Determinacy of Equilibrium in an Overlapping Generations Model with Heterogeneous Agents *

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Abstract: We study the determinacy of perfect foresight equilibrium near a steady state in an overlapping generations model with production and both altruistic and non altruistic agents having distinct utility functions. The proportions of each type of consumers are exogenously given. Our main results show that when there are positive stationary bequests, some standard assumptions on preferences and technology rule out local indeterminacy for any positive value of the proportions. In the particular case of a separable utility function for altruistic agents, we prove that the determinacy property does not depend on the size of the “infinite lived” altruistic dynasties.

Abstract: Nous étudions la détermination de l’équilibre de prévisions parfaites dans un voisinage d’un état stationnaire dans un modèle à générations imbriquées avec production et agents hétérogènes. Il existe en effet des consommateurs altruistes et non altruistes ayant des fonctions d’utilité distinctes, et dont le nombre est fixé par des proportions exogènes. Nous montrons que lorsque les legs stationnaires sont strictement positifs, des hypothèses standard sur les préférences et la technologie permettent d’éliminer l’indétermination locale pour toute valeur des proportions. Dans le cas particulier d’une fonction d’utilité séparable pour les agents altruistes, nous montrons que la propriété de détermination ne dépend pas de la “taille” des dynasties altruistes à durée de vie infinie.

Key-words: OLG models, heterogeneous agents, bequest motives, local determinacy of equilibria.

Mots-clés: modèles à générations imbriquées, agents hétérogènes, legs opérant, détermination locale des équilibres stationnaires.

Classification JEL: C62, D91, O21, O41.

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1 Introduction

This paper uses the canonical framework of an overlapping generations model with production. We consider Diamond [9] and Barro [4] formulations simultaneously and include both altruistic and non altruistic agents with distinct utility functions. When bequests are positive, families of finite lived agents who care about the welfare of their offspring behave like infinite lived households. Our model therefore combines overlapping generations of finite lived agents and infinite lived consumers. As in Michel and Pestieau [15], the proportions of each type of agents are exogenous. In this regard, our paper contrasts with four contributions which also combines both types of agents: Muller and Woodford [17] and Nourry and Venditti [18] are concerned with the issue of local indeterminacy but do not consider the constraint of operative bequest, and Aiyagari [2, 3] studies the existence of a competitive equilibrium in which the mix of agents is endogeneized.

We consider the general formulation of preferences over the life-cycle of each agent given by non separable utility functions. This is in contrast with the standard assumptions of time-additively separable utility function for infinitely lived agents “à la Ramsey”. The main distinction concerns the intertemporal substitution effects which are reinforced when the present marginal utility depends on future consumption. As shown by Michel and Venditti [16] in an OLG model with a proportion $p = 1$ of altruistic agents, this simple difference may have some important consequences for the equilibrium dynamics.

Our aim is to study the local determinacy properties of equilibrium paths for bequest and capital in an economy with both altruistic and non altruistic dynasties. A given configuration will be refered to as locally indeterminate as soon as there exists a continuum of distinct equilibrium paths starting from the same initial values for bequests and the capital stock. It is now well-known that indeterminacy of perfect foresight equilibrium is a sufficient condition for the existence of sunspot equilibria and stochastic fluctuations based upon extrinsic uncertainty.$^1$

With imperfect competition, local indeterminacy easily arises in Ramsey as well as in OLG models.$^2$ However, in the case of perfect competition, models with a finite number

$^1$See Cass and Shell [8], and Woodford [24].

$^2$See Benhabib and Nishimura [7] and the recent survey by Benhabib and Farmer [6].
of infinitely lived agents are distinguished from models with an infinite number of finitely lived agents on the grounds of the determinacy properties of their perfect foresight equilibria. In their pioneering paper, Kehoe and Levine [12] consider a pure exchange economy and prove that, unlike optimal growth models, in which equilibria are generically locally determinate, there exist robust examples of overlapping generations models with a continuum of equilibria. This indeterminacy result has been extended to production economies by Muller and Woodford [17]. They also generalize the findings of Kehoe and Levine by mixing both infinite lived consumers and overlapping generations of finite lived agents. They show that if the infinitely lived agent is “small” enough, there exist some open sets of economies with locally indeterminate equilibria. On the contrary, Kehoe, Levine and Romer [13] consider a production economy with a finite number of heterogeneous infinitely lived agents, and show that equilibria are locally unique for almost all endowments.

