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MULTICOLORED TREES IN RANDOM GRAPHS

by

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1 INTRODUCTION

Let $G = (V, E)$ be a graph in which the edges are coloured. A set $S \subseteq E$ is said to be multicoloured if each edge of $S$ is a different colour. A spanning tree of $G$ is said to be multicoloured if its edge set is. In this paper we study
the existence of a multicoloured spanning tree (MST) in a randomly coloured random graph.

In fact, our main result will concern a randomly coloured graph process. Here $e_1, e_2, \ldots, e_N$ is a random permutation of the edges of the complete graph $K_n$ and so $N = \binom{n}{2}$. Each edge $e$ independently chooses a random colour $c(e)$ from a given set of colours $W$, $|W| \geq n - 1$.

The graph process consists of the sequence of random graphs $G_m, m = 1, 2, \ldots, N$, where $G_m = ([n], E_m)$ and $E_m = \{e_1, e_2, \ldots, e_m\}$. We identify the following events:

$\mathcal{C}_m = \{G_m \text{ is connected}\}$.

$\mathcal{N}_m = \{|c(E_m)| \geq n - 1\}$, where $c(E_m)$ is the set of colours used by $E_m$.

$\mathcal{M}T_m = \{G_m \text{ has a multicoloured spanning tree}\}$.

Let $\mathcal{E}_m$ stand for one of the above three sequences of events and let

$m_\mathcal{E} = \min\{m : \mathcal{E}_m \text{ occurs}\},$

provided such an $m$ exists. Clearly, if $m_\mathcal{M}T$ is defined,

$m_\mathcal{M}T \geq \max\{m_\mathcal{C}, m_\mathcal{N}\},$

and the main result of the paper is

**Theorem 1** *In almost every (a.e.) randomly coloured graph process*

$m_\mathcal{M}T = \max\{m_\mathcal{C}, m_\mathcal{N}\}.$

\(\square\)
To establish the existence of an MST we use a result of Edmonds [2] on the matroid intersection problem. In this scenario $M_i, M_j$ are matroids over a common ground set $E$ with rank functions $r_1, r_2$ respectively. Edmonds' general theorem on this problem is

$$\max(|I| : I \text{ is independent in both matroids}) = \min_{E_1 \cap E_2 = \emptyset} (r_1(E_1) + r_2(E_2))$$

(1)

For us $M_i$ is the cycle matroid of a graph $G = G_m$ and $M_2$ is the partition matroid associated with the colours. Thus for a set of edges $E$, $r_1(E) = n - K(E)$ where $K(E)$ is the number of components of the graph $G_S = ([V], E)$ and $r_2(E)$ is the number of distinct colours occurring in $E$. If $i \in G_W$ then $C_i$ denotes the set of edges of colour $i$ and for $I \subseteq W$, $C_I = \bigcup_{i \in I} C_i$. We will use Edmonds' theorem in the following form:

**Theorem 2** Suppose $|W| = n - 1$. Then a necessary and sufficient condition for the existence of an MST is that

$$K(d) \leq n - |I|$$

for all $I \subseteq W$.  

(2)

[To see this, w.l.o.g. restrict attention in (1) to $E_2$ of the form $C_i$ and then take $I = W \setminus \text{jin} (2).$]

2 Proof of Theorem 1

Observe first that if $u = u(n) \rightarrow \infty$ slowly, then in a.e. randomly coloured graph process

$$m_e \geq m_0 = \left\lfloor \frac{1}{2} n \left( \ln n - \omega \right) \right\rfloor \text{ and } m_N \leq m_1 = \left\lceil n \left( \ln n + \omega \right) \right\rceil.$$
We will start by justifying a concentration on the case $|W| = n - 1$. We will describe a coupled process in which there are never more than $n - 1$ colours used: from $m_N$ onwards, the colours that have not yet been used are randomly changed to one of the $n - 1$ colours that have appeared so far. The relevant properties of this coupled process are

1. For each $m \in [m_0, m_1]$ the edges of $G_m$ are independently randomly coloured from a choice of $n - 1$ colours.

2. If $m_{MT} > \max\{m_C, m_N\}$ holds for the original process then it also holds for the coupled process.

Thus to prove our theorem we need only prove that

$$\Pr(m_{MT} > \max\{m_C, m_N\}) = o(1).$$

where $\Pr$ refers to the coupled process.

