Distance-Regular Graphs with \( b_t = 1 \) and Antipodal Double-Covers

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Let \( \Gamma \) be a distance-regular graph of diameter \( d \) and valency \( k \geq 2 \). If \( b_t = 1 \) and \( 2r \leq d \), then \( \Gamma \) is an antipodal double-cover. Consequently, if \( f > 2 \) is the multiplicity of an eigenvalue of the adjacency matrix of \( \Gamma \) and if \( \Gamma \) is not an antipodal double-cover then \( d \leq 2f - 3 \). This result is an improvement of Godsil’s bound.

1. Introduction

Throughout this paper, we assume \( \Gamma \) is a connected finite undirected graph without loops or multiple edges. We identify \( \Gamma \) with the set of vertices. For vertices \( \alpha \) and \( \beta \) in \( \Gamma \), let \( \bar{d}(\alpha, \beta) \) denote the distance between \( \alpha \) and \( \beta \) in \( \Gamma \), that is, the length of a shortest path connecting \( \alpha \) and \( \beta \). Let \( d = d(\Gamma) \) denote the diameter of \( \Gamma \), that is, the maximal distance between any two vertices in \( \Gamma \). Let

\[
\Gamma_i(\alpha) = \{ \gamma \in \Gamma : \bar{d}(\alpha, \gamma) = i \}.
\]

For vertices \( \alpha \) and \( \beta \) in \( \Gamma \) at distance \( i \), let

\[
C_i(\alpha, \beta) = \Gamma_{i-1}(\alpha) \cap \Gamma_i(\beta),
\]

\[
A_i(\alpha, \beta) = \Gamma_i(\alpha) \cap \Gamma_i(\beta),
\]

\[
B_i(\alpha, \beta) = \Gamma_{i+1}(\alpha) \cap \Gamma_i(\beta).
\]
For the cardinalities we use lower case letters, that is,
\[ c_i(\alpha, \beta) = |C_i(\alpha, \beta)|, \quad a_i(\alpha, \beta) = |A_i(\alpha, \beta)| \quad \text{and} \quad b_i(\alpha, \beta) = |B_i(\alpha, \beta)|. \]

Let \( j \) be an integer with \( 1 \leq j \leq d \) and \( x \) and \( y \) vertices in \( \Gamma \) at distance \( j \). For any \( w \in C_j(y, x) \), obviously \( B_{j-1}(w, y) \supseteq B_j(x, y) \) and \( C_{j-1}(w, y) \subseteq C_j(x, y) \) hold. In particular, we have
\[ |B_{j-1}(w, y) - B_j(x, y)| = b_{j-1}(w, y) - b_j(x, y), \]
\[ |C_j(x, y) - C_{j-1}(w, y)| = c_j(x, y) - c_{j-1}(w, y). \]

A connected graph \( \Gamma \) is called a distance-regular graph if for any two vertices \( \alpha \) and \( \beta \) in \( \Gamma \) at distance \( i \), the numbers \( c_i(\alpha, \beta), a_i(\alpha, \beta) \) and \( b_i(\alpha, \beta) \) depend only on the distance \( d(\alpha, \beta) = i \) rather than on individual vertices. When this is the case we denote numbers \( c_i(\alpha, \beta), a_i(\alpha, \beta) \) and \( b_i(\alpha, \beta) \) by \( c_i, a_i \) and \( b_i \), respectively, and call them the intersection numbers of \( \Gamma \). It is obvious that a distance-regular graph is a regular graph of valency \( k = b_0 \).

The basic properties of the intersection numbers of a distance-regular graph are:

1. \( k = b_0 > b_1 \geq \cdots \geq b_{d-1} \geq 1 \),
2. \( 1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k \),
3. \( c_h \leq b_{d-h} \) for \( 1 \leq h \leq d \).

For the general theory of a distance-regular graph, the reader is referred to [1] or [2].

A distance-regular graph \( \Gamma \) of diameter \( d \) is an antipodal double-cover (of its folded graph), if and only if \( b_i = c_{d-i} \), for \( i = 0, \ldots, d - 1 \). For more details on antipodal graphs see [4], and [2, §4.2]. Antipodal distance-regular graphs are imprimitive and play important role in the classification of distance-regular graphs.