These contributions are concerned with multi-dimensional OLG models with many consumption and capital goods. The coexistence of state and forward variables allows one therefore to consider local indeterminacy of perfect foresight equilibrium near steady state equilibria. On the contrary, in the present paper as well as in our previous contribution (Nourry and Venditti [18]), we consider a one-dimensional Diamond model with one state variable. Local indeterminacy is therefore not possible. However, as shown in Galor and Ryder [10], when the saving function is decreasing with respect to the interest rate, global indeterminacy of perfect foresight equilibria may emerge. One might have the intuition that in the present framework, the introduction of infinite lived dynasties does not provide any new opportunity for the existence of a continuum of equilibrium paths. This intuition is wrong as proved in Nourry and Venditti [18]. That paper considers the same basic model as the present manuscript but with standard infinitely lived agents “à la Ramsey”. While bequests of altruistic dynasties need to be positive, the wealth of infinite lived consumers may be negative. We provide an example with CES utility functions and Cobb-Douglas technology in which the perfect foresight equilibrium is globally determinate in the Diamond model, but locally indeterminate in the mixed case with stationary consumptions for infinite lived households close to zero. We show

\[3\text{This is the case when infinite-lived agents own a large fraction of the total wealth but their consumption is not too great a part of total consumption.}\]
that such a configuration is obtained if the stationary wealth of infinitely lived agents is negative.

We consider in this paper equilibria with positive bequests, in which both types of agents co-exist. According to the contributions of Abel [1] and Weil [23], the non-negativity constraint for bequests introduces a strong restriction. In particular, depending on some characteristics of the economy without the bequest motive, bequests may not be operative. If this case occurs for the stationary state, then in the long run the altruistic families behave as if they were “selfish”. We first provide a necessary and sufficient condition for operative bequests which is a mere extension to an economy with heterogeneous agents of the recent results of Thibault [21].\footnote{See also Aiyagari [2, 3] for similar results in an exchange economy.} When the long run optimal bequest is positive, the stationary capital labor ratio satisfies the modified golden rule, and does not depend on the proportion \( p \), at least as long as \( p > 0 \). There is however a discontinuity when \( p = 0 \) since the stationary capital labor ratio corresponds to a steady state of the Diamond [9] economy with non altruistic agents. The constraint of operative bequest and this discontinuity are the main ingredients of our results.

Our main conclusions are the following: as long as income effects are not too great with respect to substitution effects, the perfect foresight equilibrium cannot be indeterminate if the marginal utility of first period consumption for altruistic agents is a non-increasing function of second period consumption. This property may be interpreted as a complementarity between consumption levels. We show that our determinacy conclusions do not depend on the size of the infinitely lived dynasty. Such a result still holds when the Diamond economy without bequest motive has globally indeterminate equilibria.

If non altruistic agents save more in the long run than altruistic ones, the previous results are extended to configurations with a marginal utility of the first period consumption for altruistic households which is a highly increasing function of the second period consumption. The corresponding restrictions on fundamentals are actually identical to the saddle point stability condition provided by Michel and Venditti [16] in the case \( p = 1 \). We prove finally that when the altruistic agents save more in the long run than the non altruistic ones, a strengthening of the saddle point stability condition rules out
indeterminate perfect foresight equilibrium.

The paper is organized as follows. Section 2 presents the model. Section 3 gives some conditions for positive stationary bequests, and establishes the existence of the steady state equilibrium. Section 4 provides the conditions for determinacy of perfect foresight equilibrium near a steady state, and section 5 gives some examples. Section 6 contains some concluding comments while all the proofs are gathered in section 7.