Fix some $m$ in the range $[m_0, m_1]$. We define the event

$$A_k = \{\exists I \subseteq W, |I| = k : \kappa(C_I) \geq n - |I| + 1\}.$$

We know that if $|W| = n - 1$, $G_m$ is connected and each colour is used at least once and there is no MST then $A_k$ occurs for some $k \in [3, n - 2]$ ($A_1 \cup A_2$ cannot occur if all $n - 1$ colours are used and $A_{n-1}$ cannot occur if $G_m$ is connected.) Take a minimal $k$, corresponding set $I$ and let $S = C_I$.

Claim 1 $G_S$ has no bridges.

Proof If there is a bridge, remove it and all edges of the same colour. Clearly $A_{k-1}$ occurs, contradicting the minimality of $k$. \qed
With the notation of Claim 1 suppose then that $G_s$ has $i$ isolated vertices and $n-k+x-i$ non-trivial components, $x \geq 1$. Since non-trivial components without bridges have at least three vertices,

$$i + 3(n - k + x - i) \leq n$$

or

$$i \rightarrow \frac{3}{2}k + \frac{3}{2}\ 
\geq \ n - k + \frac{x}{2}.$$

So now let $B_k$ denote the event

$$\{ 3/ C W, \forall \lambda = k, T C \subseteq [n] : \ t = \forall \lambda < 3(k - 1)/2, \text{ all edges coloured with } / \text{ are contained in } T, \text{ there are } u \geq \max\{fc,i\} /-\text{coloured edges} \}.$$ Here $T$ is the set of vertices in the non-trivial components of $G_c$. Thus if $\forall W = n-1$,

$$\forall W \subseteq n \ A_k \subseteq \bigcup_{i=3}^k J Bi \text{ for } k \geq 3.$$

For $k \geq 9n/10$ we consider a slightly different event.

We first rephrase (2) as

$$K(CW/J) \leq \forall \lambda + 1 \text{ for all } JCW.$$

So if $\forall W = n - 1$ and there is no MST then there exist $\ell \geq 1$ colours whose deletion produces $A \geq \ell + 2$ components of sizes $n_i, \ldots, n_\lambda$.

**Claim 2** Some subsequence of the $n_i$'s sums to between $\ell + 1$ and $n/2$. 

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Proof Assume \( n_1 < r_2 < \cdots < r_A \) and 
\[ n_i \leq \ell \Rightarrow 1 \leq i \leq A. \]

If \( n_i > \ell + 1 \), one of \( n_1, \ldots, n_{i-1} \) and \( r_A \) suffices.

Suppose then that \( n_i \leq \ell \), \( 1 \leq i \leq A \).

Choose \( r \) such that
\[
 n_i - h n_r \leq n/2, \quad nH - h n_{r+1} > n/2
\]
and then
\[
 n_i - h n_r > n/2 - n_{r+1} \\
\geq n/2 - \ell \\
\geq \ell.
\]

and we can take \( n_1, \ldots, n_r \).

Note next that if \( J \) is minimal in (5) then each colour in \( J \) appears at least twice as an edge joining components of \( G_{C \cup J} \).

So if \( G_m \) is connected and there is no MST and \( Ak \) does not occur for \( k \leq 9n/10 \) then there is a set \( L \) of \( 1 \leq \ell < n/10 \) colours and a set \( S \) of size \( s \), \( \ell + 1 \leq s \leq n/2 \) such that (i) all \( t = r/2 = \gamma(S : S \cap 1 \) edges are L-coloured, \( (5* : 3) \) is the set of edges joining \( 5* \) and \( 5* = V \setminus S \), (ii) the lexicographically first \( \max\{2^\ell <, 0\} \) non-(\( 5' : 3 \)) edges joining up components (of the \( \forall \) \( L \) coloured edges) are also L-coloured. Let \( T > \ell \) denote this event.

Then
\[
 C_m \cap \left( \bigcup_{k=9n/10}^{n-2} A_k \right) \subseteq C_{PT}(V_\ell).
\]

It follows from (4) and (6) that
\text{Pr}(m_{MT} > \max\{m_N, m_C\}) \leq \\
oindent o(1) + \sum_{m = m_0}^{m_1} \left[ \sum_{k = 3}^{9n/10} \text{Pr}_m(B_k) + \sum_{t = 2}^{n/10} \text{Pr}_m(D_t) \right] + \text{Pr}\left( \bigcup_{m = m_0}^{m_1} (C_m \cap A_{n-2}) \right). \quad (7)

Here \( \text{Pr}_m \) denotes probability w.r.t. \( G_m \) and the \( o(1) \) term is the probability that \( G_{m_0} \) is connected or that \( m_N > m_1 \). (Our calculations force us to separate out \( A_{n-2} \).)

