In this paper, bifurcation trees of periodic motions to chaos in a parametric oscillator with quadratic nonlinearity are investigated analytically as one of the simplest parametric oscillators. The analytical solutions of periodic motions in such a parametric oscillator are determined through the finite Fourier series, and the corresponding stability and bifurcation analyses for periodic motions are completed. Nonlinear behaviors of such periodic motions are characterized through frequency–amplitude curves of each harmonic term in the finite Fourier series solution. From bifurcation analysis of the analytical solutions, the bifurcation trees of periodic motion to chaos are obtained analytically, and numerical illustrations of periodic motions are presented through phase trajectories and analytical spectrum. This investigation shows period-1 motions exist in parametric nonlinear systems and the corresponding bifurcation trees to chaos exist as well.

Keywords: Parametric quadratic nonlinear oscillator; period-1 motions to chaos; period-2 motions to chaos; nonlinear dynamical systems.

1. Introduction

The stability of solutions in parametric linear systems has been understood through Hill equation and Mathieu equation. Parametric nonlinear systems extensively exist in engineering. The parametric nonlinear systems possess dynamical behaviors distinguishing from periodically forced nonlinear systems. In the traditional perturbation analysis, only symmetric periodic motions in parametrically forced nonlinear systems can be determined. In this paper, one of the simplest parametric nonlinear oscillators will be investigated, and the nonlinear behaviors of such possible periodic motions will be discussed through the frequency–amplitude curves.

For parametric oscillators, one should go back to periodic solutions in the Mathieu equations. Mathieu [1868] investigated the linear Mathieu equation (also see [Mathieu, 1873; McLachlan, 1947]). The corresponding stability of periodic solutions in the linear Mathieu equation was discussed. Whittaker [1913] presented a method to find the unstable solutions for very weak excitation (also see [Whittaker & Watson, 1935]). In engineering, Sevin [1961] used the Mathieu equation to investigate the vibration-absorber with parametric excitation. Hsu [1963] discussed the first approximation analysis and stability criteria for a multiple-degree of freedom dynamical system (also see [Hsu, 1965]). Hayashi [1964] discussed the approximate periodic solutions in parametric systems and the corresponding stability by the averaging method and harmonic balance method. Tso and Caughey [1965] discussed the stability of parametric, nonlinear systems. Mond et al. [1993] did the stability...
analysis of nonlinear Mathieu equation. Zoumes and Rand [2000] discussed the transient response for the quasi-periodic Mathieu equation. Luo [2004] discussed chaotic motions in the resonant separatrix bands of the Mathieu–Duffing oscillator with a twin-well potential. Shen et al. [2008] used the incremental harmonic balance method to investigate the bifurcation route to chaos in the Mathieu–Duffing oscillator. Luo [2012] developed a generalized harmonic balance method to determine periodic motions in nonlinear dynamical systems. Luo and Huang [2012a] applied the generalized harmonic balance method to determine period-1 solutions in the Duffing oscillator (also see [Luo & Huang, 2012b, 2012c]). In [Luo & O'Connor, 2013], periodic motions in hardening Mathieu–Duffing oscillator were presented. In the traditional analysis, asymmetric periodic motions cannot be obtained. In other words, period-1 motions in parametric linear oscillator cannot be obtained. In [Luo & Yu, 2013a, 2013b, 2013c], periodically forced, quadratic nonlinear oscillators were investigated, and bifurcation trees of period-1 motion to chaos were presented. Herein, parametric quadratic nonlinear oscillators will be investigated.

In this paper, analytical solutions of parametric nonlinear oscillators will be obtained through the generalized harmonic balance. Analytical bifurcation trees of period-2 motion to chaos in such a parametric oscillator will be determined first and the bifurcation trees of period-1 to chaos will be also discussed. The corresponding stability and bifurcation of periodic motions should be presented through eigenvalue analysis. Numerical illustrations will be completed for a better understanding of periodic motions in the simplest parametric nonlinear oscillators.

2. Approximate Solutions

Consider a parametric, quadratic nonlinear oscillator as

\[ \ddot{x} + \delta \dot{x} + (\alpha + \Omega_0 \cos \Omega t) x + \beta x^2 = 0 \]  

where parameters \( \alpha \) and \( \beta \) are positive constants. The damping coefficient \( \delta \) is positive. A parametric excitation \( \Omega_0 \cos \Omega t \) has excitation amplitude \( \Omega_0 \) and frequency \( \Omega \). The standard form of Eq. (1) is written as

\[ \ddot{x} + f(x, \dot{x}, t) = 0 \]  

where

\[ f(x, \dot{x}, t) = \delta \dot{x} + (\alpha + \Omega_0 \cos \Omega t) x + \beta x^2. \]  

As in [Luo, 2012], the analytical solution of period-\( m \) motion is assumed as

\[ x^*_{m}(t) = a(Z^*_m(t)) + \sum_{k=1}^{N} b_k m(t) \cos \left( \frac{k \Omega}{m} t \right) \]

\[ + \sum_{k=1}^{N} c_k m(t) \sin \left( \frac{k \Omega}{m} t \right). \]  

From the foregoing approximate expression, the first and second order derivatives of \( x^*_{m}(t) \) are

\[ \ddot{x}^*_{m}(t) + \sum_{k=1}^{N} \left( \frac{k \Omega}{m} \right) b_k m(t) \cos \left( \frac{k \Omega}{m} t \right) \]

\[ + \left( \frac{k \Omega}{m} \right) c_k m(t) \sin \left( \frac{k \Omega}{m} t \right). \]  

Substituting Eqs. (4)–(6) into Eq. (2) and averaging all terms of \( \cos(k \Omega / m) \) and \( \sin(k \Omega / m) \) gives

\[ a_0^{(m)} + F_0^{(m)}(a_0^{(m)}, b_0^{(m)}, c_0^{(m)}, a_0^{(m)}, b_0^{(m)}, c_0^{(m)}) = 0 \]

\[ b_k = \frac{k \Omega}{m} c_k - \left( \frac{k \Omega}{m} \right)^2 b_k \quad \text{for} \quad k = 1, 2, \ldots, N. \]  

\[ c_k = 2 \frac{k \Omega}{m} b_k - \left( \frac{k \Omega}{m} \right)^2 c_k \quad \text{for} \quad k = 1, 2, \ldots, N. \]
The coefficients of constant, \( \cos(k\Omega t/m) \) and \( \sin(k\Omega t/m) \) for the function of \( f(x, \dot{x}, t) \) can be obtained in the form of

\[
f(0)^{(m)}(a_0^{(m)}, b_0^{(m)}, c_0^{(m)}, \dot{a}_0^{(m)}, \dot{b}_0^{(m)}, \dot{c}_0^{(m)})
= \frac{1}{m^2} \int_0^{mT} F(x, \dot{x}, t) dt
= \delta a_0^{(m)} + \alpha c_0^{(m)} + \beta (a_0^{(m)})^2
+ \frac{1}{2} Q \sum_{i=1}^N (\delta_{ij/m} + c_j^{(m)})
+ \frac{1}{2} \sum_{i=1}^N \left( \delta_{ij/m} + c_j^{(m)} \right) \\
= \frac{2}{m^2} \int_0^{mT} F(x, \dot{x}, t) \cos \left( \frac{k}{m} \Omega t \right) dt \\
= \delta \left( \dot{b}_1^{(m)} + c_k^{(m)} \frac{k}{m} \right) + \alpha b_k^{(m)} + \beta (b_k^{(m)})^2
+ 2 \left( b_0^{(m)} \right) b_k^{(m)} + f_k^{(r)}
\]

where

\[
f_k^{(r)} = a_0^{(m)} Q_k^{(r)} b_k^{(m)}
+ \frac{1}{2} Q \sum_{i=1}^N a_{ij/m} (\dot{s}_{ij/m} + \dot{s}_{ij/m} + \ddot{s}_{ij/m})
+ \frac{1}{2} \sum_{i=1}^N \left( \dot{s}_{ij/m} + \dot{s}_{ij/m} \right) + \ddot{s}_{ij/m}.
\]

