Abstract—Optimal resource allocation is an outstanding issue in wireless communication systems. In this work we focus on the ever challenging optimization of the tradeoff between throughput and fairness. Specifically, we propose a novel framework to analytically derive the maximum average throughput versus fairness under the assumptions that the throughput of each user i) increases if more resources are allocated to him and ii) depends on how many resources and not which resources are allocated. We achieve a general formulation of the tradeoff optimization problem and we also derive and validate a closed form solution in those scenarios where throughput linearly depends on resources, which cover several realistic cases. Besides these valuable results, the framework also lays solid basis toward a more general solution.

I. INTRODUCTION

Optimal resource allocation is a critical issue in wireless communication systems. The fact that it has a primary role in research can be appreciated counting the number of IEEE publications that in the last four years (from 2008 to 2011) include the words scheduling or resource allocation in their title: more than 1000 journal papers and almost 8000 conference papers, still not including those works edited by other publishers or handling this topic with other key terms in the title. Optimizing the resource allocation given a maximum available shared power (e.g. [1]), relating energy consumption to the achieved performance (e.g. [2]), optimally allocating resources in multiple radio access technologies (e.g. [3]), distributing resources to achieve fairness when radio conditions are unfair (e.g. [4]), allocating the resources constrained by a minimum required quality of service (e.g. [5]), are only some of the issues that are far from being fully explored.

One significant index to quantify the performance of a wireless network can be represented by the average throughput perceived by users (or, equivalently the sum throughput). However, clearly this metric alone is not adequate to describe the performance of a multi-user scenario: it is, in fact, generally unacceptable that resources are used by few users and the others are not served, even if this maximizes the throughput. To this aim, the concept of fairness was introduced, which is maximum when all users perceive the same throughput, while it is minimum when a single user gets all resources. A well-known index to quantify the fairness of an allocation scheme is represented by the Jain’s index [6]. It is applicable to any resource sharing or allocation problem, it is independent of the amount of resources and it always lies between 0 and 1. In general, the resource allocation scheme which maximizes the average throughput is not the one which maximizes the fairness. In most cases, throughput and fairness have opposite trends: the higher the average throughput is achieved, the lower the fairness becomes (and vice-versa).

An important question then arises: can a theoretical limit for the throughput-fairness tradeoff be derived? In other words, is it possible to mathematically evaluate the maximal average throughput attainable for a given fairness level? This problem can be graphically represented: if any possible resource allocation is represented by a point in a plot, where the x-axis denotes the fairness $F$ and the y-axis the average throughput $T$, and if all possible combinations are considered thus showing the working region over that plot, then answering the above question means finding a close-form expression for the superior limit of the obtained region. Despite it is reasonable to expect a not increasing curb, at the best of our knowledge no analytical work derived its behavior.

These considerations are supported by an accurate analysis of the wide literature on this topic. Several algorithms have been studied to increase $F$ taking into account $\bar{T}$ constraints. In [7] a simple scheduling algorithm to tradeoff $\bar{T}$ and $F$ is proposed and compared with maximal signal-to-noise ratio algorithm [8] and proportional fairness [9] algorithm, but without discussing optimality. In [10], a scheduling algorithm with limited complexity and Pareto optimality is proposed for joint subcarrier and power allocation in the uplink of orthogonal frequency division multiple access (OFDMA) systems, where different utility functions can be adopted to achieve specific throughput to fairness tradeoffs. In [11] a scheduling algorithm is defined for cellular systems with time division access, where the maximization of an utility function is performed and a tunable parameter $\alpha$ allows to qualitatively control the $\bar{T}$ versus $F$ tradeoff. Also in [12] the optimization problem is described using the same $\alpha$-fair utility function and a novel scheduling algorithm is proposed for long term evolution (LTE). In [13] the authors formulate the problem of throughput maximization following a fairness constraint and a minimum quality of service in interference limited networks, and propose an iterative algorithm that brings to a Pareto optimal solution. Solutions with optimality properties are also found in [14], where the authors propose a framework to tradeoff fairness and efficiency in scenarios with multiple resource types. Unfortunately, none of the results in the related works can be exploited to find the maximum $\bar{T}$ for a given $F$, exactly quantified by an index.