We must now estimate the individual probabilities in (7). It is easier to work with the independent model \( G_p, p = m/N \), where each edge occurs independently with probability \( p \) and is then randomly coloured. For any event \( \mathcal{E} \) we have (see Bollobás [1] Chapter II) the simple bound

\[ \text{Pr}_m(\mathcal{E}) \leq 3\sqrt{n \ln n} \text{Pr}_p(\mathcal{E}). \quad (8) \]

where \( \text{Pr}_p \) denotes probability w.r.t. the model \( G_p \).

Now, where \( p = \alpha \ln n/n, 1 - o(1) \leq \alpha \leq 2 + o(1), \)

\[ \text{Pr}_p(B_k) \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max(t,k)}^{(i)} \binom{n}{t} \binom{n-1}{k} \binom{\binom{i}{2}}{u} \left(1 - \frac{kp}{n-1}\right)^{\binom{i}{2}-u} \left(\frac{kp}{n-1}\right)^u \]

\[ \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max(t,k)}^{(i)} \frac{n^t e^t n^k e^k}{t^t k^k} \left(\frac{t^2 e}{2u}\right)^u n^{-\alpha k (\frac{1}{2} - o(1))} \left(\frac{\alpha k \ln n}{n^2}\right)^u. \quad (9) \]

Case 1: \( 3 \leq k \leq k_0 = n/(3 \ln n). \)

\[ \text{Pr}_p(B_k) \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max(t,k)}^{(i)} \left(\frac{e^3 n^{1-o(\frac{1}{2} - o(1))}}{k}\right)^k \left(\frac{t}{n}\right)^{2u-t} \left(\frac{\alpha e k \ln n}{2u}\right)^u \]

\[ = \sum_{t=1}^{3(k-1)/2} \sum_{u=\max(t,k)}^{(i)} \left(\frac{e^3 n^{1-o(\frac{1}{2} - o(1))}}{k}\right)^k \left(\frac{t}{n}\right)^{u-t} \left(\frac{\alpha e k \ln n}{2un}\right)^u \]

7
\[
\sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{(\frac{t}{2})} \left( \frac{e^{3n^1-\alpha(\frac{1}{2} - \alpha(1))\alpha e^k \ln n}}{2kn} \right)^k \left( \frac{t}{n} \right)^{u-t} \left( \frac{\alpha e^k \ln n}{2n} \right)^{u-k} \\
= O \left( \left( \frac{\ln n}{n^{1-\alpha(1)}} \right)^k \right).
\]

It follows from this and (8) that
\[
\sum_{m=m_0}^{m_1} \sum_{k=4}^{k_0} \Pr_m(B_k) = O((n \ln n)(\sqrt{n \ln n}((\ln n)^4/n^{2-\alpha(1)})))
\]
\[
= o(1).
\]
(10)

For \( k = 3 \) we compute \( \Pr_m(B_3) \) directly, but since now \( u = t = k = 3 \) is forced,
\[
\Pr_m(B_3) \leq \left( \frac{n}{3} \right)^2 \left( 1 - \frac{3}{n-1} \right)^m \left( \frac{3}{n-1} \right)^3 \left( \frac{N-3}{m-3} \right) \left( \frac{N}{m} \right)^m
\]
\[
= O(e^{3n^1}(\ln n)^3n^{-3/2})
\]

and so
\[
\sum_{m=m_0}^{m_1} \Pr_m(B_3) = o(1).
\]
(11)

Case 2: \( k_0 < k \leq n/2 \).