Define

\[
\dot{z}^{(m)} \equiv (a_0^{(m)}, b_0^{(m)}, c_0^{(m)})^T
= (a_0^{(m)}, b_0^{(m)}, b_0^{(m)}, \ldots, b_{N/m}^{(m)}, c_0^{(m)}, \ldots, c_{N/m}^{(m)})^T
\equiv (z_0^{(m)}, z_1^{(m)}, \ldots, z_{N/m}^{(m)})^T.
\]

\[
z_{1/m}^{(m)} = \dot{z}^{(m)} = (a_0^{(m)}, b_0^{(m)}, c_0^{(m)})^T
= (a_0^{(m)}, b_0^{(m)}, b_0^{(m)}, \ldots, b_{N/m}^{(m)}, c_0^{(m)}, \ldots, c_{N/m}^{(m)})^T
\equiv (z_0^{(m)}, z_1^{(m)}, \ldots, z_{N/m}^{(m)})^T.
\]

where

\[
b^{(m)} = (b_{1/m}, b_{2/m}, \ldots, b_{N/m})^T.
\]

\[
c^{(m)} = (c_1/m, c_2/m, \ldots, c_{N/m})^T.
\]

Equation (7) becomes

\[
\dot{z}^{(m)} = z_{1/m}^{(m)} \text{ and } \dot{z}_{1/m}^{(m)} = g^{(m)}(z^{(m)}, z_{1/m}^{(m)}).
\]

where

\[
g^{(m)}(z^{(m)}, z_{1/m}^{(m)}) = \begin{pmatrix}
-F_{1}^{(m)}(z^{(m)}, z_{1/m}^{(m)}) \\
-F_{2}^{(m)}(z^{(m)}, z_{1/m}^{(m)}) + 2k \frac{\Omega}{m} c^{(m)} + 2k \frac{\Omega}{m} c^{(m)}
\end{pmatrix}
\]
where \( m = 1, 2, \ldots, N \),

\[
F_i^{(m)} = \left[ F^{(1)}_{i1}, F^{(1)}_{i2}, \ldots, F^{(1)}_{iN} \right]^T,
\]

\[
F_2^{(m)} = \left[ F^{(2)}_{11}, F^{(2)}_{12}, \ldots, F^{(2)}_{2N} \right]^T
\]

for \( N = 1, 2, \ldots, \infty \)

and

\[
y^{(m)} = \left[ \mathbf{z}^{(m)} \right]^T \quad \text{and} \quad f^{(m)} = \left[ \mathbf{x}^{(m)} \right]^T
\]

Thus, equation (12) becomes

\[
\dot{y}^{(m)} = f^{(m)}(y^{(m)}).
\]

The steady-state solutions for periodic motions in the parametric quadratic nonlinear oscillator can be obtained by setting \( y^{(m)} = 0 \), i.e.,

\[
F_i^{(m)}(0)^{*,*}, \mathbf{b}^{(m)*}, \mathbf{c}^{(m)*}, 0, 0, 0) = 0,
\]

\[
F_1^{(m)}(0)^{*,*}, \mathbf{b}^{(m)*}, \mathbf{c}^{(m)*}, 0, 0, 0) - \frac{\Omega^2}{m} \mathbf{h}^{(m)*} = 0,
\]

\[
F_2^{(m)}(0)^{*,*}, \mathbf{b}^{(m)*}, \mathbf{c}^{(m)*}, 0, 0, 0) - \frac{12}{m^2} \mathbf{k}^{(m)*} = 0
\]

(17)

The \( (2N + 1) \) nonlinear equations in Eq. (17) are solved by the Newton–Raphson method. In Luo, (2012), the linearized equation at \( y^{(m)} = (\mathbf{x}^{(m)}, 0)^T \) is

\[
\Delta \dot{y}^{(m)} = Df^{(m)}(y^{(m)}) \Delta y^{(m)}
\]

where

\[
Df^{(m)}(y^{(m)}) = \left. \frac{\partial f^{(m)}(y^{(m)})}{\partial y^{(m)}} \right|_{y^{(m)}=m} \quad \text{or} \quad \frac{\partial f^{(m)}(y^{(m)})}{\partial y^{(m)}}
\]

(19)

The Jacobian matrix is

\[
Df^{(m)}(y^{(m)})
\]

\[
= \begin{bmatrix}
  0_{(2N+1) \times (2N+1)} & I_{(2N+1) \times (2N+1)} \\
  G_{(2N+1) \times (2N+1)} & H_{(2N+1) \times (2N+1)}
\end{bmatrix}
\]

(20)

and

\[
G = \left( \frac{\partial y^{(m)}}{\partial \mathbf{g}^{(m)}} \right) = (G^{(0)}, G^{(c)}, G^{(a)})^T
\]

(21)

for \( r = 0, 1, \ldots, 2N \).
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\[ H = \frac{\partial H^{(m)}}{\partial k} = (H^{(0)}, H^{(1)}, H^{(2)})^T \]

(28)

where

\[ H^{(0)} = (H_0^{(0)}, H_1^{(0)}, \ldots, H_{2N}^{(0)}), \]

\[ H^{(e)} = (H_0^{(e)}, H_1^{(e)}, \ldots, H_{2N}^{(e)}), \]

\[ H^{(s)} = (H_0^{(s)}, H_1^{(s)}, \ldots, H_{2N}^{(s)}), \]

(29)

for \( N = 1, 2, \ldots, \infty \), with

\[ H_k^{(e)} = (H_k^{(e)} H_k^{(e)} \ldots H_k^{(e)}), \]

\[ H_k^{(s)} = (H_k^{(s)} H_k^{(s)} \ldots H_k^{(s)}), \]

(30)

for \( k = 1, 2, \ldots, N \). The corresponding components are

\[ H_k^{(0)} = -N \delta k, \]

\[ H_k^{(e)} = -2 \frac{\delta k}{m} N k - N \delta k, \]

\[ H_k^{(s)} = 2 \frac{\delta k}{m} k - N \delta k, \]

(31)

for \( r = 0, 1, \ldots, 2N \).

