 5.1]. By using Hering’s notation [22], we have that \( \text{Fix}(\gamma) \) is of type \( (D_k) \) with \( i \geq 0 \). Furthermore \( \text{Fix}(T) \) is either of type \( (D_k) \) with \( k \leq i \) for \( i \neq 2 \), or of type \( (A) \) or \( (D_k) \) with \( k \in \{0, 2\} \) for \( i = 2 \) by [22, Table 3.5], as \( T \) is abelian. Note that the nontrivial elements in \( T \) lie in a unique conjugate class by [50, Proposition 4.2], since \( G \) is \( 2 \)-transitive on \( \mathcal{O} \). Thus each element in \( T \) fixes as many points on \( l \) as those fixed by \( \gamma \) on \( l \). If \( k = 0 \), then \( T \) is semiregular on \( l \). Then \( n = 2v \), since \( v \mid n \) and \( v < n \leq 2v \). A contradiction by [31, Theorem 13.18], since \( v \) is odd. Hence \( k \geq 1 \). Assume that \( \text{Fix}(T) \) is of type \( (A) \). Then \( \text{Fix}(\gamma) \) is of type \( (D_2) \) by [22, Table 3.5]. Then \( T \) leaves a triangle \( \Psi \) invariant, since \( T \) is abelian and \( T \) does not contain planar elements. Hence \( p = 3 \). Let \( T_0 \) be the pointwise-stabilizer of \( \Psi \). Clearly \( |T : T_0| = 3 \). Furthermore, each nontrivial element in \( T \) fixes exactly a triangle on \( \Pi \) since \( T \) does not contain planar elements. Thus \( \xi \) fixes a triangle \( \Psi_\xi \) for any \( \xi \in T - T_0 \). Clearly \( \Psi_\xi \cap \Psi = \emptyset \), since \( |T : T_0| = 3 \). Then \( |T| = 9 \) by [22, Theorem 3.12], since \( T \) does not contain planar elements. It is easily seen that the subgroup \( T' \langle \alpha \rangle \) of \( G \) acts faithfully and strongly irreducibly one the subspace generated by its perspectives, since \( \text{Fix}(T) \) is of type \( (A) \), \( T \) does not contain planar elements and \( T \) is semiregular on \( \mathcal{O} \). Then the minimal subgroup of \( T' \langle \alpha \rangle \) has order \( 9 \) by [22, Theorem 5.5]. A contradiction, since any subgroup of order \( 3 \) in \( T \) is normal in \( T' \langle \alpha \rangle \) because \( T \) is abelian and \( \alpha \) acts as the inversion on \( T \). Hence \( \text{Fix}(T) \) cannot be of type \( (A) \).

Let \( Y \in l \cap \text{Fix}(T) \). Then \( T \) is semiregular on \( XY - \{X, Y\} \), since \( \text{Fix}(T) - \{X\} \subset l \). Thus \( v \mid n - 1 \). Hence \( n = v + 1 \), since \( v < n \leq 2v \). Then \( k = 2 \), since there are \( v \) tangents and \( 2 \) external lines to \( X \) through \( X \), because \( T \) fixes \( X \) and acts regularly on \( \mathcal{O} \). Hence \( T \) fixes a triangle \( \Delta = \{X, Y, Z\} \). Then \( \mathcal{O} \cup \Delta \) is a hyperoval of \( \Pi \), since the sides of \( \Delta \) are external to \( \mathcal{O} \). Then either \( \text{Fix}(G) = \Delta \), or \( \text{Fix}(G) = \{X, Y, Z\} \), or \( \text{Fix}(G) = \emptyset \), again by [22, Table 3.5]. If \( \text{Fix}(G) = \Delta \), then \( \alpha \) is a homology. A contradiction, since \( n = v + 1 \) is even. Assume that one of the remaining cases occurs. Note that \( G_O / H \leq S_3 \), where \( H \) is the kernel of \( G_O \) on \( \Delta \). If \( H \) has even order, then \( H \) contains Baer collineations, since \( \Delta \cup \{O\} \) is a quadrangle. Thus \( n = v + 1 \), \( v = q^d - 1 \), is a square. A contradiction by [52, Result A5.1], since \( v \) is odd and \( v > 3 \). Hence, we may assume that \( H \) has odd order. Thus \( G_O \) is solvable, since \( G_O / H \leq S_3 \) and \( H \) is solvable. Furthermore, \( G_O = H_1 \langle \alpha \rangle \) with \( H \leq H_1 \leq G_O \) and \( |H_1 : H| \leq 3 \). Actually \( H_1 = H \), since \( \alpha \in Z(G_O) \). Thus the case \( \text{Fix}(G) = \emptyset \) is ruled out. Hence \( \text{Fix}(G) = \{X, Y, Z\} \). In particular, \( G_O \leq \Gamma L(1, v) \) by [32]. Furthermore \( v \equiv 3 \mod 4 \), since \( 2 \mid |G_O| \). If \( n = 2^t \), with \( t \geq 2 \), then \( v \) is a Mersenne prime by [52, Result B1.1], since \( n = v + 1 \) and \( v \) is odd, \( v > 3 \). Thus the assertion (1). \( \square \)