Deriving this limit curb is important for at least the two following reasons. First, knowing the relationship between
$T$ and $F$ is required if an optimum solution is targeted following any criterion that does not force a fixed $F$. This would help algorithms design since it answers the following kind of questions: Reducing $F$ a little makes the maximum $T$ just a little higher or rapidly increasing? Second, evaluating the performance of any new algorithm needs the knowledge of how distant it is from the maximum, otherwise only comparisons with previous solutions or the verification of some properties, like the Pareto optimality, are possible.

In this work a multiuser scenario is addressed with the assumptions that i) throughput of an user increases if the amount of resources allocated to him increases, and that ii) the multiuser diversity cannot be exploited to improve the system performance. Both these assumptions are deepened in the following. Although not covering all possible scenarios, we obtain a solution to the above mentioned optimization problem of the tradeoff between $T$ and $F$ in closed form, which applies in several practical cases. Moreover, and most important, the described framework poses a solid basis for further investigations.

The rest of the paper is organized as follows: The assumptions and the problem formulation are described in Section II; The general analytical framework is proposed in Section III; In Section IV equations are specified for linear relationships between throughput and resource allocation; In Section V example results are shown and commented. Finally, in Section VI conclusions are drawn and future work is discussed.

II. System Model

A. Assumptions

The scenario of interest consists of a finite number of users $N$ and a finite amount of resources $S$ to be allocated. Denoting by $r_i$ the amount of resources allocated to the $i$th user, with $i \in [0, N - 1]$, we have

$$\sum_{i=0}^{N-1} r_i \leq S$$

where the equality stands when all resources are allocated. For simplicity, we denote with

$$\delta_i \triangleq \frac{r_i}{S}$$

the normalized ratio of resources assigned to user $i$, with $\delta_i \in [0, 1]$. We develop our framework under the following assumptions:

Assumption 1. Throughput $T_i$ perceived by the $i$th user is a function of how many allocated resources and not of which resources.

Assumption 2. Throughput $T_i$ strictly increases if more resources are allocated to the $i$th user (i.e., increasing $\delta_i$).

Hence we can write

$$T_i = g_i(\delta_i)$$

where $g_i(\cdot)$ is a strictly increasing (thus invertible) function. We assume $g_i(0) = 0$, thus $T_i = 0$ when no resources are allocated to user $i$. At the opposite, when all resources are allocated to him, the $i$th user perceives its maximum throughput $\hat{T}_i \triangleq g_i(1)$. Depending on the radio link conditions (e.g., the distance from a base station), $\hat{T}_i$ can be different from $\hat{T}_j$ for $i \neq j$. Also remark that channel conditions impact the performance through the functions $g_i(\cdot)$, which depend on $i$.

The given assumptions are valid in several realistic scenarios. For example, assumption 1 is valid anytime that channel state information cannot be used at the transmitter to exploit multiuser diversity, while assumption 2 is valid anytime that the medium access mechanism is based on orthogonal resources.

Proposition. Given the assumptions, it can be demonstrated that the average throughput never decreases as the total amount of allocated resources increases. Hence, throughput is maximized when all resources are allocated, that is when

$$\sum_{i=0}^{N-1} \delta_i = \sum_{i=0}^{N-1} \chi_i(T_i) = 1$$

where

$$\chi_i(T_i) \triangleq g_i^{-1}(T_i).$$

B. Problem formulation

Given the constrain in (4), the objective of this framework is to maximize the average throughput $\bar{T}$ defined as

$$\bar{T} \triangleq \frac{\sum_{i \in N} T_i}{N}$$

given a specified value of fairness $F$, where $F$ is described by the Jain’s index [6]

$$F = \frac{(\sum_{i \in N} T_i)^2}{N \sum_{i \in N} T_i^2}$$

and $N' \triangleq \{0, 1, ..., N - 1\}$.