The corresponding eigenvalues are determined by

\[ |DF^{(m)}(y^{(m)}) - \lambda_{2N+1}(2N+1)| = 0. \]

(32)

From Luo [2012], the eigenvalues of \( DF^{(m)}(y^{(m)}) \) are classified as

\[ (n_1, n_2, n_3 | n_4, n_5, n_6). \]

(33)

The corresponding boundary between the stable and unstable solutions is given by the saddle-node bifurcation and Hopf bifurcation.

3. Frequency–Amplitude Curves

The curves of harmonic amplitude varying with excitation frequency \( \Omega \) for each harmonic term are illustrated. The corresponding solution in Eq. (4) can be rewritten as

\[ a^*(t) = a_0^{(m)} + \sum_{k=1}^{N} A_k/m \cos \left( \frac{k \Omega t}{m} - \varphi_k/m \right), \]

(34)

where the harmonic amplitude and phase are defined by

\[ A_k/m = \sqrt{\delta k/m + \varphi_k/m}, \]

(35)

\[ \varphi_k/m = \arctan \frac{\delta k/m}{\delta k/m}, \]

The system parameters are

\[ \delta = 0.5, \ \alpha = 5, \ \beta = 20. \]

(36)

In all frequency-amplitude curves, the acronyms “SN” and “HB” represent the saddle-node and Hopf bifurcations, respectively. Solid curves represent stable period-\( m \) motions. Long dashed, short dashed and chain curves represent unstable period-1, period-2 and period-4 motions, respectively.

Consider a bifurcation tree of period-2 motion to chaos through period-2 to period-4 motion in parametrically excited, quadratic nonlinear oscillator. Using the parameters in Eq. (36), the frequency–amplitude curves based on 40 harmonic terms of period-2 to period-4 motion are presented in Figs. 1-3 for \( Q_0 = 10 \). In Fig. 1, a global view of frequency–amplitude curve for period-2 to period-4 motion is presented. In Fig. 1(a), constant \( a_0^{(m)} \) versus excitation frequency \( \Omega \) is presented, and all the values of \( a_0^{(m)} \) are less than zero. \( a_0^{(m)} \in (-0.17, 0.0) \) is for period-2 and period-4 motion. The saddle-node bifurcations of period-2 motions are their onset points at \( \Omega_2 \approx 2.6, 6.4 \). The Hopf bifurcations of period-2 motions yield the onset of period-4 motions at \( \Omega_4 \approx 2.59, 5.95 \) for the large branch and \( \Omega_4 \approx 2.92, 3.0 \) for the small branch. In Fig. 1(b), the harmonic amplitude \( A_{1/4} \) versus excitation frequency \( \Omega \) is presented for period-4 motion. We have \( A_{1/4} \in (0.0, 0.2) \). The Hopf bifurcation points of period-4 motions are the saddle-node bifurcation points for period-8 motions. Since the stable period-8 motion exists in the short range, period-8 motions will not be presented herein. In Fig. 1(c), the harmonic amplitudes \( A_{1/2} \) versus excitation frequency \( \Omega \) are presented for period-2 and period-4 motions, as for constant \( a_0^{(m)} \) in Fig. 1(a). We have \( A_{1/2} \in (0.0, 0.3) \). In Fig. 1(d), the harmonic amplitude \( A_{1/4} \) versus excitation frequency \( \Omega \) is presented for period-4 motion in the range of \( A_{1/4} \in (0.0, 0.12) \). The frequency–amplitude curves in \( (\Omega, A_1) \) are illustrated in Fig. 1(e) for period-2 and period-4 motions in the range of
Fig. 1. Frequency-amplitude curves ($Q_0 = 10$) based on 40 harmonic terms (HB40) of period-2 motion to period-4 motion in the parametric, nonlinear quadratic oscillator: (a) constant term $a_0/m$, (b)–(h) harmonic amplitude $A_k/m$ ($k = 1, 2, \ldots, 4, 8, 12, 40, m = 4$). ($\delta = 0.5$, $\alpha = 5$, $\beta = 20$).
A_1 \in (0.0, 0.3). To save space, the harmonic amplitude A_{1/4} (\text{mod}(k, 4) \neq 0) will not be presented. To further show effects of harmonic terms, the harmonic amplitudes A_2, A_3, A_{10} are presented in Figs. 1(f)-1(h). A_2 < 10^{-1}, A_3 < 10^{-2} and A_{10} < 10^{-3} are observed. Thus, effects of the higher order harmonic terms on period-2 and period-4 motions can be ignored.

To clearly illustrate the bifurcation trees of period-2 motions, the bifurcation tree-1 of period-2 motion to period-4 motion based on 40 harmonic terms is presented in Fig. 2 for Q_0 = 10 within the range of \Omega \in (5.7, 6.0). In Fig. 2(a), the constant versus excitation frequency is presented. The bifurcation tree is very clearly presented. In Fig. 2(b), the amplitude versus excitation frequency for period-4 motion in the bifurcation tree is presented. We have A_{1/4} = 0 for period-2 motion. The bifurcation tree-1 for harmonic amplitude A_{1/2} is presented in Fig. 2(c). For the zoomed range, the quantity level of the harmonic amplitudes is A_{1/2} \sim 10^{-1}. Similar to A_{1/4}, the harmonic amplitude A_{1/2} for period-4 motion in bifurcation tree-1 is presented in Fig. 2(d). The harmonic amplitude A_1 for the bifurcation tree-1 of period-2 to period-4 motion is given in Fig. 2(e). For the zoomed range, the quantity level of the harmonic amplitude is A_1 \sim 7 \times 10^{-3}. For the period-2 motion, the harmonic amplitude A_{k/2} (k = 2l - 1, l = 1, 2, ...) is very important. Thus, the harmonic amplitude A_{1/2}
Fig. 2. (Continued)
Fig. 3. Bifurcation tree-2 of period-2 motion ($Q_0 = 10$) to period-4 motion based on 40 harmonic terms ($HB_{40}$) in the parametric, nonlinear quadratic oscillator: (a) constant term $a_{0}^{(m)}$, (b)–(h) harmonic amplitude $A_{k/m}$ ($k = 1, 2, \ldots, 4, 6, 8, 24$, $m = 4$). ($\delta = 0.5$, $\alpha = 5$, $\beta = 20$).
for the bifurcation tree-1 is presented in Fig. 2(f). For the zoomed range, the quantity level of the harmonic amplitudes is $A_{3/2} \sim 10^{-2}$. The harmonic amplitude with the range of $A_2 \sim 10^{-4}$ is presented in Fig. 2(g). For reduction of abundant illustrations, the harmonic amplitude $A_6 \sim 3 \times 10^{-11}$ for the bifurcation is presented in Fig. 2(h). Other harmonic amplitudes can be similarly presented.