We now write (9) as
\[
\Pr_p(B_k) \leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{(\frac{t}{2})} \left( \frac{e^{3n^1-\alpha(\frac{1}{2} - \alpha(1))\alpha e^k \ln n}}{2kn} \right)^k \left( \frac{t}{n} \right)^{u-t} \left( \frac{\alpha e^k \ln n}{2un} \right)^u
\]
\[
\leq \sum_{t=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{(\frac{t}{2})} \left( \frac{e^{3n^1-\alpha(\frac{1}{2} - \alpha(1))\alpha e^k \ln n}}{k} \right)^k \left( \frac{t}{n} \right)^{u-t} \frac{\alpha e^k \ln n}{n^{2\alpha k}}
\]

8
(after maximising the last term over \( u \))

\[
\begin{align*}
&= \sum_{i=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{(t)} \left( \frac{e^{3\alpha n^{1-\frac{\alpha}{2}(1-\frac{1}{2}-\alpha(1))}}}{k} \right)^{k} \left( \frac{t}{n} \right)^{u-1} \\
&\leq \sum_{i=1}^{3(k-1)/2} \sum_{u=\max\{t,k\}}^{(t)} \left( \frac{e^{3\alpha n^{1-\frac{\alpha}{2}(1-\frac{1}{2}-\alpha(1))}}}{k} \right)^{k} \tag{13}
\end{align*}
\]

since \( t \leq 3(\text{fc} - 1)/2 \leq 3n/4 \).

(13) and (8) clearly imply

\[
\sum_{m=m_0}^{n/2} \sum_{k=k_0}^{n/2} \Pr_m(B_k) = o(1). \tag{14}
\]

Case 3: \( n/2 < A \leq 9n/10 \)

**Claim 3** Choose any constant \( A > 0 \). Then, in a.e. process, simultaneously for each \( m \in [m_0, m_i] \), the sets of \( s \leq A \) vertices of \( G_m \) which span at least \( s \) edges together contain at most \((\ln n)^{A+1}\) vertices.

**Proof** We need only prove this for \( G_{m_i} \) and since the property is monotone decreasing we need only prove it for \( G_{p_l} \), \( p_l = m\hat{y}N \) ([1], Chapter II.)

But

\[
E_{p_l}(\text{number of vertices}) \leq \frac{\ln^A n}{k} \left( \frac{\alpha}{k} \right)^k \left( \frac{2A}{k} \right)^k \tag{15}
\]

Now use the Markov inequality. Q

It follows that we may rewrite (3) as

\[
i + 3(\ln n)^{A+1} + (A + 1)(n - k + x - i) \leq n
\]
and so
\[
i \geq n - \frac{A + 1}{A} k - O((\ln n)^{A+1})
\]
\[
\geq n - \frac{A}{A - 1} k.
\]

By making \( A \) sufficiently large we see that if \( k < 9n/10 \) then \( t < 19n/20 \) in (12) and consequently
\[
\sum_{m=m_0}^{9n/10} m(B_k) = o(i). \tag{15}
\]

Case 4: \( A; \geq 9n/10 \)
\[
\Pr_{p}(D_t) \leq
\]
\[
\sum_{n/2}^{n/10} (n \setminus n - 1 \setminus s(n-s) \setminus s(n-s) \setminus t \setminus p \setminus V \setminus \max(2t - t_0, 0))
\]

Let \( u(s, t, t) \) denote the summand in the above and let \( p = a \ln n/n \) and note that \( a \in [1 - \ln n, 2 + \ln n/n] \).

Case 4.1: \( i \leq 2\ell \)

It will generally be convenient to split \( s \) into two ranges:

Case 4.1.1: \( s \leq n^{1/10} \)
\[
u(s, \ell, t) = \binom{n}{s} \left( \frac{n-1}{\ell} \right)^{s(n-s)} \left( \frac{\ell}{n-1} \right)^{2\ell}
\]
\[
\leq \left( \frac{n^{1-\alpha + s/n}}{s} \right)^s \left( \frac{\ell e}{n-1} \right)^{\ell} \left( \frac{e^{s(n-s)}(\ln n)^t}{tn} \right)^{t} \left( \frac{\ell}{n-1} \right)^{2\ell}
\]
\[
\leq \left( \frac{n^{1-\alpha + s/n}}{s} \right)^s \left( \frac{\ell e}{n-1} \right)^{\ell} \left( \frac{e^{s(n-s)}(\ln n)^t}{tn} \right)^{t} \left( \frac{\ell}{n-1} \right)^{2\ell}
\]
\[
\leq \left( \frac{n^{1-\alpha + s/n}}{s} \right)^s \left( \frac{e^2(n-s)^2(\ln n)^2}{n^3 \ell} \right)^{\ell}. \tag{16}
\]
Now
\[ n^{t-a+as/n} \leq \left( \frac{n}{s} \right)^{\alpha} o(1) e^{\omega} \]  
(17)
where \( a > 1 \) — \( u > \ln n \) and \( u > \rightarrow \infty \) slowly.