Since $\bar{T}$ cannot be negative, the following inequalities must be taken into account

$$T_i \geq 0 \quad \forall i \in N'.$$

Our objective is the maximization of $\bar{T}$. For convenience we define

$$f(T_0, ..., T_{N-1}) \triangleq -\bar{T} = -\frac{\sum_{i \in N} T_i}{N}$$

and we minimize (9) taking into account the constraints given

1For instance, it applies in OFDMA systems like WiMAX when distributed subcarriers allocation is adopted; in such a case only the average signal quality is known and throughput is mainly related to the number of resource units allocated and not to their position. Assumption 1 is valid also for Wi-Fi systems: in fact, although link adaptation is used to adapt the data rate to the channel gain, no mechanism is implemented at the transmitter to have instantaneous pictures of the channel, thus $T_i$ is essentially proportional to the percentage of time the $i$th user can access the medium.
by (4), (7), and (8) as

\[
\text{Minimize } f(T_0, \ldots, T_{N-1}) = -\frac{\sum_{i \in N} T_i}{N} \quad (10)
\]

subject to,

\[
\frac{\sum_{i \in N} T_i^2}{N} - F = 0, \quad (11)
\]

\[
\sum_{i \in N} \chi_i(T_i) - 1 = 0, \quad (12)
\]

\[-T_i \leq 0 \quad \forall i \in N. \quad (13)
\]

where (10) is the objective function, (11) is the first equality constraint (given by the target value of \(F\)), (12) is the second equality constraint (given by the resource constraint) and (13) are the inequality constraints (to avoid negative throughput values). The necessary requirements for the solution in \(T_i\) of this maximization problem are provided by the Karush-Kuhn-Tucker (KKT) conditions\(^2\) that can be applied as follows

\[
\begin{align*}
\frac{2\lambda_2}{N} \left( \frac{\sum_{i \in N} T_i \sum_{i \in N} T_i^2}{N} - N \frac{\sum_{i \in N} T_i^2}{N} \right) - \frac{1}{N} - \mu_j + \lambda_1 \frac{d \chi(T_j)}{d T_j} &= 0, \quad \forall j \in N \\
T_j &\leq 0, \quad \forall j \in N \\
\sum_{i \in N} \chi_i(T_i) - 1 &= 0 \\
\frac{\sum_{i \in N} T_i^2}{N} - F &= 0 \\
\mu_j &\geq 0, \quad \forall j \in N \\
\mu_j T_j &= 0, \quad \forall j \in N
\end{align*}
\]

(14)

where the first line represents the \(N\) stationarity conditions, the second, third, and fourth ones the \(N + 2\) primal feasibility conditions, the fifth one the \(N\) dual feasibility conditions and the sixth one the complementary slackness conditions.

### III. Solution of the Maximization Problem

The solution to (14) can be obtained through the following five steps.

**Step 1.** Defining

\[
\chi_j(T_j) \triangleq \frac{d \chi_j(T_j)}{d T_j} \quad (15)
\]

and exploiting the fourth equation of (14), the first equation becomes

\[
\mu_j = \lambda_1 \chi_j(T_j) - \frac{1}{N} + 2\lambda_2 \frac{F}{N^2T} \left[ 1 - \frac{T_j}{T} \right], \quad \forall j \in N. \quad (17)
\]

Multiplying both terms in (17) by \(T_j\), performing the sum in \(j\) from 0 to \(N - 1\), and using (6) and the fourth equation in (14), we obtain

\[
\sum_{j \in N} \mu_j T_j = \lambda_1 \sum_{j \in N} T_j \chi_j(T_j) - \frac{1}{N} \sum_{j \in N} T_j \\
+ 2\lambda_2 \frac{F}{N^2T} \sum_{j \in N} T_j - 2\lambda_2 \frac{F^2}{N^2T^2} \sum_{j \in N} T_j^2 \\
= \lambda_1 \sum_{j \in N} T_j \chi_j(T_j) - T. \quad (18)
\]

From the slackness condition it follows that \(\sum_{j \in N} \mu_j T_j = 0\); hence

\[
\lambda_1 = \frac{T}{\sum_{j \in N} T_j \chi_j(T_j)}. \quad (19)
\]