The second bifurcation tree of period-2 motion to period-4 motion is in the narrow range of $\Omega \in (2.90, 3.02)$, as shown in Fig. 3. In this bifurcation tree, the period-4 motion has four parts of stable solutions and three parts of unstable solution on the same curve. In Fig. 3(a), constant $a_0(m) \sim -3 \times 10^{-2}$ is presented. The four segments of stable solutions and three segments of unstable solutions of period-4 motions are clearly observed. The harmonic amplitude $A_{1/4} \sim 2 \times 10^{-2}$ is presented in Fig. 3(b) and the two segments of stable solutions in the middle of frequency ranges are very tiny, which is difficult to be observed. The harmonic amplitude $A_{1/2} \sim 0.3$ is presented in Fig. 3(c) for the bifurcation of period-2 to period-4 motion. The harmonic amplitude $A_{3/4} \sim 10^{-1}$ in Fig. 3(d) is similar to the harmonic amplitude $A_{1/4}$. The harmonic amplitude $A_1 \sim 0.24$ is similar to $A_{1/2}$, as shown in Fig. 3(e). To avoid abundant illustrations, the harmonic amplitudes $A_{k/4}$ ($k = 4l + 1, l = 1, 2, \ldots$) will not be presented, which are similar to $A_{1/4}$. The harmonic amplitude $A_{3/2} \sim 5 \times 10^{-2}$ is presented in Fig. 3(f),

![Fig. 3. (Continued)](https://example.com/fig3continued.png)

![Fig. 4. Global view for frequency-amplitude curves ($Q_0 = 15$) based on 80 harmonic terms (HB80) of period-1 motion to period-4 motion in the parametric, nonlinear quadratic oscillator: (a) constant term $a_0^{(m)}$, (b) harmonic amplitude $A_k/m$ ($k = 1, 2, \ldots, 4, 8, 12, 80, m = 4$). ($\delta = 0.5$, $\alpha = 5$, $\beta = 20$)](https://example.com/fig4.png)
Fig. 4. (Continued)
which is different from $A_{1/2}$. The harmonic amplitude $A_2 \sim 2.5 \times 10^{-2}$ is similar to $A_1$, as shown in Fig. 3(g). For this bifurcation tree, the harmonic amplitude $A_6 \sim 2 \times 10^{-7}$ is presented in Fig. 3(h).

For a linear parametric system, one can find period-2 motions instead of period-1 motion. However, for a nonlinear parametric system, period-1 motions can be found. The global view of the frequency–amplitude for bifurcation trees of period-1 motion to period-4 motion is presented for $Q_0 = 15$. Such illustrations of constants and harmonic amplitude ($a_0^{(m)}, A_{1/4}, A_{1/2}, A_{1/4}, A_1, A_2, A_3, A_2, A_2$) are presented for the frequency range of $\Omega \in (0, 3.0)$ in Figs. 4(a)–4(h), respectively. In Fig. 4(a), the constant $a_0^{(m)} \in (-0.4, 0.2)$ versus excitation frequency is presented. There are a few branches of bifurcation trees and the stable and unstable solutions of period-1 to period-4 motions are crowded together. The saddle-node bifurcation of period-1 motion occurs between the stable and unstable period-1 motions without the onset of a new periodic motion. In addition to unstable period-1 motion, the Hopf bifurcation of stable period-1 motion will generate the onset of period-2 motion. Continually, the saddle-node bifurcation of period-1 motion is the Hopf bifurcation of period-1 motion, and the Hopf bifurcation of period-2 motion is the onset of period-4 motion with a saddle-node bifurcation. For period-4 motion, the harmonic amplitude $A_{1/4} \sim 0.25$ is presented in Fig. 4(b). Most of the solutions are unstable, and the stable solutions are in a few short ranges. In addition, independent unstable period-4 motions are observed. The harmonic amplitude $A_{1/2} \sim 0.4$ is presented in Fig. 4(c). The onset of period-2 motion is at the Hopf bifurcation of the period-1 motion. Nonindependent period-4 motions are generated by the Hopf bifurcations of period-2 motions, but the independent period-4 motion is separate from period-2 motions. For the period-4 motion, the harmonic amplitude $A_{1/4} \sim 0.4$ is presented in Fig. 4(d), which is similar to $A_{1/4}$. The harmonic amplitude $A_1 \sim 0.7$ is presented in Fig. 4(e). To show the quantity levels of harmonic amplitudes, the harmonic amplitude $A_2 \sim 0.25$ is presented in Fig. 4(f). The harmonic amplitude $A_3 \sim 0.25$ is presented in Fig. 4(g), which is different from $A_2$. For lower frequency, the quantity levels of $A_1$ and $A_2$ are the same, but for higher frequency, the quantity level of $A_1$ is much higher than the quantity level of $A_2$. To illustrate such change, the harmonic amplitude $A_{20}$ is presented with a common logarithmic scale. The quantity level of the harmonic amplitude $A_{20}$ decreases with excitation frequency with a power law. For stable solutions, the harmonic amplitude $A_{20} \sim 10^{-13}$ is observed.

Since the global view of the bifurcation trees from period-1 motion to period-4 motion is very crowded, it is very difficult to observe the nonlinear characteristics. The local view of frequency–amplitude curves is presented in Fig. 5 with all possible solutions together. In Figs. 5(a) and 5(b), two local views of constant term $a_0^{(m)}$ versus excitation frequency are presented for $\Omega \in (1.6, 2.1)$ and