So if \( 5 \leq 3e^n \) then (16) implies that
\[ u(s, \ell, t) \leq n^{-(1-\alpha(1))\ell}, \]
and if \( s > 3e^n \)
\[ U(SJA) \leq \left( e^{\frac{6e^\alpha - s}{2}(\ln n)^2} \right)^{\ell} \]
\[ = O \left( \left( \frac{s}{n^{1-\alpha(1)}} \right)^{\ell} \right). \]

Case 4.1.2: \( 5 > n^{1/10} \).

Claim 4 /n a.e. process, every \( G_{m_0, m} \in [m_0, m] \) such that \( TJ(S) \geq -y|S| \ln n \)
/or all \( n^{1/10} \leq |S| \leq n/2 \), where \( y > 0 \) is some absolute constant.

Proof (outline) For \( |S| \geq n^{2/3} \) one can use the Chernoff bounds on the
tails of the binomial \( r_j(S) \). If \( |S| \leq n^{2/3} \) we use the fact that with high
probability (i) \( G_{m_0} \) has \( n^{e^r} \) vertices of degree \( \leq 6\ln n \) where \( e^r = \gamma(e) \rightarrow 0 \)
with 6, and (ii) in \( G_{m_i} \) no set \( S \) of size \( \leq n/(\ln n)^2 \) contains 3151 edges.

So if \( s \geq n^{1/10} \) then we can take \( t \geq 7s\ln n > 2\ell \) for some constant \( y > 0 \)
and this case is vacuous.

Case 4.2 : \( t > 2\ell \).

\[ u(s, \ell, t) \leq \left( \frac{ne}{s} \right)^{\ell} \left( \frac{(n-1)e}{\ell} \right) \left( \frac{s(n-s)e^{1+s\alpha\ell \ln n}}{m(n-l)} \right)^{\ell} n^{-\alpha s(n-s)/n} \]
\[
\left( \frac{n^{1-\alpha+\alpha/\epsilon}e^{\epsilon}}{V} \right) \left( \frac{1}{\lambda} \right) \left( \frac{f(n-l)e^{\epsilon}f_{s}(n-s)e^{\epsilon} + 'annn'f_{s}(n-s)j(t)k(t)}{m(n-l)} \right) \tag{18}
\]

Case 4.2.1: \( t \ll 2n \) and so \( ((n - \sqrt{t})/t)^{l} \leq (2n/e)^{l/2} \).

\[
u(s, \ell, t) \leq \left( \frac{n^{1-\alpha+\alpha/\epsilon}e^{\epsilon}}{s} \right)^{l} \left( \frac{20\ell \ln n \sqrt{n}}{t^{3/2}n^{1/2}} \right). \tag{19}\]

Case 4.2.1.1: \( s < n^{1/10} \). Now (17) gives

\[
\left( \frac{n^{1-\alpha+\alpha/\epsilon}e^{\epsilon}}{s} \right)^{l} \leq \left( \frac{(1 + o(1))e^{\epsilon}e^{1/2}}{s} \right)^{l}
\leq \frac{e^{(1+o(1))e^{\epsilon}}}{s} = e^{\omega}, \text{ say,}
\]

and so (19) implies

\[
u(s, \ell, t) \leq \left( \frac{n^{1-\alpha+\alpha/\epsilon}e^{\epsilon}}{s} \right)^{l} \tag{20}\]

Case 4.2.1.2: \( s \geq n^{1/10} \).

Using Claim 4 and (19),

\[
u(s, \ell, t) \leq n^{s/11} \left( \frac{1}{n^{2-\alpha(1)} \sqrt{s}} \right)^{l}.
\]

Case 4.2.2: \( t \geq 2n \) and so \((ne/\ell)^{s} \leq e^{n} \leq e^{l/2}\).

From (18),

\[
u(sj, t) \leq \left( \frac{(1 + o(1))e^{\epsilon}e^{1/2}}{s} \right)^{l} \left( 20\ell \ln n \sqrt{n} \right)^{l}.
\]

Case 4.2.2.1: \( s < n^{1/10} \).
Arguing as in (20),

\[ u(s, \ell, t) \leq \left( \frac{s}{n^{1-o(1)}} \right)^t. \]

**Case 4.2.2.2: \( s \geq n^{1/10} \).**

From Claim 4

\[ u(s, \ell, t) \leq \left( \frac{(1 + o(1))e^{\ell+1}}{s} \right)^s \left( \frac{A\ell}{n} \right)^t. \]

for some constant \( A > 0 \). Now this clearly implies

\[ u(s, \ell, t) = O(2^{-n}) \]  \hspace{1cm} (21)

for \( \ell \leq n/(3A) \). For \( \ell > n/(3A) \) we have \( s \geq \ell \) and

\[ u(s, \ell, t) \leq n^{-s/2} A^n \]

and so (21) holds here also.