**Step 2.** Performing the sum in \(j\) from 0 to \(N - 1\) of (17) (differently from step 1, we do not multiply by \(T_j\)) we obtain

\[
\sum_{j \in N} \mu_j = \lambda_1 \sum_{j \in N} \chi_j(T_j) - 1 + 2\lambda_2 \frac{F}{T} - 2\lambda_2 \frac{F^2}{N^2T^2} \sum_{j \in N} T_j \\
= \lambda_1 \sum_{j \in N} \chi_j(T_j) - 1 + 2\lambda_2 \frac{F}{T} (1 - F). \quad (20)
\]

Therefore by substituting (19) in (20) we have

\[
\lambda_2 = \frac{T}{2F(1 - F)} \left[ 1 + \sum_{j \in N} \mu_j - T \sum_{j \in N} \chi_j(T_j) \right], \quad (21)
\]

Thus, defining

\[
\alpha(T) \triangleq \sum_{i \in N} \chi_i(T_i) \quad (22)
\]

\[
\beta(T) \triangleq \sum_{i \in N} \chi_i(T_i) \quad (23)
\]

\[
\gamma(T) \triangleq \sum_{i \in N} T_i \chi_i(T_i) \quad (24)
\]

and substituting (19) and (21) in (17), (14) becomes

\[
\begin{align*}
\mu_j &= \frac{T \chi_j(T_j)}{\gamma(T)} - \frac{1}{N} + \frac{1}{N^2(1 - F)} \left[ 1 + \sum_{i=0}^{N-1} \mu_i - T \frac{\chi_j(T_j)}{\gamma(T)} \right], \quad \forall j \in N \\
T_j &\geq 0, \quad \forall j \in N \\
\alpha(T) &= 1, \\
\mu_j &\geq 0, \quad \forall j \in N \\
\mu_j &= 0 \lor T_j = 0, \quad \forall j \in N
\end{align*}
\]

(25)

where the fourth equation of (14) is not reported since it was exploited to achieve (19) and (21).

**Step 3.** From the slackness condition it follows that for any \(j \in N\) either \(\mu_j\) or \(T_j\) has to be imposed equal to zero. By defining the partitions

\[
T \triangleq \{ i \in [0, \ldots, N - 1] : T_i = 0 \} \quad (26)
\]
\[ \mathcal{M} \triangleq \{ i \in [0, \ldots, N-1] : \mu_i = 0 \} \]  
(27)

with

\[ \mathcal{T} \cap \mathcal{M} = \emptyset \quad \text{and} \quad \mathcal{T} \cup \mathcal{M} = [0, \ldots, N-1] \]  
(28)

we obtain \(2^N\) systems of equations, one for each possible value of \(\mathcal{T}\) and \(\mathcal{M}\), and each one given by

\[
\frac{\mathcal{T}_{\chi_j}(T_j)}{\gamma(T_j)} - \frac{1}{N} + \frac{1}{N(1-F)} \left[ 1 - F \frac{T_j}{\gamma(T_j)} \right] \times \left[ 1 + \sum_{i \in \mathcal{T}} \mu_i - \frac{T_j \beta(T_j)}{\gamma(T_j)} \right] = 0, \quad \forall j \in \mathcal{M}
\]

\[
\mu_j = \frac{\mathcal{T}_{\chi_j}(0)}{\gamma(T_j)} - \frac{1}{N} + \frac{1}{N(1-F)} \times \left[ 1 + \sum_{i \in \mathcal{T}} \mu_i - \frac{T_j \beta(T_j)}{\gamma(T_j)} \right], \quad \forall j \in \mathcal{T}
\]

\[ T_j \geq 0, \quad \forall j \in \mathcal{T} \]

\[ \alpha(T) = 1, \]

\[ \mu_j \geq 0, \quad \forall j \in \mathcal{T}. \]

In (29), the first and second equations derive from the first equation of (25) simply distinguishing \(j \in \mathcal{T}\) from \(j \in \mathcal{M}\).