![Fig. 5. Local views for frequency–amplitude curves ($Q_0 = 15$) based on 80 harmonic terms (HB80) of period-1 motion to period-4 motion in the parametric, nonlinear quadratic oscillator: (a) and (b) constant term $a_0^{(m)}$; (c)–(n) harmonic amplitude $A_{1/4}^m$ ($k = 1, 2, \ldots, 4, 6, 8, 12, m = 4$). ($\delta = 0.5, \alpha = 5, \beta = 20$).](image-url)
Fig. 5. (Continued)
Fig. 5. (Continued)
Fig. 6. Bifurcation tree-1 of period-1 motion to period-4 motion \((Q_0 = 15)\) based on 80 harmonic terms \((\eta_{80})\) in the parametric, nonlinear quadratic oscillator: (a) constant term \(a_0\), (b)–(f) harmonic amplitude \(A_k/m\) \((k = 1, 4, \ldots, 16, 36, m = 4)\). \((\delta = 0.5, \alpha = 5, \beta = 20)\).
Fig. 7. Bifurcation tree-2 of period-1 motion to period-2 motion (\(Q_0 = 15\)) based on 80 harmonic terms (HB80) in the parametric, nonlinear quadratic oscillator: (a) constant term \(a_0^m\), (b)–(h) harmonic amplitude \(A_{m/m}^k\) \((k = 1, 2, \ldots, 8, 18, m = 2)\). (\(\delta = 0.5, \alpha = 5, \beta = 20\)).
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Fig. 8. Bifurcation tree-3 of period-1 motion to period-4 motion ($Q_0 = 15$) based on 80 harmonic terms (HB80) in the parametric, nonlinear quadratic oscillator: (a) constant term $a_0^m$, (b)–(f) harmonic amplitude $A_k/m$ ($k = 1, 4, \ldots, 16, 36$, $m = 4$). ($\delta = 0.5$, $\alpha = 5$, $\beta = 20$).
Fig. 9. Bifurcation tree-4 of period-1 motion to period-4 motion ($Q_0 = 15$) based on 80 harmonic terms (HB80) in the parametric, nonlinear quadratic oscillator: (a) constant term $a_{0/m}$, (b)-(f) harmonic amplitude $A_{k/m}$ ($k = 1, 4, \ldots, 16, 36, m = 4$). ($\delta = 0.5, \alpha = 5, \beta = 20$).
\( \Omega \in (2.4, 2.75) \), respectively. From the local view, the relation of frequency–amplitude is very clearly presented. The frequency–amplitude curves are complicated, which means multiple coexisting solutions exist. In Fig. 5(a), the lower part relative to stable solutions are presented, and the solution varying with excitation frequency is observed. In Fig. 5(b), the upper part relative to stable solutions are presented. In Fig. 5(c), the harmonic amplitude \( A_{1/3} \) relative to the stable solution of period-4 motion is zoomed. To observe period-2 motion and period-4 motion, two local views of harmonic amplitudes \( A_{1/3} \) are presented for \( \Omega \in (1.6, 2.1) \) and \( \Omega \in (2.4, 2.7) \) in Figs. 5(d) and 5(e), respectively. The stability and bifurcations of period-2 and period-4 motions are observed clearly. The harmonic amplitude \( A_{1/3} \) relative to the stable solution of period-4 motion is zoomed in Fig. 5(f), which is similar to the harmonic amplitude \( A_{1/3} \) for period-4 motion only herein. The two zoomed areas for harmonic amplitude \( A_1 \) are presented in Figs. 5(g) and 5(h) for \( \Omega \in (1.6, 2.1) \) and \( \Omega \in (2.4, 2.75) \). Again, the frequency–amplitude relations of coexisting solutions of period-1, period-2 and period-4 motions are very crowded in the two zoomed local views. Similarly, the two zoomed areas for the harmonic amplitude \( A_{1/2} \) are presented for \( \Omega \in (1.6, 2.1) \) and \( \Omega \in (2.4, 2.7) \), in Figs. 5(i) and 5(j). The frequency–amplitude curves for period-2 and period-4 motions are still fully occupied. To look into higher order harmonic terms effects, the harmonic amplitudes \( A_2 \) and \( A_3 \) are presented in Figs. 5(k)–5(n). In Figs. 5(k) and 5(l), two local views for harmonic amplitude \( A_2 \) are presented for \( \Omega \in (2.4, 2.75) \) and \( \Omega \in (1.6, 2.1) \), respectively. The more detailed view is observed. Similarly, two local views for harmonic amplitude \( A_3 \) are presented in Figs. 5(m) and 5(n) for \( \Omega \in (2.4, 2.75) \) and \( \Omega \in (1.6, 2.1) \), respectively. In similar fashion, other higher order harmonic terms can be illustrated. To avoid abundant illustrations, they will not be presented. From the local views, the bifurcation trees of period-1 to period-4 motions are not clear because of too many coexisting stable and unstable solutions. The further refined local views should be presented to demonstrate the bifurcation trees of period-1 to period-4 motion.

To show bifurcation trees of period-1 to period-4 motion, the harmonic amplitudes \( A_{1/4}^{(m)} \) and \( A_{k/4}^{(m)} \) \( m = 4, k = 4l, l = 1, 2, 3, 4, 9 \) are presented for the bifurcation tree-1 with \( \Omega \in (2.55, 2.71) \) in Figs. 6(a)–6(f), respectively. The bifurcation tree-2 for \( \Omega \in (2.40, 2.7) \) is based on the period-1 motion to period-2 motion, and the harmonic amplitudes \( A_{1/2}^{(m)} \) and \( A_{k/2}^{(m)} \) \( k = 2l, l = 1, 2, 3, 4, 9 \) are illustrated in Figs. 7(a)–7(f). In Figs. 8(a)–8(f), the harmonic amplitudes \( A_{1/4}^{(m)} \) and \( A_{k/4}^{(m)} \) \( m = 4, k = 4l, l = 1, 2, 3, 4, 9 \) for the bifurcation tree-3 are presented for \( \Omega \in (1.98, 2.06) \). The bifurcation tree-4 for \( \Omega \in (1.665, 1.685) \) is presented in Figs. 9(a)–9(f) through the harmonic amplitudes \( A_{1/4}^{(m)} \) \( m = 4, k = 4l, l = 1, 2, 3, 4, 9 \).

4. Numerical Simulations

In this section, numerical simulations of periodic motions in parametric quadratic nonlinear oscillator will be completed for illustrations of periodic motions. The initial conditions for numerical simulations are computed from approximate analytical solutions of periodic solutions. The numerical and analytical results can be compared. In all plots, circular symbols give approximate analytical solutions, and solid curves give numerical results. The numerical solutions of periodic motions are generated via the midpoint scheme.

For system parameters \( \delta = 0.5, \alpha = 5, \beta = 20 \), a period-2 motion \( (\Omega = 6.8) \) is presented in Fig. 10 for \( Q_0 = 15 \) and the initial condition \( (x_0 \approx -0.174477, y_0 \approx 0.065932) \) is computed from the analytical solution with ten harmonic terms \((BH10)\). Displacement and velocity responses are presented in Figs. 10(a) and 10(b), respectively. The displacement and velocity curves for such a periodic motion are not simple sinusoidal curves. In Fig. 10(c), the trajectory of period-2 motion is presented. The asymmetry of periodic motion with one cycle is observed but not a simple cycle. The amplitude spectrum based on five harmonic terms is presented in Fig. 10(d). The main harmonic amplitudes are \( A_1^{(2)} \approx -0.029046, A_{1/2} \approx 0.130685, A_2 \approx 5.859010E-3, A_{1/2} \approx 9.633124E-3 \). The other harmonic amplitudes are \( A_3 \approx 10^{-4}, A_{1/2} \approx 2.5 \times 10^{-4}, A_3 \approx 3 \times 10^{-6}, A_{1/2} \approx 3.2 \times 10^{-6}, A_4 \approx 10^{-7}, A_{1/2} \approx 2.6 \times 10^{-8}, \) and \( A_4 \approx 2 \times 10^{-9} \). For this periodic motion, the harmonic amplitude \( A_{1/2} \approx 0.130685 \) plays an important role in period-2 motion. For \( k \geq 6 \), the harmonic amplitudes \( A_{1/2} \) can be ignored.