2 The model

Consider an economy in which there are a proportion \(0 < p \leq 1\) of altruistic agents \((a)\), and a proportion \(q = 1 - p\) of non altruistic agents \((e)\). Each agent lives for two periods: he works during the first (with one unit of labor supplied) and he has preferences for his consumption \((c^i, i = a, e, \text{ when he is young}, \text{and } d^i, i = a, e, \text{ when he is old})\) which are summarized by the utility function \(u^i(c^i, d^i), i = a, e\). We assume that:

Assumption 1. Each instantaneous utility function \(u^i(c^i, d^i), i = a, e\), is strictly increasing with respect to each argument \((u^1_1(c^i, d^i) > 0 \text{ and } u^1_2(c^i, d^i) > 0)\), strictly concave and \(C^2\) over the set \(\mathbb{R}^2_{++} = [0, +\infty] \times [0, +\infty]\). Moreover, for all consumption levels \(c^i, d^i > 0, \lim_{\xi \to 0} u^1_1(\xi, d^i) = \infty = \lim_{\psi \to 0} u^1_2(c^i, \psi)\).

Assumption 2. The consumptions \(c^i_t\) and \(d^i_{t+1}\) are both normal goods for \(i = a, e\).

Each agent of type \(i\) is assumed to have \(1 + n\) children of type \(i\). We assume perfect foresight. A non altruistic agent \(e\) maximizes his life-cycle utility function as follows:

\[
\max_{c^e_t, d^e_{t+1}} u^e(c^e_t, d^e_{t+1})
\]

\[\text{s.t. } w_t = c^e_t + s^e_{t+1} (1 + r_{t+1})s^e_t = d^e_{t+1}\]

with \(w_t\) the wage rate and \(r_{t+1}\) the interest rate. The first order condition gives the saving of each agent \(e\) as a function of \(w_t\) and the interest factor \(R_{t+1} = 1 + r_{t+1}\):

\[
s^e_t = s^e(w_t, R_{t+1})
\]
The assumption of altruism with perfect foresight can be formulated as follows: an agent born at time $t$ maximizes the sum of his life-cycle utility and of the welfare of his immediate descendants, discounted at rate $\delta$, namely:

$$\mathcal{V}_t(x_t) = \max_{c_t^a, d_{t+1}^a = x_{t+1}} \{ u^a(c_t^a, d_{t+1}^a) + \delta(1 + n)\mathcal{V}_{t+1}(x_{t+1}) \}$$

s.t.  

$$w_t + x_t = c_t^a + s_t^a$$

$$(1 + r_{t+1})s_t^a = d_{t+1}^a + (1 + n)x_{t+1}$$

$$x_{t+1} \geq 0$$

(3)

$\delta$ is the degree of altruism. We assume that the discount factor $\beta = \delta(1 + n)$ satisfies $\beta < 1$. Bequest $x_t$ is received by each young member born in an altruistic family at period $t$. The proceeds of his saving, $R_{t+1}s_t^a$, is allocated between consumption $d_{t+1}^a$ and bequest $(1 + n)x_{t+1}$ to his children. Considering the budget constraints, given in system (3), and solving for $s_t^a$ give the following dynamic equation for bequests:

$$x_{t+1} = \frac{1}{1 + n} \left[ R_{t+1} (x_t + w_t - c_t^a) - d_{t+1}^a \right]$$

(4)

We may thus associate implicit prices $q_t, q_{t+1}$ to bequest $x_t, x_{t+1}$ and define the following Lagrangian function $\mathcal{L}$ which corresponds to the sum of the utility of generation $t$ with the increase of the value of bequest:

$$\mathcal{L} = u^a(c_t^a, d_{t+1}^a) + \frac{\beta}{1 + n} q_{t+1} \left[ R_{t+1} (x_t + w_t - c_t^a) - d_{t+1}^a \right] - q_t x_t$$

(5)

Maximizing $\mathcal{L}$ with respect to $c_t^a, d_{t+1}^a$ and $x_t$ subject to the non negativity constraint for bequests gives the following necessary and sufficient conditions:\footnote{See Michel [14].}

$$u_1^a \left( c_t^a, d_{t+1}^a \right) = \frac{\beta}{1 + n} q_{t+1} R_{t+1}$$

(6)

$$u_2^a \left( c_t^a, d_{t+1}^a \right) = \frac{\beta}{1 + n} q_{t+1}$$

(7)

$$\frac{\beta}{1 + n} q_{t+1} R_{t+1} \leq q_t \quad (= \text{if } x_t > 0)$$

(8)

$$\lim_{t \to \infty} \beta^t q_t x_t = 0$$

(9)