The main result of this paper is the following:

**Theorem 1.** Let \( \Gamma \) be a distance-regular graph of diameter \( d \) and valency \( k > 2 \). If \( b_1 = 1 \) and \( 2 t \leq d \) then \( \Gamma \) is an antipodal double-cover.

This result improves the main result of [11], generalizes Proposition (2.8) from [9] (cf. [3]) and gives a partial answer to Problem (ii) from [2, p. 182]. As an immediate consequence we have the following result:

**Corollary 2.** Let \( \Gamma \) be a distance-regular graph of diameter \( d \). Let \( f > 2 \) be the multiplicity of an eigenvalue of the adjacency matrix of \( \Gamma \). If \( \Gamma \) is not an antipodal double-cover, then we have \( d \leq 2f - 3 \).
Proof. Straightforward application of [2, Proposition 4.4.8 or Theorem 5.3.2], [5] or [6, Lemma 13.5.1].

This result is very important for the classification of distance-regular graphs with an eigenvalue of small multiplicity (as oppose to a dual classification of distance-regular graphs with small valency). So far the distance-regular graphs with an eigenvalue of multiplicity seven or less have been determined [12] and [10], while in the valency case only the distance-regular graphs of valency at most four have been determined.

2. Proof of Theorem 1

A box is a quadruple \((u, v, x, y)\) of vertices that satisfy

\[
\bar{d}(u, y) = \bar{d}(v, x) = d, \quad \bar{d}(u, x) = \bar{d}(v, y) = d - 1, \quad \bar{d}(u, v) = \bar{d}(x, y) = 1,
\]
as illustrated below:

![Diagram of a box]

Notice that there are many boxes in an antipodal distance-regular graph \(\Gamma\) of diameter \(d \geq 3\). Namely, given \(u, y\) with \(\bar{d}(u, y) = d\), there is a one to one correspondence between vertices of \(\Gamma_d(u)\) and vertices of \(\Gamma_d(y)\) such that for \(v \in \Gamma_1(u)\) and its correspondent vertex \(x \in \Gamma_1(y)\) the quadruple \((u, v, x, y)\) is a box.

Now we shall derive some results about boxes.

Lemma 3. Let \(\Gamma\) be a distance-regular graph of diameter \(d\). If \(\Gamma\) has no boxes, then we have

\[
e_d \times b_{d-1} \leq a_d^2.
\]

Proof. Let \(x\) and \(\beta\) be vertices in \(\Gamma\) at distance \(d\). Let \(N = C_d(x, \beta)\), \(T = C_d(\beta, x)\) and \(S = A_d(\beta, x)\). Note that \(|N| = |T| = e_d\), \(|S| = a_d\) and \(\Gamma_1(x) = T \cup S\), since \(k = a_d + e_d\), i.e., \(b_d = 0\). Let \(P = \{z, w\} \mid z \in S, w \in N, \bar{d}(z, w) = d\}\). We count the elements of \(P\) in two ways. Take \(w \in N\) and consider the set \(\Gamma_d(w) \cap \Gamma_1(x)\). Since \(\bar{d}(x, w) = d - 1\) and \(\Gamma_1(x) = T \cup S\), we have \(b_{d-1} = |\Gamma_d(w) \cap \Gamma_1(x)| = |\Gamma_d(w) \cap T| + |\Gamma_d(w) \cap S|\). Suppose there
exists a vertex $x$ in $\Gamma_d(w) \cap T$. Then the quadruple $(x, x, w, \beta)$ is a box, which
contradicts our assumption. Hence we have $b_{d-1} = |\Gamma_d(w) \cap S|$, and therefore $|P| = |N| \times b_{d-1} = c_d \times b_{d-1}$. On the other hand, take $z \in S$. Since
$\partial(z, \beta) = d$ and $N \subseteq \Gamma_d(\beta)$, we have $a_d = |\Gamma_d(z) \cap \Gamma_d(\beta)| \geq |\Gamma_d(z) \cap N|$. This
implies $|P| \leq |S| \times a_d = a_d^2$.

The following result provides an important inequality for a graph with
$b_{d-1} = 1$ if it contains a box. For adjacent vertices $u$ and $v$ of $\Gamma$ we write $u \sim v$.

