Rationalized procedures of FDWT/IDWT basic operations execution for DB4, DB6 and coif6 filters

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Abstract: In paper a rationalized approaches to Discrete Wavelet Transform (DWT) coefficients computing, based on modified algorithm of DWT base operation execution, with less multiplication operations are presented. Those several intuitive ways allow to decrease computational complexity of hardware operational units for DWT basic operation computation and provide convenient conditions for wavelet data treatment processors in reprogrammable circuits effective realization.

Keywords: Discrete Wavelet Transform, Computational methods, Daubechies filters, Efficient algorithms, Signal processing algorithms, Wavelets

1. INTRODUCTION

The base of Discrete Wavelet Transform implementation is the division of the initial signal by means of a pair of filters (high and low pass) on two components and then hierarchic decomposition one or two of these components in analogical way.

This approach received a name of “fast wavelet transform” or “Mallat’s Pyramid”. The described algorithm, by rejection of redundant data in wavelet signal representation, allows to significantly minimize the number of indispensable, to whole DWT procedure execution, arithmetic operations. [see Mallat (1989),Daubechies (1992)]

“The core” of this whole procedure is “DWT base operation” multiplication of data vector by DWT base matrix, which describes the filters coefficients [see Daubechies (1992)].

In the matrix-vector form this operation can be defined in following way:

\[ Y_{2 \times 1}^{(l)} = F_{2 \times L} X_{L \times 1}^{(l)} \]  \hspace{1cm} (1)

while the reverse operation used to recreate signal from DWT coefficients (Inverse DWT base operation) looks as follows:

\[ \tilde{X}_{L \times 1}^{(l)} = (F_{2 \times L})^T Y_{2 \times 1}^{(l)} \]  \hspace{1cm} (2)\n
where \( N \) – number of original signal samples, \( L \) – moving window size that defines the part of the signal processed by the given base operation, \( k = 0, \ldots, K-1 \) – decomposition step number indicating the granularity degree of wavelet analysis, \( K \) – the total decomposition steps number.

\[ F_{2 \times L} = \begin{bmatrix} h_0 & h_1 & \ldots & h_{L-1} \\ g_0 & g_1 & \ldots & g_{L-1} \end{bmatrix} \]  \hspace{1cm} (3)

Vector \( X_{L \times 1}^{(l)} = [x_{2l}, x_{2l+1}, \ldots, x_{2(l+1)-1}, \ldots, x_{L+2l-1}]^T \) in (1,2) represents a column vector length of \( L \) whose elements represent the corresponding set of signal samples.

Vector \( \tilde{X}_{L \times 1}^{(l)} = [\tilde{x}_{2l}, \tilde{x}_{2l+1}, \ldots, \tilde{x}_{2(l+1)-1}, \ldots, \tilde{x}_{L+2l-1}]^T \).

Vector \( Y_{2 \times 1}^{(l)} = [y_{2l}, y_{2l+1}]^T \) is a two-element output data vector for each base operation (1,2), where \( y_{2l} \) is corresponding coefficient of DWT, and \( y_{2l+1} \) – result of approximation subject, along with other odd-indices results, to the further transformation.

The calculation of formula (1) requires \( 2L \) multiplications and \( 2(L-1) \) additions, reverse operation (2) the same number of multiplication and \( L \) additions. At present there are no solutions to reduce the number of arithmetic operations required in the DWT base operation to decrease above-mentioned values.

It turns out, however, that this possibility exists. Taking into account the specific characteristics of DWT base matrix, it is possible to construct DWT base operation algorithms with at least multiplications number reduced.

2. SYNTHESIS OF RATIONALIZED ALGORITHMS FOR DWT BASE OPERATION

Based on the \( F_{2 \times L} \) matrix elements let us build a new matrix \( D_{2L} = \text{diag}(h_0, h_1, \ldots, h_{L-1}, g_0, g_1, \ldots, g_{L-1}) \) of
represented by the following matrix coefficients:
\[ F \]

It is easy to see that the matrix needed to determine the DWT base operation results.

of factorization minimizing the number of multiplications that this matrix can be effectively subjected to a process after taking into account these values, we can also notice the operation implementation process can be summarized as follows:

\[ Y_{2 \times 1}^{(l)} = A_{2 \times 2L} D_{2L} P_{2 \times 1} X_{L \times 1}^{(l)} \]  
\[ \tilde{X}_{2 \times 1}^{(l)} = (P_{2 \times 1}^T) D_{2L} (A_{2 \times 2L}^T) Y_{2 \times 1}^{(l)} \]

In Figure 1 a graph representing the FDWT and IDWT base operation computational process for any size \( L \) filters is shown. With straight lines data transfer operations are here marked, with circles – operations of multiplication by an entered constant.

Easy to notice that the realization of this operation for the general case requires, as already mentioned, execution of 2\( L \) multiplication operations and respectively 2(\( L - 1 \)) and \( L \) addition operations.