Together equations (6), (7) and (8) give the Euler-Lagrange equation:

$$-u_2^a \left( c_{t-1}^a, d_t^a \right) + \delta u_1^a \left( c_t^a, d_{t+1}^a \right) \leq 0 \quad (= \text{if } x_t > 0)$$

(10)
The saving function is derived from equations (6), (7):

\[ s_t = s^a \left( w_t + x_t, -(1 + n)x_{t+1}, R_{t+1} \right) \]  \hspace{1cm} (11)

The production function of a representative firm, denoted \( F(K, L) \), depends on the stock of capital \( K \) and labor \( L \). We assume that \( F(\ldots) \) satisfies:

**Assumption 3**. The production function \( F : \mathbb{R}_+^2 \to \mathbb{R}_+ \) is continuous, \( C^2 \) over \( \mathbb{R}_+^2 \), and satisfies the following properties:

i) \( F(\ldots) \) is increasing, homogeneous of degree one and concave;

ii) \( F_{11}(\ldots) < 0 \) for all \( L > 0 \);

iii) \( \lim_{K \to 0} F_1(K, L) = \lim_{L \to 0} F_2(K, L) = \infty \) and \( \lim_{K \to \infty} F_1(K, L) = \lim_{L \to \infty} F_2(K, L) = 0 \).

We denote by \( \mu \in [0, 1] \) the rate of depreciation of the capital stock. Let \( k = K/L \) be the capital stock per young agent, and \( f(k) = F(k, 1) + (1 - \mu)k \). The competitive equilibrium conditions imply that the interest factor \( R_t \) and the wage rate \( w_t \) satisfy:

\[ R_t = f'(k_t) \equiv R(k_t) \]  \hspace{1cm} (12)

\[ w_t = f(k_t) - k_t f'(k_t) \equiv w(k_t) \]  \hspace{1cm} (13)

In this economy, the dynamics of capital and bequests are given by the Euler-Lagrange equation (10), and the following capital accumulation condition:

\[ (1 + n)k_{t+1} = (1 - p)s^c(w_t, R_{t+1}) + ps^a \left( w_t + x_t, -(1 + n)x_{t+1}, R_{t+1} \right) \]  \hspace{1cm} (14)

for all \( t \geq 0 \). At the initial date 0, we assume that there exist some generations of altruistic and non altruistic agents, born at time \(-1\). The consumption levels \( c^e_{-1}, c^a_{-1} \), and the savings \( s^e_{-1}, s^a_{-1} \) are given. The initial capital labor ratio is \( (1 + n)k_0 = (1 - p)s^c_{-1} + ps^a_{-1} \) and the prices are determined. We thus have \( R_0s^c_{-1} = d_0^c \) and \( R_0s^a_{-1} = d_0^a + (1 + n)x_0 \). Denoting \( \delta_0 \equiv (1 - p)d_0^c + pd_0^a \), the total consumption of the old generations, it follows that \( R_0(1 + n)k_0 = \delta_0 + p(1 + n)x_0 \). The initial value for bequest \( x_0 \) therefore needs to satisfy the following consistency condition:

\[ R_0k_0 > p(1 + n)x_0 \]

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\(^6\)Under Assumption 3, the production function in its intensive formulation \( f(k) \) is positive valued, \( C^2 \), strictly increasing, strictly concave over \( \mathbb{R}_+^+ \), and satisfies \( \lim_{k \to 0} f'(k) = \infty \) and \( \lim_{k \to \infty} f'(k) = 1 - \mu \).
3 Steady state and comparative statics

First consider a pure Diamond [9] economy with heterogeneous agents and without bequest motive. Restricting attention to steady states, and using the competitive equilibrium prices, we may define the following equilibrium condition which corresponds to the capital accumulation equation (14) with $x_t = x_{t+1} = 0$:

$$
(1 - p)s^e\left(w(k), R(k)\right) + ps^a\left(w(k), 0, R(k)\right) - (1 + n)k = 0
$$