**Proposition 4.** Let $\Gamma$ be a distance-regular graph of diameter $d$ with
$b_{d-1} = 1$ for some $t \geq 1$. If $\Gamma$ contains a box, then we have

$$b_{d-h} - c_h \leq b_{d-(h-1)} - c_{h-1} \quad \text{for} \quad t + 1 \leq h \leq d.$$  

Proof. Let $u, v, x, y$ be vertices in $\Gamma$ such that the quadruple $(u, v, x, y)$ is a box. Take any $z \in A_{d-1}(x, y)$, then $z \in C_d(v, x)$, i.e., $\partial(v, z) \neq d$ (as otherwise $\{z, x\} \subseteq B_{d-1}(v, y)$ is contradicting $b_{d-1} = 1$), and $\partial(v, z) \geq \partial(v, x) = h_1 = d - 1$. By symmetry we have $\partial(u, z) = d - 1$ and therefore

$$\{y\} \cup A_{d-1}(y, x) \subseteq C_d(v, x) - C_{d-1}(u, x).$$

Thus we have $b_{d-1} = 1 + a_1 \leq c_d - c_{d-1}$. Hence our assertion holds for $h = d$ and we may assume now that $h < d$. Take a vertex $p \in \Gamma_{d-1}(u) \cap \Gamma_{d-1}(x)$. By $c_{d-1} \leq b_{d-1} = 1$, we set $\{w\} = C_{d-1}(x, p)$. It is clear that $\partial(y, p) = d - h + 1$, $\partial(p, v) = h$, $\partial(v, w) = d - h$, $\partial(u, w) = h$ and $\partial(v, w) = h + 1$. In order to prove the statement, it is sufficient to show

$$B_{d-h}(x, p) - B_{d-(h+1)}(y, p) \subseteq C_d(v, p) - C_{d-1}(u, p).$$

Take any $z \in B_{d-h}(x, p) - B_{d-(h+1)}(y, p)$. Note that $\partial(x, z) = d - h + 1$ and $\partial(y, z) = d - h + 2$. Since $\partial(x, w) = d - h - 1$, we have $z \neq w$.

Claim 1. $\partial(y, z) = d - h + 1$.

Since $\partial(y, x) = 1$ and $\partial(x, z) = d - h + 1$, we have $\partial(y, z) \in \{d - h, d - h + 1\}$. If $\partial(y, z) = d - h$, then $\{w, z\} \subseteq C_{d-1}(y, p)$. This contradicts the inequality $c_{d-1} = b_{d-1} = 1$. Hence we have $\partial(y, z) = d - h + 1$.

Claim 2. $\partial(u, z) = h - 1$.

Since $\partial(u, p) = h - 1$ and $\partial(p, z) = 1$, we have $\partial(u, z) \in \{h - 2, h - 1, h\}$.

Suppose $\partial(u, z) = h$. Then we have $\{z, w\} \subseteq B_{h-1}(u, p)$, which contradicts $b_{h-1} = 1$. Suppose $\partial(u, z) = h - 2$. Then we have

$$d = \partial(u, y) \leq \partial(u, z) + \partial(z, y) = (h - 2) + (d - h + 1) = d - 1.$$
This is a contradiction. Hence we have \( \partial(u, z) = h - 1 \) and therefore also \( z \not\in C_{d-1}(u, p) \).

**Claim 3.** \( \partial(v, z) = h - 1 \), i.e., \( z \in C_d(v, p) \).

Since \( \partial(v, p) = h \) and \( \partial(p, z) = 1 \), we have \( \partial(v, z) \in \{ h - 1, h, h + 1 \} \). Suppose \( \partial(v, z) = h + 1 \). Then we have \( \{ z, w \} \subseteq B_d(v, p) \), which contradicts \( b_h = 1 \). Let \( z = z_h \sim z_1 \sim \cdots \sim z_{d - h + 1} = y \) be a shortest path connecting \( z \) and \( y \). It is clear that \( \partial(u, z_j) = h + 1 + j \) for \( 0 \leq j \leq d - h + 1 \). Since \( b_{h - 1, j} = 1 \), we have \( B_{h - 1, j}(u, z_j) = \{ z_{j+1} \} \) for \( 0 \leq j \leq d - h - 1 \). Suppose \( \partial(v, z_j) = h \). Then \( u \in C_j(z, v) \) and \( B_d(v, z) \subseteq B_{j+1}(u, z) = \{ z_1 \} \), which implies \( B_j(v, z) = B_{h - 1}(u, z) \). Inductively, we have \( u \in C_j(z_j, v) \) and \( B_{h - 1, j}(u, z_j) = \{ z_{j+1} \} \) for \( 0 \leq j \leq d - h - 1 \). In particular, we have \( \partial(v, z_{d - h}) = d \). Note that \( z_{d - h} \neq x \), since \( \partial(x, z) = d - h + 1 \). Then we have \( \{ z_{d - h}, x \} \subseteq B_{d - 1}(v, y) \). This contradicts \( b_{d - 1} = 1 \).