**Example 25.** In \( PG(2, 8) \) there exists a 7-arc on which the group \( G \cong AGL(1, 7) \) acts \( 2 \)-transitively. In particular, this arc is contained in a conic.

Let \( \mathcal{C} \) be a conic of \( \Pi \cong PG(2, 8) \). Clearly \( \mathcal{C} \) is left invariant by \( PGL(2, 8) \). Pick any \( Z_7 \) inside \( PGL(2, 8) \). Then \( Z_7 \) fixes two points \( P_1 \) and \( P_2 \) on \( \mathcal{C} \) and \( Z_7 \) acts regularly on \( C = \{P_1, P_2\} \). Fur-
Moreover $N_{PGL(2,8)}(Z_7) \cong D_{14,\langle \sigma \rangle}$, where $\sigma$ is the collineation of $\Pi$ “induced” by a Frobenius automorphism of $GF(8)$. Set $O = C - \{P_1, P_2\}$ and $G = N_{PGL(2,8)}(Z_7)$. Then $G \cong AGL(1,7)$ acts 2-transitively on $O$.

**Proposition 26.** Let $\Pi$ be a finite projective plane of order $n$ and let $O$ be a 2-transitive $G$-arc of length $p^2$, with $n > p^2 > n/2$ and $p$ odd. If $G_O$ contains an involutory dilatation, then $T$ does not fix any point of $\Pi$, $p \neq 2 \mod 3$ and $p \neq 7$ and 19.