2.1 Rationalized DWT computational procedures for DB-4 filters

Let us look at the specific examples and Consider the case of Daubechies-4 filters [see Daubechies (1992)], where \( L = 4 \)

Remember that for this case
\[ F_{2 \times 4} = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_1 & -c_2 & c_1 & -c_0 \end{bmatrix} \]

wherein \( c_0 = \frac{1 + \sqrt{3}}{4 \sqrt{2}} \), \( c_1 = \frac{3 + \sqrt{3}}{4 \sqrt{2}} \), \( c_2 = \frac{3 - \sqrt{3}}{4 \sqrt{2}} \), \( c_3 = \frac{1 - \sqrt{3}}{4 \sqrt{2}} \).

After taking into account these values, we can also notice that this matrix can be effectively subjected to a process of factorization minimizing the number of multiplications needed to determine the DWT base operation results.

It is easy to see that the matrix \( F_{2 \times L} \) can now be represented by the following matrix coefficients:

\[ P_{6 \times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \end{bmatrix}, \quad A_{4 \times 6} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ \end{bmatrix}, \quad D_6 = diag (1, 3, 1, 1, 3, 1), \quad H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D_4 = diag \left( \frac{\sqrt{3}}{4 \sqrt{2}}, \frac{1}{4 \sqrt{2}}, \frac{1}{4 \sqrt{2}}, \frac{\sqrt{3}}{4 \sqrt{2}} \right). \]

Then the modified implementation algorithm of the FDWT and respectively IDWT base operation with a reduced number of multiplications operations can be summarized as follows:

\[ Y_{2 \times 1}^{(l)} = H_2 A_{2 \times 4} D_4 A_{4 \times 6} D_6 P_{6 \times 4} X_{1 \times 1}^{(l)} \]

\[ \tilde{X}_{2 \times 1}^{(l)} = \tilde{A}_{4 \times 4} D_6 A_{6 \times 4} D_4 P_{4 \times 2} H_2 Y_{2 \times 1}^{(l)} \]

where \( P_{4 \times 2} = (A_{2 \times 4})^T, A_{6 \times 4} = (A_{4 \times 6})^T, \tilde{A}_{4 \times 6} = (P_{6 \times 4})^T. \)

In Figure 2 and 3, graphs representing the computational process of the FDWT and IDWT base operations for the Daubechies-4 filters, according to the proposed method (6,7), are shown.

Dashed lines shows here and further the subtraction operations.

2.2 Rationalized DWT computational procedures for DB-6 filters

Let us now consider another example of the wavelet base operation implementation, with use of the Daubechies-6 filters.

In this case \( L = 6 \) and FDWT base matrix takes the following form:
The modified implementation algorithm of the coif-6 filters IDWT base operation with reduced number of multiplication operations can be summarized as follows:

\[
\mathbf{F}_{2 \times 6} = \begin{bmatrix}
  c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\
  c_5 & -c_4 & -c_3 & -c_2 & -c_1 & -c_0
\end{bmatrix}
\]

where:

\[
c_0 = \frac{1 + \sqrt{10} + \sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}}, \quad c_1 = \frac{5 + \sqrt{10} + 3\sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}},
\]

\[
c_2 = \frac{10 - 2\sqrt{10} + 2\sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}}, \quad c_3 = \frac{10 - 2\sqrt{10} - 2\sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}},
\]

\[
c_4 = \frac{5 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}}, \quad c_5 = \frac{1 + \sqrt{10} - 3\sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}}.
\]

It turns out that also this matrix can be effectively subjected to the factorization process. Matrices in this case look as follows:

\[
\mathbf{P}_{12 \times 6} = \begin{bmatrix}
  1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 \\
  1 & -1 & -1 & -1 & -1 & -1 \\
  1 & -1 & 1 & 1 & 1 & 1 \\
  1 & 1 & -1 & -1 & -1 & -1 \\
  1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}, \quad \mathbf{A}_{6 \times 12} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},
\]

\[
\mathbf{D}_{12} = \text{diag}(1, 3, 5, 1, 2, 10, 10, 2, 1, 5, 3, 1),
\]

\[
\mathbf{D}_6 = \text{diag}\left(\frac{\sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}}, \frac{\sqrt{10}}{16\sqrt{2}}, \frac{1}{16\sqrt{2}}, \frac{\sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}}\right).
\]

Then the modified implementation algorithm of the FDWT and IDWT base operation with reduced number of multiplication operations can be summarized as follows:

\[
\mathbf{Y}^{(1)}_{2 \times 1} = \mathbf{H}_2 \mathbf{A}_{2 \times 6} \mathbf{D}_6 \mathbf{A}_{6 \times 12} \mathbf{D}_{12} \mathbf{P}_{12 \times 6} \mathbf{X}^{(1)}_{6 \times 1} \quad (8)
\]

\[
\tilde{\mathbf{X}}^{(1)}_{6 \times 1} = \tilde{\mathbf{A}}_{6 \times 12} \mathbf{D}_{12} \mathbf{A}_{12 \times 6} \mathbf{D}_6 \mathbf{P}_{6 \times 2} \mathbf{Y}^{(1)}_{2 \times 1} \quad (9)
\]

where

\[
\mathbf{P}_{6 \times 2} = (\mathbf{A}_{2 \times 6})^T, \quad \mathbf{A}_{12 \times 6} = (\mathbf{A}_{6 \times 12})^T, \quad \tilde{\mathbf{A}}_{6 \times 12} = (\mathbf{P}_{12 \times 6})^T.
\]

In figure 4, a graph representing the computational process of the FDWT base operation for the Daubechies-6 filters (8), according to the proposed method, is shown. In figure 5 the reverse operation (9) is presented.