Therefore, we obtain (1).

The next lemma shows that a graph which satisfies the assumption of Theorem 1 contains a box.

**Lemma 5.** Let \( \Gamma \) be a distance-regular graph of diameter \( d \) and valency \( k > 2 \). If \( b_1 = 1 \) and \( 2t \leq d \), then the following statements hold.

1. \( a_t \leq 1 \) for \( 2t \leq h \leq d \).
2. \( \Gamma \) contains a box.

**Proof.** (1) Let \( 2t \leq h \leq d \). Let \( u \) and \( v \) be vertices of \( \Gamma \) at distance \( h \) and take \( x = \Gamma_{h-1}(u) \cap \Gamma_t(v) \). Let \( A = A_{d}(u, v) \). Now we show that \( A \supseteq \Gamma_{t+1} \setminus (x) \cap \Gamma_t(v) \). Let \( z \in A \). It is clear that \( \partial(x, z) \in \{ t, t + 1 \} \). Suppose \( \partial(x, z) = t \). Let \( v = v_0 \sim v_1 \sim \cdots \sim v_t = z \) be a path connecting \( x \) and \( z \) and \( x = z_0 \sim z_1 \sim \cdots \sim z_j = z \) such a path connecting \( x \) and \( z \). It is easy to see that the vertices \( v_i \) and \( z_i \) are in \( \Gamma_{h-1}(u) \) for \( 0 \leq i \leq t \). By monotonicity of the sequence \( \{ b_i \} \) and the condition \( t \leq h - t \), we have \( b_1 = b_{t+1} = \cdots = b_{d-1} = 1 \). Thus we have \( z_i = v_i \) for \( 0 \leq i \leq t \). This contradicts \( v \neq z \). Hence we have \( A \supseteq \Gamma_{t+1} \setminus (x) \cap \Gamma_t(v) \). Therefore \( a_t \leq b_t = 1 \).

(2) Note that \( a_d \leq 1 \). Suppose \( \Gamma \) contains no boxes. Then from Lemma 3 we have \( c_d \leq a_d^2 \leq 1 \). This contradicts \( a_d + c_d = k > 2 \).

**Proof of Theorem 1.** Let \( \Gamma \) be a distance-regular graph with valency \( k > 2 \), \( b_1 = 1 \) and diameter \( d \geq 2t \). We need to show that

\[
b_h = c_{d - h}
\]

holds for \( 0 \leq h \leq d - 1 \). We have \( 1 = b_1 = \cdots = b_{d-1} \). By \( c_i \leq b_{d-i} \) for \( i < d \), we also have \( 1 = c_1 = \cdots = c_{d-1} \). Hence (2) holds for \( t \leq h \leq d - 1 \). Since,
by Lemma 5, the graph $\Gamma$ contains a box, it follows from Proposition 4 that
\[ 0 \leq b_0 - c_d \leq b_1 - c_{d-1} \leq \cdots \leq b_t - c_{d-t}. \]
But $b_t = 1 = c_{d-t}$ and the equalities must hold.

Remarks. (1) Hiraki [7] has already proved that $d < 2t + 20$ if $b_t = 1$.
(2) Recently the authors have solved Problem (ii) of [2, p. 182].
(3) Koolen [8, Theorem 7.17] used standard representation to show that for a graph $\Gamma$ with an eigenvalue of multiplicity $f \geq 3$ and diameter $d$ we have $d \leq 2f - 1$, with equality if and only if $\Gamma$ is the dodecahedron.

REFERENCES
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