**Proof.** Assume there exists a point $P$ of $\Pi - O$ fixed by $T$. If there is a secant to $O$ through $P$, then there are exactly $p^2/2$ secants to $O$ through $P$, since $T$ is regular on $O$. A contradiction, since $p$ is odd. Hence $[P]$ consists of $p^2$ tangents and $n + 1 - p^2$ external lines to $O$. Denote by $E$ the set of external lines to $O$ through $P$. Assume there exists a $T$-orbit of length $p^2$ on $E$. Then either $n = 2p^2 - 1$ or $n = 2p^2$, since $n < p^2$. Actually $n = 2p^2$ is ruled out by [31, Theorem 13.18]. Hence $|E| = p^2$ and $n = 2p^2 - 1$. Thus $T$ is semisimple on $[P]$. If $P\alpha \neq P$, then $T$ fixes the line $P\alpha$, since $T$ fixes $P$ and $\alpha$ normalizes $T$. A contradiction, since $T$ is semisimple on $[P]$. Hence $\alpha$ fixes $P$. Clearly $\alpha$ leaves each $T$-orbit on $[P]$ invariant. Furthermore, $\alpha$ fixes exactly one line on each $T$-orbit on $[P]$ by [50, Proposition 4.2], since $T$ is semisimple on $[P]$ and $n + 1 = 2p^2$. Hence $\alpha$ fixes exactly 2 lines of $[P]$. A contradiction, since $\alpha$ is a Baer collineation of $\Pi$ by Lemma 24 being $v \equiv 1 \mod 4$. Hence we may assume that each $T$-orbit on $E$ has length less than $p^2$. Note that each nontrivial element in $T$ fixes the same number $k$ of lines on $E$ by [50, Proposition 4.2], since $G$ is a 2-transitive group fixing $P$. Then $|T| |((T| - 1)k + n + 1 - p^2$ by [31, Result 1.4]. Hence $p^2 | n + 1 - k$. Then $n = p^2 + k - 1$, since $n < 2p^2$. Therefore $T$ fixes all the external lines to $O$ through $P$. Note that $k \geq 2p + 2$, since $n \geq (p + 1)^2$. Let $c \equiv n \mod p$ such that $0 \leq c < p - 1$. Assume that $c \geq 2$. Any nontrivial element $\tau$ in $T$ fixes at least 3 points on $l$ for each line $l \in [P] \cap Fix(T)$. Then $\tau$ fixes a subplane of order at least $2p + 1$, since $|[P] \cap Fix(T)| = k$ and $k \geq 2p + 2$. A contradiction by [31, Theorem 3.7], since $n < 2p^2$. Hence, we may assume that $c < 1$. If $c = 1$ then $n + 1 \equiv 2 \mod p$, since $p$ is odd. Then there are at least 2 lines $l_1, l_2 \in [P]$ fixed by $T$. If $T$ is semisimple on $l_1 - \{P\}$ or $l_2 - \{P\}$, then $n = 2p^2 - 1$ or $n = 2p^2$. A contradiction in any case by similar arguments to that used above, since $\alpha$ fixes exactly 3 points on $l_1$ or $l_2$. As a consequence, we may assume that $T$ is not semisimple on $l_1 - \{P\}$, with $i = 1$ or 2. Then there exists a nontrivial element $\phi$ in $T$ fixing at least 2 points on $l_1 - \{P\}$ and on $l_2 - \{P\}$, since $c = 1$. Thus $\phi$ fixes a subplane of $\Pi$ of order at least $2p + 1$, since $k \geq 2p + 2$. Again a contradiction. Hence $c = 0$. Then $p | n$. Actually $p^2 | n$, since $n$ is a square. Then $n = 2p^2$, since $p^2 < n \leq 2p^2$. A contradiction by [31, Theorem 13.18]. Hence $T$ does not fix any point on $\Pi$. In particular, $p \neq 2 \mod 3$ by [49, Theorem 94]. It remains to prove that the cases $p = 7$ and $p = 19$ cannot occur. Since $n$ must be a square and $p^2 < n \leq 2p^2$, we have that $(p, n) = (7, 81)$ by filtering the list $p^2 < n \leq 2p^2$, with $p = 7$ or 19, with respect to the conditions $p | n^2 + n + 1$, $n$ a square and to Theorem 13.18 of [31]. Assume that $T$ fixes a line. Then $T$ fixes at least two lines, since $p | n^2 + n + 1$. Thus $T$ fixes the intersection points of these lines. A contradiction. Hence, we may assume that $T$ does not fix any line of $\Pi$. Clearly $T$ does not leave any triangle invariant, since $p = 7$. Hence $T$ is irreducible on $\Pi$. Nevertheless $T$ is not semisimple on $\Pi$, since $n = 81$ while $|T| = 7^2$. Then $T$ contains a planar element $\delta$ by [22, Theorem 3.13]. Let $\Pi_0 = Fix(\delta)$ and let $m = o(\Pi_0)$. Let $J$ be the kernel of $T \langle \alpha \rangle$ on $\Pi_0$. Assume that $J = T \langle \alpha \rangle$. Then $Fix(\delta) = \Pi_0$ is a Baer subplane of $\Pi$. Let $Y \in O$. Then there exists a line $r \in \Pi_0$ such that $Y \in r$ by the Baer property. Then $\delta$ fixes $r \cap O$ pointwise, since $Fix(\delta) = \Pi_0$, $|r \cap O| \leq 2$ and $o(\delta) = 7$. A contradiction. Therefore $J < T \langle \alpha \rangle$ and $m < 9$. Clearly either $J = T$ or $J = \langle \delta \rangle$. Assume that $J = T$. Thus $T$ must be semisimple on $s \in (s \cap \Pi_0)$ where $s$ is a secant.
of $\Pi_0$. Hence $7^2 | 81 - m$. A contradiction, since $m < 9$. As a consequence $J = (\delta)$. Then $T_{(\delta)}(\alpha)$ acts not trivially on $\Pi_0$. Since $\alpha$ acts as an inversion on $T_{(\delta)}$, then $\alpha$ fixes exactly one point on each invariant $T_{(\delta)}$-orbit on $\Pi_0$. Note that $T_{(\delta)}$ does not fix any point on $\Pi_0$, since $T$ does not fix any point on $\Pi$. Hence $7 \mid m^2 + m + 1$. Then either $m = 2$ or 4 by [31, Theorem 3.7], since $m < 9$. Then $T_{(\delta)}$ has exactly 1 or 3 orbits on $\Pi_0$, respectively. Then $\alpha$ fixes 1 or 3 points on $\Pi_0$ by [50, Proposition 4.2], respectively, since $\alpha$ acts as the inversion on $T_{(\delta)}$. A contradiction. \qed

**Theorem 27.** Let $\Pi$ be a projective plane of order $n$ and let $\mathcal{O}$ be a 2-transitive G-arc of odd length $v$, with $n > v \geq n/2$. If $G$ is of affine type, then $G \leq A\Gamma L(1, v)$.