As in the standard case with homogeneous households considered by Galor and Ryder [10], a Diamond economy with heterogeneous non altruistic agents may have multiple steady states.\(^7\) Assume therefore that the number of stationary equilibria, denoted $k^D_i$, is equal to $m \geq 0$. Let us also denote the total stationary savings of this Diamond economy by

$$
(1 - p)s^{De}(k) + ps^{Da}(k) \equiv (1 - p)s^e\left(w(k), R(k)\right) + ps^a\left(w(k), 0, R(k)\right)
$$

A simple extension of the results provided by Thibault [21] in the case of a proportion $p = 1$ of altruistic agents gives:

Proposition 1. Under Assumptions 1, 2, and 3, let the Diamond economy with heterogeneous agents have $m \geq 0$ stationary equilibria. Then the economy with a proportion $p > 0$ of altruistic agents has $1 \leq g \leq m + 1$ steady states. Moreover, there exists a steady state $(k^*, x^*)$ with $k^* > 0$ such that $f'(k^*) = \delta^{-1}$ and $x^* > 0$ if and only if

$$
(1 - p)s^{De}(k^*) + ps^{Da}(k^*) < (1 + n)k^*
$$

The intuition of this result is simple: let us consider the modified golden rule $k^*$. The above condition implies that the total savings of the Diamond economy is not sufficient to reach $k^*$ as a steady state. The assumption of altruism allows some agents to provide positive bequests and thus to locate the economy at the golden rule. On the contrary, if the inequality is reversed, $k^*$ can be a steady state only if negative bequests are allowed. Such a possibility is ruled out by the constraint of operative bequests.

\(^7\)This result comes from the fact that the saving functions are in general not concave.
Abel [1] and Weil [23] assume that the steady state of the Diamond economy, \( k^D \), is unique and stable. Under these assumptions, the condition for positive bequests of Proposition 1 is equivalent to their well-known restriction on the degree of altruism, namely \( \delta > 1/R^D \), with \( R^D = f'(k^D) \). This condition rules out the dynamically inefficient case of overaccumulation in which the steady state \( k^D \) is greater than the golden rule. On the contrary, Proposition 1 shows that positive stationary bequests may be obtained even if there exist many steady states in the Diamond economy with some of them greater than the golden rule. It is also possible to show as in Thibault [21] that all the steady states of the Diamond economy which are greater than the modified golden rule are also equilibria of the economy with altruistic agents, but with stationary bequests equal to zero.\(^8\) Furthermore, the following Corollary shows that the existence of a steady state in the Diamond economy is not necessary:

**Corollary 1.** Under Assumptions 1, 2, and 3, if \( m = 0 \) the economy with a proportion \( p > 0 \) of altruistic agents has a unique steady state \((k^*, x^*)\) such that \( f'(k^*) = \delta^{-1} \) and \( x^* > 0 \).

Consider now the complete model. We introduce the following assumption:

**Assumption 4.** The modified golden rule \( k^* = (f')^{-1}(\delta^{-1}) \) satisfies
\[
(1 - p)s^{De}(k^*) + ps^{Da}(k^*) < (1 + n)k^*.
\]

We may now prove uniqueness of the stationary values for capital, consumptions and positive bequest:

**Proposition 2.** Under Assumptions 1, 2, 3 and 4, if the proportion \( p \) of altruistic agents satisfies \( 0 < p \leq 1 \), then there exist:

i) a unique stationary capital stock \( k^* \) which satisfies the modified golden rule, i.e. \( f'(k^*) = \delta^{-1} \);

ii) a unique stationary optimal bequest \( x^* > 0 \);

iii) for \((k^*, x^*)\), some unique interior stationary values for the consumption levels of altruistic and non altruistic agents, \( \bar{c}^a, \bar{d}^a, \bar{c}^e, \bar{d}^e \);

iv) a unique stationary value for the implicit price of bequest \( q^* \).

\( ^8\)See Aiyagari [2, 3] for similar results in a pure exchange economy.
As long as there exists a positive proportion of altruistic agents, the stationary capital stock does not depend on the decisions of the non altruistic agents. This result is not surprising: in an economy with infinitely lived agents, the long run capital accumulation follows from the saving decisions of the more patient consumers (see Ramsey [20] and Becker [5]). In an economy with altruistic and non altruistic agents, the altruistic dynasties are more “patient”. The stationary capital stock is thus a discontinuous function of the proportion $p$ of altruistic agents for $p = 0$.