**Step 4.** Defining now

\[
\tau(T) \triangleq \sum_{i \in \mathcal{T}} \chi_i(T_i) = \sum_{i \in \mathcal{T}} \chi_i(0)
\]

(30)

\[
\tau_{\mathcal{M}}(T) \triangleq \sum_{i \in \mathcal{T}} \chi_i(T_i)
\]

(31)

\[ N_T \triangleq \#\mathcal{T} \]

(32)

\[ N_{\mathcal{M}} \triangleq \#\mathcal{M} = N - N_T \]

(33)

where \(\#\mathcal{X}\) stands for cardinality of \(\mathcal{X}\), and summing for every \(j \in \mathcal{T}\) the second equation of (29) we obtain

\[
\sum_{j \in \mathcal{T}} \mu_j = \frac{\mathcal{T}_{\tau}(T_j)}{\gamma(T_j)} - \frac{N_T}{N} + \frac{N_T}{N(1-F)} \left[ 1 - \frac{T_j \beta(T_j)}{\gamma(T_j)} \right] \times \left[ 1 - \frac{N_T}{N(1-F)} \right].
\]

(34)

Then, defining

\[
\zeta_{\tau}(T) = 1 + \frac{\mathcal{T}_{\tau}(T)}{\gamma(T)} - \frac{N_T}{N} + \frac{N_T}{N(1-F)} \left[ 1 - \frac{T_j \beta(T_j)}{\gamma(T)} \right] - \frac{T_j \beta(T_j)}{\gamma(T)} = \frac{N_{\mathcal{M}} - \Gamma_{\mathcal{M}}(T)}{N_{\mathcal{M}} - NF} \quad \forall j \in \mathcal{T}
\]

(35)

and substituting (34) in (29) we have

\[
\begin{cases}
\frac{\mathcal{T}_{\chi_j}(T_j)}{\gamma(T_j)} - \frac{1}{N} + \frac{\zeta_{\tau}(T_j)}{N(1-F)} \left[ 1 - F \frac{T_j}{\gamma(T_j)} \right] = 0, \quad \forall j \in \mathcal{M} \\
\mu_j = \frac{\mathcal{T}_{\chi_j}(0)}{\gamma(T_j)} - \frac{1}{N} + \frac{\zeta_{\tau}(0)}{N(1-F)}, \quad \forall j \in \mathcal{T} \\
T_j \geq 0, \quad \forall j \in \mathcal{T} \\
\alpha(T) = 1, \\
\mu_j \geq 0, \quad \forall j \in \mathcal{T}.
\end{cases}
\]

(36)

Note that in (36) the first equation does not depend on any \(\mu_i\), with \(i \in \mathcal{N}\).

**Step 5.** Rearranging the first equation of (36) we have

\[ T_j = \frac{\mathcal{T}_j}{\gamma(T_j)} \left[ \frac{N(1-F)}{\zeta_{\tau}(T_j)} - \frac{1}{\zeta_{\tau}(T_j)} \right] \cdot \frac{1}{\zeta_{\tau}(T_j)} + 1, \quad \forall j \in \mathcal{M}. \]

(37)

Multiplying (37) by \(\chi_j(T_j)\), summing for \(j \in \mathcal{M}\), reminding (24) and (31), and defining

\[ \Gamma_{\mathcal{M}}(T) \triangleq \sum_{j \in \mathcal{M}} (\chi_j(T_j))^2 \]

(38)

we finally obtain

\[ \frac{N(1-F)}{\zeta_{\tau}(T)} \Gamma_{\mathcal{M}}(T) \frac{T_j}{\gamma(T)} \quad \left( 1 - \frac{1 - F}{\zeta_{\tau}(T)} \right) \cdot \frac{\gamma(T)}{\zeta_{\tau}(T)} = T_j \]

(39)

If \(\zeta_{\tau}(T), \Gamma(T), \gamma(T), \) and \(\tau_{\mathcal{M}}(T)\) do only depend on \(\mathcal{T}\) (and not on each single \(T_i\), with \(i \in \mathcal{N}\)), (39) represents a compact solution to our maximization problem. Otherwise, the solution must be obtained solving (36).