Easy to notice that in this case the number of multiplications in the base operation implementation is the same as in the “classic” case, but a coefficients set is much more convenient. Half of them are integers, so multiplying blocks can be easily implemented in hardware.

2.3 Rationalized DWT computational procedures for coif-6 filters

In case of coiflet-6 filters the reduction of multiplication operation according to DB-6 base operation is even bigger. Length of filters remains the same ($L = 6$), also the structure of base matrix $\mathbf{F}_{2 \times 6}$. Coif-6 coefficients of this matrix:

\[
c_0 = \frac{1 - \sqrt{7}}{16\sqrt{2}}, \quad c_1 = \frac{5 + \sqrt{7}}{16\sqrt{2}}, \quad c_2 = \frac{14 + 2\sqrt{7}}{16\sqrt{2}},
\]

\[
c_3 = \frac{14 - 2\sqrt{7}}{16\sqrt{2}}, \quad c_4 = \frac{1 - \sqrt{7}}{16\sqrt{2}}, \quad c_5 = \frac{5 - \sqrt{7}}{16\sqrt{2}}.
\]

Factorization of coif-6 matrix is similar to the previous cases. Let us define transform matrices:

\[
\tilde{\mathbf{P}}_{12 \times 6} = \begin{bmatrix}
  1 & 1 & 1 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 \\
  1 & -1 & -1 & -1 & -1 & -1 \\
  1 & -1 & 1 & 1 & 1 & 1 \\
  1 & 1 & -1 & -1 & -1 & -1 \\
  1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix},
\]

\[
\tilde{d}_4 = \frac{\sqrt{5 + 2\sqrt{10}}}{16\sqrt{2}}, \quad \tilde{d}_5 = \frac{\sqrt{10}}{16\sqrt{2}}, \quad \tilde{d}_6 = \frac{1}{16\sqrt{2}}.
\]
Fig. 6. A graph-structural model of new algorithm implementation for coiflet-6 filters FDWT base operation.

\[
\begin{bmatrix}
-1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
\]

\[
A_{12} = \begin{bmatrix}
-1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
\]

\[
\tilde{D}_{12} = \text{diag}(1, 3, 5, 1, 2, 14, 14, 2, 1, 5, 3, 1),
\]

\[
A_{4 \times 12} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix},
\]

\[
\tilde{D}_{4} = \text{diag}(\sqrt{7}/16\sqrt{2}, 1/16\sqrt{2}, 1/16\sqrt{2}, \sqrt{7}/16\sqrt{2}).
\]

Then the rationalized algorithm of the FDWT/IDWT base operation for coiflet-6 filters can be summarized as in (10,11)

\[
Y_{2 \times 1}^{(l)} = A_{2 \times 4} A_{4 \times 12} A_{12 \times 2 \tilde{D}_{12}} X_{6 \times 1}^{(l)} \quad (10)
\]

\[
X_{6 \times 1}^{(l)} = B_{6 \times 12} A_{12 \times 2}^T P_{12 \times 4} D_{4 \times 2} X_{2 \times 1}^{(l)} \quad (11)
\]

where

\[
P_{12 \times 4} = (A_{4 \times 12})^T, \quad B_{6 \times 12} = (\tilde{P}_{12 \times 6})^T.
\]

In figure 6, a graph representing the computational process of the FDWT base operation for the coiflet-6 filters (10), according to the proposed method, is shown. In figure 7 the reverse operation (11) is presented.

3. HARDWARE AND TIME LOSS ASSESSMENT

As you can see, the approach proposed in the paper allows to reduce the number of multiplication operations (or in case of hardware implementation multiplication blocks) near 2 times compared to the classical method at the cost of some increased number (10% - 50%) of adding operations. This operations however are less complicated than multiplication operations, which in total gives a gain while implementation. Estimation of this profit for each case it is not trivial and requires further research. Mostly this is related to the large number of the DWT base matrix factorization variants for different lengths and filters modes.

4. SUMMARY

We have shown here a new original approach to optimized computational process implementation of discrete wavelet transform coefficients calculation, based on FDWT/IDWT base matrix factorization.

In a similar manner algorithms for a large number of filters types (Daubechies, Coiflets, Symlets) and other weight coefficients sets can be constructed.

Issues of their synthesis for different coefficients sets deserve for separate study and will be developed in future projects of the authors.

REFERENCES