For the aforementioned period-2 motion, once the Hopf bifurcation occurs, the period-1 motion
can be observed in such a bifurcation tree. Thus, with system parameters ($\delta = 0.5$, $\alpha = 5$, $\beta = 20$), a period-4 motion ($\Omega = 6.78$) is illustrated in Fig. 11 for $Q_0 = 15$ and the initial condition ($x_0 \approx -0.159366$, $y_0 \approx 0.065932$) is computed from the analytical solution with 24 harmonic terms (HB24). The time-histories of displacement and velocity are presented in Figs. 11(a) and 11(b), respectively. The displacement and velocity curves for such a periodic motion are periodic with $4T$. In Fig. 11(c), the trajectory of period-4 motion is presented, and there are two cycles instead of one cycle in the period-2 motion. The amplitude spectrum based on 24 harmonic terms is presented in Fig. 11(d). The main harmonic amplitudes are $A_{01} \approx -0.030139$, $A_{1/4} \approx 0.013961$, $A_{1/2} \approx 0.132269$, $A_{1/4} \approx 2.824103E-3$, $A_{1} \approx 6.130666E-3$, $A_{3/4} \approx 1.471135E-3$, $A_{1/2} \approx 9.793038E-3$, $A_{5/4} \approx 1.048286E-4$. The other harmonic amplitudes are $A_{2} \approx 1.3 \times 10^{-4}$, $A_{9/4} \approx 4 \times 10^{-5}$, $A_{11/2} \approx 2.5 \times 10^{-4}$, $A_{13/4} \approx 9.6 \times 10^{-5}$, $A_{3} \approx 3 \times 10^{-4}$, $A_{13/4} \approx 5.4 \times 10^{-5}$, $A_{7/2} \approx 3.3 \times 10^{-4}$, $A_{15/4} \approx 1.8 \times 10^{-4}$, $A_{4} \approx 10^{-5}$, $A_{17/4} \approx 4 \times 10^{-5}$, $A_{13/2} \approx 2.6 \times 10^{-4}$, $A_{19/4} \approx 5.2 \times 10^{-5}$, $A_{5} \approx 1.8 \times 10^{-9}$, $A_{21/2} \approx 2.1 \times 10^{-11}$, $A_{11/2} \approx 1.6 \times 10^{-10}$, $A_{19/4} \approx 7 \times 10^{-12}$, and $A_{6} \approx 2.2 \times 10^{-11}$. For this periodic motion, the harmonic amplitudes of $A_{1/4}$ make important contributions on the period-4 motion. For $k \geq 12$, the harmonic amplitudes $A_{1/4}$ can be ignored.

For another branch of period-2 motion to period-4 motion, to save space, only phase
Periodic Motions to Chaos in a Parametric, Quadratic Nonlinear Oscillator

Fig. 11. Period-4 motion ($\Omega = 6.78$): (a) displacement, (b) velocity, (c) trajectory and (d) amplitude. Initial condition ($x_0 \approx -0.159366, y_0 \approx 0.065306$). ($\beta = 0.5, \alpha = 5, \beta = 20, Q_0 = 15$).

trajectories and spectrums for period-2 motions will be presented in Fig. 12 for $\Omega = 3.042402$ with $(x_0 \approx -0.427224, y_0 \approx 0.753339)$ and $\Omega = 2.89$ with $(x_0 \approx -0.000109, y_0 \approx 0.429608)$. In Fig. 12(a), the trajectory of period-2 motion with $\Omega = 3.042402$ is presented. The analytical solutions possess 20 harmonic terms ($H_{20}$).

The harmonic amplitude in spectrum is shown in Fig. 12(b). The main harmonic amplitudes are $A_{10} \approx -0.043146, A_{1/2} \approx 0.382101, A_1 \approx 0.299341, A_{1/2} \approx 0.057045$ and $A_2 \approx 0.054044$. The other harmonic amplitudes are $A_{3/2} \approx 6.2 \times 10^{-3}$, $A_3 \approx 1.1 \times 10^{-4}$, $A_{7/2} \approx 6.4 \times 10^{-4}$, $A_4 \approx 2.8 \times 10^{-3}$, $A_{9/2} \approx 4.5 \times 10^{-5}$, and $A_5 \approx 1 \times 10^{-5}$. $A_{k/2} \in (10^{-10}, 10^{-5})$ for $k = 11, 12, \ldots, 20$. For this periodic motion, the harmonic amplitudes $A_{1/2}$ to $A_2$ make significant contributions on period-2 motion rather than majorly from $A_{1/2}$, $A_1$ and $A$ are two most important terms. For $k \geq 10$, the harmonic amplitudes $A_{k/2}$ can be ignored. In Fig. 12(c), the phase trajectory of period-2 motion for $\Omega = 2.89$ is illustrated, which is different from the one in Fig. 12(a). The corresponding spectrum of period-2 motion is presented in Fig. 12(d). The main harmonic amplitudes are $A_{10} \approx -3.292999E-3, A_{1/2} \approx 0.158649, A_1 \approx 0.047177, A_{1/2} \approx 0.084358, A_2 \approx 0.020375$ and $A_{3/2} \approx 0.010720$. The other harmonic amplitudes are $A_3 \approx 3.4 \times 10^{-3}, A_{7/2} \approx 3.0 \times 10^{-4}, A_4 \approx 2.9 \times 10^{-3}, A_{9/2} \approx 5.7 \times 10^{-5}, A_5 \approx 1.1 \times 10^{-5}$. $A_{k/2} \in (10^{-11}, 10^{-5})$ for $k = 11, 12, \ldots, 20$. For this period-2 motion, the harmonic amplitudes ($A_{1/2}$ to $A_{3/2}$) make significant
Fig. 12. Period-2 motion ($\Omega = 3.042402$) with initial condition ($x_0 \approx -0.427224$, $y_0 \approx 0.733339$). (a) trajectory and (b) amplitude. Period-4 motion ($\Omega = 2.89$) with initial condition ($x_0 \approx -0.001019$, $y_0 \approx 0.625068$). (c) trajectory and (d) amplitude. ($\delta = 0.5$, $\alpha = 5$, $\beta = 20$, $Q_0 = 15$).

Next illustrations are trajectories for period-1 motion to period-4 motions. In linear parametric oscillators, no such period-1 motions can be observed. However, the nonlinear parametric oscillator possesses bifurcation tree of period-1 motion to chaos. For different branches, nonlinear dynamical behaviors are different, as shown in Figs. 13-15. To save space, only phase trajectories and spectrums are illustrated, and the initial conditions are listed in Table 1.