Contrary to what happens with standard infinite lived agents having different discount rates, the stationary consumption levels are strictly positive for both types of agents. At the modified golden rule, we have $r^* > n$. The discounted life-cycle stationary consumption of altruistic agents is thus greater than that of non altruistic agents, namely $\bar{c} + \bar{d}/R^* = w^* + (r^* - n)x^*/R^* \geq \bar{c}^* + \bar{d}^*/R^* = w^*$. Considering now the stationary utility level of altruistic agents as an information on their long run welfare gives:

**Proposition 3**. Under Assumptions 1, 2, 3 and 4, the optimal stationary bequest and the long run welfare of altruistic agents increase with the proportion of non altruistic agents if and only if $s^a(w^* + x^*, -(1 + n)x^*, R^*) \geq s^e(w^*, R^*)$. A sufficient condition for this property to hold is that both type of agents have the same utility function.

If altruistic agents save more in the long run than the non altruistic ones, the bequests of the former actually offset the insufficient saving of the latter. In this regard, the more the altruistic agents are, the lower the bequest necessary to reach the modified golden rule is.

In the presence of a bequest motive, Barro’s [4] debt neutrality theorem is applicable to an economy with homogeneous agents ($p = 1$) if and only if bequests are operative at all dates in the absence of government debt. As in Michel and Pestieau [15], it is possible to show that when $p < 1$, public debt does not affect the stationary capital stock but tends to increase the welfare of altruists (via increased bequests) at the expenses of the non altruists.\(^9\)

\(^9\)See also Phelps and Shell [19] for a discussion of the Ricardian equivalence in a Solow-growth model.
4 Determinacy of equilibria

Consider the economy with positive bequest. We need to precisely define the system of difference equations which characterize the economic dynamics. From the first order conditions (6) and (7) we obtain:

**Lemma 1.** Under Assumption 1-4, if \( \tilde{H}_a (\tilde{c}, \tilde{d}) = u_{11}^a (\tilde{c}, \tilde{d}) u_{22}^a (\tilde{c}, \tilde{d}) - \left[ u_{12}^a (\tilde{c}, \tilde{d}) \right]^2 > 0 \), then the consumption levels \( c_t^a, d_{t+1}^a \) may be locally expressed by some differentiable functions of \((q_t, R_{t+1})\), i.e. \( c_t^a = c^a (q_t, R_{t+1}) \) and \( d_{t+1}^a = d^a (q_t, R_{t+1}) \).

Using Lemma 1 and the budget constraint of the first period of life for altruistic agents, the capital accumulation condition (14) becomes:

\[
(1 + n) k_{t+1} = (1 - p) s^e (w_t, R_{t+1}) + p \left( w_t + x_t - c^a (q_t, R_{t+1}) \right)
\]  

(17)

This is an implicit difference equation of order one. The following lemma gives an explicit formulation.

**Lemma 2.** Let the Assumptions of Lemma 1 hold, and let \((k^*, q^*, x^*)\) be the stationary equilibrium. If

\[
\psi_4 (k^*, q^*, x^*, k^*) = f''(k^*) \left[ (1 - p) s^e_2 (w^*, R^*) - p \frac{u_{12}^a q^*}{\tilde{H}_a^a (R^*)^2} \right] - (1 + n) \neq 0
\]

then the capital accumulation equation may be locally expressed as a differentiable function of \((k_t, q_t, x_t)\), i.e. \( k_{t+1} = \phi (k_t, q_t, x_t) \).

We finally obtain the dynamical system which describes the equilibrium paths in a neighborhood of the steady state \((k^*, q^*, x^*)\):

\[
k_{t+1} = \phi (k_t, q_t, x_t)
\]

(18)

\[
q_{t+1} = \frac{(1 + n) q_t}{\beta f' \left( \phi (k_t, q_t, x_t) \right)}
\]

(19)

\[
x_{t+1} = \frac{f' \left( \phi (k_t, q_t, x_t) \right)}{1 + n} \left[ f(k_t) - k_t f'(k_t) + x_t - c^a (q_t, f' \left( \phi (k_t, q_t, x_t) \right)) \right]
\]

\[ - \frac{1}{1 + n} d^a (q_t, f' \left( \phi (k_t, q_t, x_t) \right)) \]

(20)
This is a three dimensional dynamical system with two predetermined variables, $k_t$, $x_t$, and one forward variable, $q_t$. Let us now give the definition of indeterminacy that will be used in our analysis.