### IV. Closed Form Solution for the Case of Throughput Linearly Dependent on Resources

In this section, we derive a closed form solution for the maximum achievable throughput as a function of \(F\) in case of linear dependence of \(T_i\) on the available resources.\(^3\) In this case, (3) can be written as

\[ T_i = \delta_i \tilde{T}_i. \]

(40)

As a consequence, it can be derived that \(\chi_i(T_i) = \delta_i = T_i/\tilde{T}_i\), \(\alpha(T) = \gamma(T) = 1\), and \(\beta(T) = \sum_{i \in \mathcal{N}} 1/\tilde{T}_i\). In addition

\[ \tau_{\mathcal{T}}(T) = \frac{\sum_{i \in \mathcal{T}} 1}{\tilde{T}_i} = \tau_{\mathcal{T}} \]

(41)

\[ \tau_{\mathcal{M}}(T) = \frac{\sum_{i \in \mathcal{M}} 1}{\tilde{T}_i} = \tau_{\mathcal{M}} \]

(42)

\[ \Gamma_{\mathcal{M}}(T) = \sum_{j \in \mathcal{T}} \frac{1}{T_j^2} = \Gamma_{\mathcal{M}}. \]

(43)

All the above functions are thus independent from any \(T_i\), \(i \in \mathcal{N}\). Moreover, (35) becomes

\[ \zeta_{\tau}(T) = \frac{N_{\mathcal{M}} - NF}{N_{\mathcal{M}} - NF} \cdot \frac{\tau_{\mathcal{M}}(T)}{\tau_{\mathcal{M}}(T)} (1 - F) \]

(44)

thus only depending on \(T_i\), \(i \in \mathcal{N}\), through their average \(\tilde{T}\).

Hence, in this case the solution can be found starting from (39), as discussed in Step 5 of Section III. Exploiting the above relationships and after some algebra, (39) can be written as

\[ \frac{T_j^2}{F} \left[ \frac{\Gamma_{\mathcal{M}} - \frac{N_{\mathcal{M}}}{N_{\mathcal{M}} - NF}}{\frac{N_{\mathcal{M}}}{N_{\mathcal{M}} - NF} - F} + \frac{2\tau_{\mathcal{M}}}{N_{\mathcal{M}} - NF} \right] = 0 \]

(45)

\(^3\)This particular case is of main interest since it applies in several real scenarios, for instance whenever a time division or a frequency division MAC is assumed (e.g., time division multiple access, frequency division multiple access, OFDMA).
TABLE I
CASE STUDIES.

<table>
<thead>
<tr>
<th>Case</th>
<th>N</th>
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</table>

that results solved in closed form by

$$
\bar{T} = \frac{F}{\bar{\tau}_M} \left\{ 1 + \sqrt{1 - \frac{1}{F}\frac{N_M}{N} \left[ 1 + (NF - N_M)\frac{\bar{\tau}_M}{\bar{\tau}_M} \right]} \right\} + \frac{1}{1 + (NF - N_M)\frac{\bar{\tau}_M}{\bar{\tau}_M}}.
$$

(46)

For each one of the \(2^N\) possible combinations given by the slackness condition, (46) provides a possible solution of \(\bar{T}\). The validity of each candidate solution has to be verified through the inequalities in (36). In particular, in this case of linear dependence between \(\bar{T}\) and resources, the last equation of (36) becomes a tautology and the others can be disjointed as

$$
\begin{align*}
T_j &= \frac{N\bar{T}(N_M-NF)}{F(N_M-N\bar{T}\bar{\tau}_M)} \left[ \frac{\bar{T}}{T_j} - \frac{1}{N} \right] + \frac{\bar{T}}{F} \\
T_j &\geq 0 \quad \forall j \in M \\
\mu_j &= \frac{\bar{T}}{T_j} - \frac{1}{N} + \frac{N_M-N\bar{T}\bar{\tau}_M}{N(N_M-NF)} \\
\mu_j &\geq 0 \quad \forall j \in T.
\end{align*}
$$

(47)

(48)

Once obtained all the \(2^N\) possible solutions through (46), we have to compare those respecting the inequalities in (47) and (48) to find out the one giving the maximum value of \(\bar{T}\). Thus, the maximization problem results solved finding out for every \(F\) the correct partition of \(N\) (i.e., \(M\) and \(T\)) and then applying (46). Ordering the users without loss of generality such that \(\hat{T}_i \leq \hat{T}_j\) if \(i > j\) (with \(i, j \in N\)), and defining

$$
\begin{align*}
F_w &= \frac{(N_{M,w} - \sum_{i=0}^{w} T_i)^2}{N(N_{M,w} + \sum_{j=0}^{w} T_j)^2} \quad 0 \leq w \leq N - 1 \\
F_N &= 1
\end{align*}
$$