**Proof.** Recall that $|\mathcal{O}| = p^d$, $d \geq 1$. By [20] a structure of $d^*$-dimensional vector space $V$ over a field $L \cong GF(p^d)$, $h \mid d$, $d = hd^*$, may be defined on $\mathcal{O}$ in such a way that $G \leq A\Gamma L(V)$ and $\mathcal{O}$ is identified with the zero-vector of $V$. By [20], by the classification of the finite 2-transitive groups of affine type and by proposition 26, we have that:

1. $G_{O} \leq A\Gamma L(1, p^d)$, $d^* = 1$;
2. $SL(d^*, p^h) \leq G_{O}$, $d^* \geq 2$;
3. $Sp(2m, p^h) \leq G_{O}$, $d^* = 2m$, and $m \geq 2$;
4. $D_8 \circ Q_8 \leq G_{O}$, $d = d^* = 4$, $p = 3$, and $G_{O}/(D_8 \circ Q_8) \leq S_5$;
5. $SL(2, 5) \leq G_{O}$, $d = d^* = 4$ and $p = 3$;
6. $SL(2, 13) \leq G_{O}$, $d = d^* = 6$, and $p = 3$.

**Cases (2) and (3).** Let $\tau \in G_{O}$, $\tau \neq 1$, inducing a transvection on $V$. Then $Fix(\tau) \cap \mathcal{O}$ is a subspace of both $V$ and $\mathcal{O}$ of order $p^{h(d^*-1)}$. Hence $\tau$ is planar with $o(Fix(\tau)) \geq p^{d-h} - 2$. Then $(p^{d-h} - 2)^2 \leq n \leq 2p^d$ by [31, Theorem 3.7]. This yields $2(d - h) \leq d$. Hence $d \leq 2h$. Then $d^* = 2$, since $d = hd^*$ and $d^* \geq 2$. Thus $SL(2, p^{d/2}) \leq G_{O}$. Therefore $PSL(2, p^{d/2}) \leq \bar{G}_{O}$, where $\bar{G}_{O}$ is the group induced by $G_{O}$ on $Fix(\alpha)$ and $\alpha$ is the involutory $O$-dilatation of $G$. Let $\bar{H} \leq \bar{G}_{O}$ such that $\bar{H} \cong PSL(2, p^{d/2})$. Note that $\bar{H}$ acts not trivially on $Fix(\alpha) \cap \mathcal{O}$, since $\bar{H}$ contains $p$-elements whose inverse images in $G_{O}$ fix $p^{d/2}$ points on $\mathcal{O}$. Assume there exists $X \in Fix(\alpha)$ such that $\bar{H}_X = (1)$. Then $|\bar{H}| \leq n + \sqrt{n} + 1$, since $|\bar{H}| = |X^{\bar{H}}|$. That is $p^{d/2}(p^{d/2} - 1)/2 \leq n + \sqrt{n} + 1$. It is a straightforward calculation to show that $p^{d/2} = 5$, since $n \leq 2p^d$ and $p^{d/2} \geq 5$. Then $n = 36$ or 49, since $G$ contains Baer involutions and $25 < n \leq 50$. Nevertheless, these cases are ruled out by [31, Theorem 3.6] and [26, Theorem A], respectively. Hence $\bar{H}_Y \neq (1)$ for each $Y \in Fix(\alpha)$. That is $\bar{H}$ is totally irregular on $Fix(\alpha)$. Now, assume that $\bar{H}$ contains perspectivities of $Fix(\alpha)$. The $\bar{H} \cong PSL(2, 2^j)$, with $j \geq 2$, by [28, Theorems 1 and 2], since $\bar{H}$ is totally irregular on $Fix(\alpha)$ and $\bar{H}$ fixes the point $O$ in $Fix(\alpha)$. Then $j = 2$ and $\bar{H} \cong PSL(2, 5)$, since $v = p^d$ with $p$ odd. Again a contradiction. Thus each involution in $\bar{H}$ is a Baer collineation of $Fix(\alpha)$ and hence $\sqrt{n}$ must be a square. Since $\bar{H}$ acts not trivially on $Fix(\alpha) \cap \mathcal{O}$, then $|\bar{H}| > 1$ for some $l \in Fix(\alpha) \cap \mathcal{O}$. Clearly $|\bar{H}| = \mu d_j(\bar{H})$, where $\mu$ is a positive integer and $d_j(\bar{H})$ denotes a primitive permutation representation degree of $\bar{H}$. Assume that $p^{d/2} \notin \{5, 7, 9, 11\}$. Then $\mu = 1$, $j = 0$, and hence $|\bar{H}| = p^{d/2} + 1$ by Lemma 7, since $n \leq 2p^d$. Set $\mathcal{W} = Fix(\alpha) \cap [\mathcal{O}] - \bar{H}$. Again by $n \leq 2p^d$, we have that $\bar{H}$ fixes $\mathcal{W}$ elementwise. Then $\sqrt{n} + 1 = p^{d/2} + 1 + k$, where $k = |\mathcal{W}|$. Hence $\sqrt{n} = p^{d/2} + k$. Actually $k > 0$, since $n > p^d$ by our assumption. Assume that $k < 2$. If $p^{d/2} \equiv 3 \bmod 4$, then each involution in $\bar{H}$ fixes exactly $k$ lines of $Fix(\alpha) \cap [\mathcal{O}]$, since $Fix(\alpha) \cap [\mathcal{O}] = l\bar{H} \cup \mathcal{W}$, the group $\bar{H}$ fixes $\mathcal{W}$ elementwise.
and acts on $l^H$ in its 2-transitive permutation representation of degree $p^{d/2} + 1$. A contradiction, since each involution in $H$ must be a Baer collineation of $\text{Fix}(\alpha)$ while $k \leq 2$. As consequence, $p^{d/2} \equiv 1 \mod 4$. Then $\sqrt{n} \equiv 2 \mod 4$ for $\sqrt{n} = p^{d/2} + 1$, and $\sqrt{n} \equiv 3 \mod 4$ for $\sqrt{n} = p^{d/2} + 2$. A contradiction in any case, since $\sqrt{n}$ must be a square. Thus $k \geq 3$. Arguing as above, it is easily seen that the action of $H$ on any line of $W$ and the action of $H$ on $\text{Fix}(\alpha) \cap |O|$ are the same, since $n \leq 2p^d$. Thus $H$ fixes a subplane $\text{Fix}(\alpha)$ of order $k - 1$, since $k \geq 3$. Furthermore $H_l$ fixes a subplane of order $k$ containing $\text{Fix}(\bar{H})$, since $H_l$ acts 2-transitively on $l^H$. A contradiction by [31, Theorem 3.7], since $\text{Fix}(\bar{H}) \subsetneq \text{Fix}(\bar{H})$. Hence $p^{d/2} \in \{5, 7, 9, 11\}$. Actually, the case $p^{d/2} = 5$ is ruled out by the above arguments. Since $\sqrt{n}$ must be an integer and $p^d < n \leq 2p^d$, the unique admissible case is $p^{d/2} = 7$ and $n = 81$. Nevertheless this case cannot occur by [26, Theorem A], applied to the group $H_l$ acting on $\text{Fix}(\alpha)$.

Cases (4) and (5). As above, let $\tilde{G}_O$ denote the group induced by $G_O$ on $\text{Fix}(\alpha)$. Note that $\sqrt{n} \in \{10, 11, 12\}$, since $p^d < n \leq 2p^d$ and $p^d = 81$. Nevertheless the cases $\sqrt{n} = 10$, $\sqrt{n} = 11$ and $\sqrt{n} = 12$ are ruled out by [31, Theorem 13.18], by [26, Theorem A], and by [34], respectively.