Summarising,

\[
\Pr(\mathcal{D}_t) = O \left( \sum_{t=1}^{2\ell} \sum_{s=t+1}^{n^{1/10}} \left( \frac{s}{n^{1-o(1)}} \right)^t + \sum_{t=2\ell+1}^{2n} \sum_{s=t+1}^{n^{1/10}} \left( \frac{s}{n^{1/2-o(1)}} \right)^t \right)
\]

\[
+ \sum_{t=2\ell+1}^{2n} \sum_{s=n^{1/10}}^{n/2} \left( \frac{s}{\sqrt{s}} \right)^t + \sum_{s=1}^{n^{1/10}} \sum_{t=2n+1}^{n^{1/2-o(1)}} \left( \frac{s}{n^{1/2-o(1)}} \right)^t
\]

\[
+ \sum_{s=n^{1/10}}^{n/2} \sum_{t=2n+1}^{n^{1/2-o(1)}} 2^{-n} \right).
\]

where the double summations correspond to the five cases enumerated above.

Thus, we see that

\[
\sum_{m=m_0}^{m_1} \sum_{t=2}^{n/10} \Pr_m(\mathcal{D}_t) = O((n \ln n)(\sqrt{n \ln n})n^{-1.7})
\]

\[ = o(1). \]  \hspace{1cm} (22)
We are thus left with \( \Pr \left( \bigcup_{m=m_0}^{m_1} (C_m \cap \mathcal{A}_{n-2}) \right) \).

We consider \( G_{m_0} \). We know that a.e. \( G_{m_0} \) consists of a giant connected component \( C \) plus \( O(e^{\omega}) \) isolated vertices \( T \). If \( \bigcup_{m=m_0}^{m_1} (C_m \cap \mathcal{A}_{n-2}) \) occurs at some time during the process then either

\((i)\) there exist \( u, v \in T \) such that the first edges of the process that are incident with each of \( u \) and \( v \) are the same colour,

OR

\((ii)\) there exists a colour \( c \) and a set \( S, 2 \leq |S| \leq n/2 \) such that in \( G_{m_0} \) the \( t > 2 (S : \bar{S}) \) edges are all of colour \( c \).

(Suppose that deleting the edges of colour \( c \) from \( G_m \) produces at least three components. If colour \( k \) has not occurred by time \( m_0 \) then two of these components must be vertices from \( T \), contradicting \( (i) \). If \( G_{m_0} \) has edges of colour \( c \) then deleting these edges must beak \( C \) into at least three pieces.)

Clearly

\[ \Pr((i)) = o(1) + O(e^{2\omega}/n) = o(1). \]

Furthermore

\[
\Pr_p((ii)) \leq \sum_{s=2}^{n/2} \binom{n}{s} n^{s(n-s)} \sum_{t=2}^{s(n-s)} \binom{s(n-s)}{t} \left( \frac{p}{n} \right)^t \left( 1 - p \right)^{s(n-s)-t}
\]

\[
\leq 2 \sum_{s=2}^{n/2} \binom{n}{s} n^{10 \ln n} \sum_{t=2}^{s(n-s)} \frac{(s(n-s))^t}{t!} \left( \frac{\alpha \ln n}{n^2} \right)^t n^{-\alpha s}
\]

\[
\leq n \sum_{s=2}^{n/2} \binom{n^{1-\alpha}}{s} \sum_{t=2}^{s \ln n} \left( s \ln n \right)^t
\]

\[
= O\left(n^{-1-o(1)}\right).
\]
The upper bound is good enough to apply (8) and so \( \Pr_{m_0}((ii)) = o(1) \). Thus

\[
\Pr \left( \bigcup_{m = m_0}^{m_1} (C_m \cap A_{n-2}) \right) = o(1). \tag{23}
\]

Our theorem now follows from (7),(10),(11),(14),(15),(22) and (23).

References


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