(49)

it can be proved that the maximum value of \(\bar{T}\) for a given value of \(F\) can be achieved through (46), for \(F \in [F_w, F_{w+1}]\), \(w = [0, \ldots, N - 1]\), with

$$
\begin{align*}
M &= M_w = \{ j \in N : j \leq w \} \\
T &= [0, \ldots, N - 1] - M_w.
\end{align*}
$$

(50)

V. EXAMPLE RESULTS

In this section, results referring to the case of linear dependence between \(\bar{T}\) and \(F\) illustrated in Section IV are plotted for different values of \(N\) and \(\hat{T}_i\), with \(i \in [0, N - 1]\). The considered case studies are listed in Table I. Since the unit of measurement is not significant, a normalized value is given for the throughput.

In Figs. 1 and 2 the maximum \(\bar{T}\) given by (46) conditioned to (49) and (50) is plotted as a function of \(F\) for cases A and E, respectively, and compared with all possible allocations of the full resource. In particular, light blue points refer to \(\bar{T}\) and \(F\) following any distribution of resources giving \(\sum_{i=0}^{N-1} \delta_i = 1\), whereas the red curve follows the proposed analysis. The perfect superimposing of the evaluated curve to the maximum values of \(\bar{T}\) varying \(F\) clearly confirms the correctness of the framework.

In Figs. 1 and 2, \(\bar{T}\) and \(F\) that follow a round robin resource allocation is also shown for comparison; in this case, the same amount of resources is given to all users. This kind of allocation corresponds to a proportional fair scheduling, since each user perceives a throughput proportional to the one that was possible using all resources. In this case, the generic user \(i\) perceives a throughput \(T_i = \hat{T}_i/N\), thus \(\bar{T} = (\sum_{i=0}^{N-1} T_i)/N^2\) and \(F = (\sum_{i=0}^{N-1} T_i)^2/(N \cdot \sum_{i=0}^{N-1} \hat{T}_i^2)\). The limit given by (46) allows to appreciate that the adoption of the round robin algorithm is not optimal: the achieved \(\bar{T}\) is not the highest possible for that \(F\), meaning that there are other possible allocations leading to both a higher \(\bar{T}\) and a higher \(F\).
In Figs. 3 and 4 the maximum \( T \) versus \( F \) is plotted for various case studies of Table I. In Fig. 3 cases A, B, C, and D are plotted, corresponding to the same number of users, varying \( T_i \). In Fig. 4 cases A, E, F, and G are plotted, corresponding to a variable number of users. In each curve of both figures, two points are highlighted, corresponding to the two following optimization criteria:

- **Criterion 1**: \( \max \{ F + T / T_{\text{max}} \} \) (marked with a circle)
- **Criterion 2**: \( \max \{ F \cdot T / T_{\text{max}} \} \) (marked with a star).

Both in Figs. 3 and 4 there are cases where the two optimization criteria do not lead to the same result; as an example, focusing to case A in Fig. 3, the maximum is achieved with \( F = 0.980 \) and \( T = 0.698 \) adopting criterion 1, while it corresponds to \( F = 0.696 \) and \( T = 1.355 \) in the other case.

**VI. Conclusion and Future Work**

In this paper, we proposed a novel analytical framework for the assessment of the tradeoff between average throughput and fairness, under the assumptions that throughput increases as the allocated resources increase and that throughput do not depend on which resources are allocated. The correctness of the solution was assessed assuming a linear relationship between resource allocation and throughput; a closed form was derived and its validity was confirmed through simulations. The proposed framework applies to many realistic scenarios and poses the basis for a more general solution to the tradeoff optimization problem. Future work will be devoted to obtain a closed form in other conditions and to relax the two highlighted assumptions, thus also considering the exploitation of multiuser diversity and deepening what happens if throughput can decrease with the growth of allocated resources.

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