Case (6). Clearly $27 < \sqrt{n} \leq 38$, since $p^d < n \leq 2p^d$ with $p^d = 3^6$. Clearly $G_O$ acts on $\text{Fix}(\alpha)$, where $\alpha$ is the involutory $O$-dilatation. Let $\tilde{G}_O$ be the group induced by $G_O$ on $\text{Fix}(\alpha)$. Then $\text{PSL}(2, 13) \leq \tilde{G}_O$. Furthermore $\tilde{G}_O$ acts on $\text{Fix}(\alpha) \cap |O|$. Hence $\sqrt{n} + 1 = 14\lambda + \delta$ by [15, Appendix B], where $\delta$ denotes the number of lines fixed by $\tilde{G}_O$ on $\text{Fix}(\alpha) \cap |O|$, $\lambda \in \{0, 1, 2\}$ and $d_0(\tilde{G}_O) = 14$. If $\delta \geq 3$, then any involution in $\tilde{G}_O$ is a Baer collineation of $\text{Fix}(\alpha)$ and hence $\sqrt{n} = 6$, since $27 < \sqrt{n} \leq 38$. A contradiction by [31, Theorem 3.6]. Then $\delta \leq 2$. Hence $\lambda = 2$, since $27 < \sqrt{n} \leq 38$. It is easily seen that each involution in $\tilde{G}_O$ is a Baer collineation fixing at least 4 lines on $\text{Fix}(\alpha) \cap |O|$, since $d_0(\tilde{G}_O) = 14$. A contradiction. Hence $d_0^* = 1$ and we have the assertion. □

The Theorem 20 easily follows by Theorems 22 and 27. Now, the proof of Theorem 1 follows by combining the results of [5] and [43] for $v > n$ with the results of Theorems 5 and 20 for $n > v \geq n/2$.

Example 28. There exists a complete 7-arc in $PG(2, 9)$ on which the group $G \cong AGL(1, 7)$ acts 2-transitively.

Example 29. There exists a complete 13-arc in $PG(2, 16)$ on which the group $G \cong AGL(1, 13)$ acts 2-transitively.

Let $\Pi = PG(2, q^2)$ where $q = p^r$, $r \geq 1$ and $q > 2$. Set $G_1 = PGL(3, q^2)$ and $G_2 = P\Gamma L(3, q^2)$. Let $S$ be a Singer subgroup of $G_1$. Then $S$ is a cyclic group of order $q^4 + q^2 + 1$ acting regularly on the points of $\Pi$. In particular, there exists a unique conjugate class of $S$ in $G_1$ and hence in $G_2$ (see [24, Corollary 4.7]). Then $[G_1 : N_{G_1}(S)] = [G_2 : N_{G_2}(S)]$. Thus $|N_{G_2}(S)| = 2r|N_{G_1}(S)|$, since $[G_2 : G_1] = 2r$. Then $|N_{G_2}(S)| = 6r|S|$, since $|N_{G_1}(S)| = 3|S|$ by [33, Satz II.7.3]. Since $S$ is regular on the points of $\Pi$, then $N_G(S) = S.H_P$ for some point $P$ of $\Pi$. Let $K$ be the subgroup of order $q^2 - q + 1$ of $S$ and let $G = K.H_P$. By [24, Theorem 4.41], each $K$-orbit in $PG(2, q^2)$ is a complete arc of length $q^2 - q + 1$. Let $O$ be the arc containing $P$. Clearly $O$ is $G$-invariant. Furthermore $G$ is faithful on $O$. If $G$ is 2-transitive on $O$, then $|O - \{P\}| | |H_P|$, and hence $p^{2r} - p^r \leq 6r$. Then $p^{2r} - p^r \leq 6r$ and hence $p^{2r} - p^r \leq 6p^r$ as $r \leq p^r$. Thus $p^r \leq 7$. Hence either $r = 2$ and $p = 2$, or $r = 1$ and $p \in \{3, 5, 7\}$, since $q > 2$. Actually the cases $q = 5$ and $q = 7$ cannot occur, since $q^2 - q$ does not divide 6. Hence, the
admissible cases are $q = 3$ or 4. Then $q^2 - q + 1 = 7$ and 13, respectively. Hence $O$ has length a prime number in any of these cases. Then $K$ has prime order, since $K$ is regular on $O$. Hence each nontrivial element in $K$ is a generator of $K$. Pick $\alpha \in H_P$ and suppose that $\alpha$ centralizes an element $\gamma \in K$, $\gamma \neq 1$. Then $\alpha$ fixes $O$ pointwise by [50, Proposition 4.2], since $\langle \gamma \rangle$ is regular on $O$ being $K = \langle \gamma \rangle$. So $\alpha = 1$, since $G$ is faithful on $O$. Thus $G$ is an automorphism group with kernel $K$ and complement $H_P$ in its natural action on $O$, by [50, Proposition 17.2]. Thus $G$ is 2-transitive on $O$. In particular, $G \cong AGL(1, v)$ with $v = q^2 - q + 1$. Hence the Examples 28 and 29. Note that these examples were omitted in [54, Proposition 16]. Nevertheless, the case $q = 4$ is already contained in [6], while the case $q = 3$ is new.

References


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