On the bifurcation tree relative to period-1 motion with $\Omega = 1.665$, the trajectories and amplitude spectrums of the period-1 motion ($\Omega = 1.665$), period-2 motion ($\Omega = 1.67$) and period-4 motion ($\Omega = 1.673$) are presented in Figs. 13(a)-13(f), respectively. In Fig. 13(a), the trajectory of period-1 motion for $\Omega = 1.665$ possesses two cycles because the second harmonic term ($A_2$) plays an important role on the period-1 motion. The main harmonic amplitudes are $A_0 = 5.583469 \times 10^{-3}$, $A_1 = 1.734440 \times 10^{-3}$, $A_2 = 9.002613 \times 10^{-3}$, $A_3 = 3.923093 \times 10^{-3}$, $A_4 = 7.304942 \times 10^{-4}$, $A_5 = 7.274669 \times 10^{-5}$. The other harmonic amplitudes are $A_6 \approx 2.5 \times 10^{-6}$, $A_7 \approx 4.1 \times 10^{-7}$, $A_8 \approx 8.3 \times 10^{-9}$, $A_9 \approx 3.8 \times 10^{-10}$. With increasing excitation frequency, period-2 motion can be observed. In Fig. 13(c), the trajectory of period-2 motion for $\Omega = 1.67$ possesses four cycles. The distribution of harmonic amplitudes
Fig. 13. Period-1 motion (Ω = 1.665) with \((x_0 \approx -0.012807, y_0 \approx 2.85163E^{-3})\): (a) trajectory and (b) amplitude. Period-2 motion (Ω = 1.67) with \((x_0 \approx -0.019150, y_0 \approx 5.13417E^{-3})\): (c) trajectory and (d) amplitude. Period-4 motion (Ω = 1.673) with \((x_0 \approx -0.021881, y_0 \approx 6.07529E^{-3})\): (e) trajectory and (f) amplitude. \((\delta = 0.5, \alpha = 5, \beta = 20, Q_0 = 15)\).
Fig. 14. Period-1 motion ($\Omega = 2.06$) with $(x_0 \approx -0.298897, y_0 \approx 0.120757)$: (a) trajectory and (b) amplitude. Period-2 motion ($\Omega = 2.02$) with $(x_0 \approx -0.369674, y_0 \approx 0.302141)$: (c) trajectory and (d) amplitude. Period-4 motion ($\Omega = 2.0$) with $(x_0 \approx -0.354555, y_0 \approx 0.267075)$: (e) trajectory and (f) amplitude. ($\delta = 0.5, \alpha = 5, \beta = 20, Q_0 = 15$).
Displacement, \( x \)
-0.16 0.02 0.12 0.26

Velocity, \( y \)
-0.6 -0.2 0.2 0.6 1.0

I.C.

Harmonics Order, \( k \)
0 1 2 3 4 5 6

Amplitude, \( A \)
0 0.05 0.10 0.15 0.20

6 \( \times \) 7 \( \times \) 8 \( \times \) 9 \( \times \) 10 \( \times \)
1e-12 1e-11 1e-10 1e-9 1e-8 1e-7 1e-6 1e-5 1e-4

\( (a) \) (b) \( (c) \) (d) \( (e) \) (f)

Fig. 15. Period-1 motion (\( \Omega = 2.625 \)) with \( (x_0 \approx -0.050637, y_0 \approx 0.679436) \): (a) trajectory and (b) amplitude. Period-2 motion (\( \Omega = 2.613 \)) with \( (x_0 \approx -0.040570, y_0 \approx 0.606281) \): (c) trajectory and (d) amplitude. Period-4 motion (\( \Omega = 2.608 \)) with \( (x_0 \approx -0.098209, y_0 \approx 0.849246) \): (e) trajectory and (f) amplitude. \( (\delta = 0.5, \alpha = 5, \beta = 20, Q_0 = 15)\).
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Table 1. Input data for numerical simulations ($\delta = 0.5, \alpha = 5, \beta = 20$).

<table>
<thead>
<tr>
<th>Frequency $\Omega$</th>
<th>Initial Conditions $((x_0, y_0))$</th>
<th>Type and Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figs. 13(a), 13(b)</td>
<td>1.665 ($(-0.012807, 2.851630E-3)$)</td>
<td>Period-1 motion (stable)</td>
</tr>
<tr>
<td>Figs. 13(c), 13(d)</td>
<td>1.670 ($(-0.035110, 5.134178E-3)$)</td>
<td>Period-2 motion (stable)</td>
</tr>
<tr>
<td>Figs. 13(e), 13(f)</td>
<td>1.673 ($(-0.021881, 6.075296E-3)$)</td>
<td>Period-4 motion (stable)</td>
</tr>
<tr>
<td>Figs. 14(a), 14(b)</td>
<td>2.606 ($(-0.288997, 0.130757)$)</td>
<td>Period-1 motion (stable)</td>
</tr>
<tr>
<td>Figs. 14(c), 14(d)</td>
<td>2.602 ($(-0.306764, 0.302141)$)</td>
<td>Period-2 motion (stable)</td>
</tr>
<tr>
<td>Figs. 14(e), 14(f)</td>
<td>2.600 ($(-0.354555, 0.267075)$)</td>
<td>Period-4 motion (stable)</td>
</tr>
<tr>
<td>Figs. 15(a), 15(b)</td>
<td>2.625 ($(-0.056265, 0.659436)$)</td>
<td>Period-1 motion (stable)</td>
</tr>
<tr>
<td>Figs. 15(c), 15(d)</td>
<td>2.613 ($(-0.040570, 0.606281)$)</td>
<td>Period-2 motion (stable)</td>
</tr>
<tr>
<td>Figs. 15(e), 15(f)</td>
<td>2.608 ($(-0.092420, 0.849246)$)</td>
<td>Period-4 motion (stable)</td>
</tr>
</tbody>
</table>

The harmonic amplitudes are $A_{\Omega} \in (10^{-9}, 10^{-5})$ ($k = 11, 12, \ldots, 20$). For the bifurcation tree relative to period-1 motion with $\Omega = 2.06$, the trajectories and amplitude spectrums of the period-1 motion ($\Omega = 2.06$), period-2 motion ($\Omega = 2.02$) and period-4 motion ($\Omega = 2.0$) are presented in Figs. 14(a)–14(f), respectively. In Fig. 14(a), the phase trajectory of period-1 motion for $\Omega = 2.06$ possesses two cycles because the second harmonic terms ($A_1$ and $A_2$) play an important role on the period-1 motion. In Fig. 14(b), the main harmonic amplitudes are $A_0 = 0.121232$, $A_1 = 0.242535$, $A_2 = 0.175221$, $A_3 = 0.016871$, $A_4 = 4.587799E-4$, $A_5 = 7.471614E-4$, and $A_6 = 8.436413E-5$. The other harmonic amplitudes are $A_2 \sim 2.2 \times 10^{-5}$, $A_3 \sim 1.2 \times 10^{-6}$, $A_7 \sim 5.3 \times 10^{-7}$, $A_{10} \sim 2.9 \times 10^{-9}$, $A_{11} \sim 1.1 \times 10^{-9}$, $A_{12} \sim 9.6 \times 10^{-10}$. With decreasing excitation frequency, period-2 motion can be observed. In Fig. 14(c), the trajectory of period-2 motion for $\Omega = 2.02$ possesses three cycles. The distribution of harmonic amplitudes is presented in Fig. 14(d). The main harmonic amplitudes for the period-2 motion are

$A_0 \approx 0.099980$, $A_{12} \approx 0.115545$, $A_{13} \approx 0.026705$, $A_{14} \approx 0.152269$, $A_{15} \approx 0.041774$, $A_{16} \approx 6.635328E-3$, $A_{17} \approx 3.946919E-3$, $A_{18} \approx 1.854786E-3$, $A_{19} \approx 4.907318E-4$, $A_{20} \approx 1.425724E-4$, and $A_{21} \approx 1.136800E-4$. The other harmonic amplitudes are $A_{1/2} \in (10^{-9}, 10^{-5})$ ($k = 13, 14, \ldots, 24$). For period-4 motion ($\Omega = 2.0$, the corresponding trajectory with six cycles is presented in Fig. 14(e), and the harmonic amplitude distribution in spectrum is presented in
Fig. 14(f). The main harmonic amplitudes for the period-4 motion are

\[ a_0^{(4)} \approx 0.090440, \quad A_{1/4} \approx 0.012614, \]

\[ A_{1/2} \approx 0.035542, \quad A_{1/4} \approx 5.508805 \times 10^{-3}, \]

\[ A_{1/2} \approx 0.177238, \quad A_{1/4} \approx 0.018558, \]

\[ A_{1/2} \approx 0.120652, \quad A_{1/4} \approx 8.798937 \times 10^{-3}, \]

\[ A_2 \approx 0.143285, \quad A_{2/4} \approx 9.725779 \times 10^{-3}, \]

\[ A_{1/2} \approx 0.031754, \quad A_{1/4} \approx 2.888349 \times 10^{-3}, \]

\[ A_4 \approx 0.015434, \quad A_{4/4} \approx 6.862216 \times 10^{-4}, \]

\[ A_{1/2} \approx 5.685201 \times 10^{-3}, \quad A_{1/4} \approx 4.777341 \times 10^{-4}, \]

\[ A_4 \approx 3.839157 \times 10^{-3}, \quad A_{1/4} \approx 3.436165 \times 10^{-4}, \]

\[ A_{1/2} \approx 2.045764 \times 10^{-3}, \quad A_{1/4} \approx 1.051130 \times 10^{-4}, \]

\[ A_4 \approx 5.735349 \times 10^{-4}, \quad A_{2/4} \approx 2.831308 \times 10^{-5}, \]

\[ A_{1/2} \approx 8.602605 \times 10^{-5}, \quad A_{2/4} \approx 3.397351 \times 10^{-5}, \]

\[ A_6 \approx 1.089921 \times 10^{-4}. \]

The other harmonic amplitudes are \( A_{k/4} \in (10^{-11}, 10^{-5}) \) for \( k = 13, 14, \ldots, 48 \).

For the bifurcation tree relative to period-1 motion with \( \Omega = 2.625 \), the trajectories and amplitude spectra of the period-1 motion (\( \Omega = 2.625 \)) are presented in Figs. 15(a)–15(f), respectively. In Fig. 15(a), the phase trajectory of period-1 motion for \( \Omega = 2.625 \) possesses one cycle because the first harmonic term \( (A_1) \) plays an important role on the period-1 motion. In Fig. 15(b), the main harmonic amplitudes are \( A_0 = -0.030284, \quad A_{1/4} = 0.142279, \quad A_{1/2} = 0.047170, \quad A_2 = 6.332490 \times 10^{-5}, \quad A_4 = 5.863546 \times 10^{-5}, \quad A_{1/2} = 5.990229 \times 10^{-6} \). The other harmonic amplitudes are \( A_2 \approx 5.0 \times 10^{-7}, \quad A_8 \approx 4.5 \times 10^{-6}, \quad A_{10} \approx 3.9 \times 10^{-7}, \quad A_{10} \approx 2.4 \times 10^{-10}, \quad A_{11} \approx 2.9 \times 10^{-11} \). With decreasing excitation frequency, period-2 motion can be observed. In Fig. 15(c), the trajectory of period-2 motion for \( \Omega = 2.613 \) possesses two cycles. The distribution of harmonic amplitudes is presented in Fig. 15(d). The main harmonic amplitudes for the period-2 motion are

\[ a_0^{(2)} \approx -0.022574, \]

\[ A_{1/2} \approx 0.024025, \quad A_1 \approx 0.153918, \]

\[ A_{1/2} \approx 0.016735, \quad A_2 \approx 0.051052, \]

\[ A_{1/2} \approx 3.589764 \times 10^{-3}, \quad A_1 \approx 6.870026 \times 10^{-3}, \]

\[ A_{1/2} \approx 5.394541 \times 10^{-4}, \quad A_1 \approx 6.614716 \times 10^{-4}, \]

\[ A_{1/2} \approx 7.394803 \times 10^{-5}, \quad A_1 \approx 6.848459 \times 10^{-5}, \]

\[ A_{1/2} \approx 8.867566 \times 10^{-6}, \quad A_1 \approx 6.699010 \times 10^{-6}. \]

The other harmonic amplitudes are \( A_{k/2} \in (10^{-9}, 10^{-7}) \) for \( k = 13, 14, \ldots, 22 \). For period-4 motion (\( \Omega = 2.608 \)), the corresponding trajectory with four cycles is presented in Fig. 15(e), and the harmonic amplitude distribution in spectrum is presented in Fig. 15(f). The main harmonic amplitudes for the period-4 motion are

\[ a_0^{(4)} \approx -0.031233, \quad A_{1/4} \approx 2.682261 \times 10^{-4}, \]

\[ A_{1/2} \approx 0.033327, \quad A_{1/4} \approx 5.144045 \times 10^{-3}, \]

\[ A_1 \approx 0.155966, \quad A_{1/4} \approx 3.383796 \times 10^{-3}, \]

\[ A_{1/2} \approx 0.023219, \quad A_{1/4} \approx 2.560205 \times 10^{-3}, \]

\[ A_2 \approx 0.051662, \quad A_{1/4} \approx 7.967204 \times 10^{-3}, \]

\[ A_{1/2} \approx 4.893564 \times 10^{-3}, \quad A_{1/4} \approx 4.856891 \times 10^{-3}, \]

\[ A_3 \approx 6.929255 \times 10^{-3}, \quad A_{1/4} \approx 1.001560 \times 10^{-3}, \]

\[ A_{1/2} \approx 7.514113 \times 10^{-4}, \quad A_{1/4} \approx 7.001600 \times 10^{-4}, \]

\[ A_4 \approx 6.697876 \times 10^{-4}, \quad A_{1/4} \approx 1.153707 \times 10^{-3}, \]

\[ A_{1/2} \approx 1.033680 \times 10^{-4}, \quad A_{1/4} \approx 9.237744 \times 10^{-5}, \]

\[ A_5 \approx 7.010572 \times 10^{-5}, \quad A_{1/4} \approx 1.138380 \times 10^{-5}, \]

\[ A_{1/2} \approx 1.242875 \times 10^{-5}, \quad A_{1/4} \approx 1.087214 \times 10^{-6}, \]

\[ A_6 \approx 6.901329 \times 10^{-6}. \]

The other harmonic amplitudes are \( A_{k/4} \in (10^{-11}, 10^{-7}) \) for \( k = 13, 14, \ldots, 48 \).

5. Conclusions

In this paper, the analytical solutions of periodic motions in a parametric quadratic nonlinear oscillator are determined through the finite Fourier series, and the corresponding stability and bifurcations of periodic motions are discussed by eigenvalue analysis. The analytical bifurcation trees of periodic motions to chaos in such a parametric oscillator were obtained. Nonlinear behaviors of such periodic motions are characterized through frequency–amplitude curves of each harmonic term in the finite Fourier series solution. Numerical illustrations of periodic motions were carried out through phase trajectories and analytical spectrum. This investigation shows period-1 motions exist in parametric nonlinear systems and the corresponding bifurcation trees of period-1 and period-2 motions to chaos exist as well.
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