**Definition 1.** Let $\{(k_t,q_t,x_t)\}_{t=0}^{\infty}$ denote an equilibrium for an economy with initial condition $(k_0,q_0,x_0)$. We say that it is an indeterminate equilibrium if for every $\epsilon > 0$ there exists another sequence $\{(k'_t,q'_t,x'_t)\}_{t=0}^{\infty}$, with $0 < |q'_0 - q_0| < \epsilon$ and $(k'_0,x'_0) = (k_0,x_0)$, which is also an equilibrium.

If an equilibrium is not indeterminate, then we call it determinate. The dimension of indeterminacy cannot be greater than one. Actually, the steady state $(k^*,q^*,x^*)$ is indeterminate if and only if the local stable manifold is three-dimensional.

First consider the simple case of a “small finite lived population”. In our framework such a configuration is obtained when the proportion $p$ of altruistic agents is close to one. We first need to introduce the following definition:

**Definition 2.** A steady state $(k^*,q^*,x^*)$ of the three dimensional dynamical system is a regular saddle point if and only if the dimension of the local stable manifold is equal to 2.

Building on the case with $p = 1$, we can extend the results provided by Michel and Venditti [16] to the case of an increasing population with a small proportion of non altruistic agents:

**Proposition 4.** Under Assumptions 1, 2, 3 and 4, if the proportion $p$ of altruistic agents is close enough to one, then local indeterminacy of perfect foresight equilibrium is impossible. Moreover, the steady state $(k^*,q^*,x^*)$ is a regular saddle point if and only if one of the following sets of conditions is satisfied:

\begin{enumerate}
  \item $f''(k^*)u_{12}^a u_2^a + f'(k^*) \bar{H}_u^a (1 + n) > 0$
  \item $f''(k^*)u_{12}^a u_2^a + f'(k^*) \bar{H}_u^a (1 + n) < 0$ and
  \[ 2\left(\frac{1 + \beta}{\beta}\right) \bar{H}_u^a (1 + n) + f''(k^*) u_2^a \left( u_{22}^a + \frac{1 + \beta}{1 + n} u_{12}^a + \frac{\beta}{(1 + n)^2} u_{11}^a \right) > 0 \]
\end{enumerate}
This result is similar to Proposition 7 of Muller and Woodford ([17], p.282). Note however that we provide some necessary and sufficient conditions for the saddle point property of the steady state: for a given production function, the utility function $u^a(c,d)$ needs to be sufficiently concave. Note also that if neither of these conditions is satisfied, the steady state is totally unstable and there exist some cycles of period two.\textsuperscript{10}

Consider now the general configuration of a proportion $0 < p \leq 1$. We introduce the following restriction:

**Assumption 5.** The saving of non altruistic agents, $s^e(w,R)$, is a non decreasing function of the interest factor $R$ when evaluated at the steady state $(w^*, R^*)$.

Such a property holds if and only if the substitution effect created by an increase in the interest rate is not smaller (in absolute value) than the income effect, i.e. if and only if the utility function of non altruistic agents satisfies $u^e_1 u^e_2 \geq (u^e_2 w_{12}^e - u^e_1 w_{22}^e) R^* s^e(w^*, R^*)$.

Depending on the sign of the derivative $\psi_4^* \equiv \psi_4(k^*, q^*, x^*, k^*)$, different results may be obtained on the ground of the determinacy properties of equilibria. Let us first begin with the simplest case $\psi_4^* < 0$.

**Theorem 1.** Under Assumptions 1-5, let $0 < p \leq 1$. If the utility function of altruistic agents satisfies $u^a_{12}(c,d) \leq 0$, then the steady state $(k^*, q^*, x^*)$ cannot be locally indeterminate.

Remark. This result still holds if $s^e_2(w^*, R^*) < 0$ and/or $u^a_{12}(c,d) > 0$, as long as $\psi_4^* = f''(k^*) \left[ (1-p)s^e_2(w^*, R^*) - p \frac{\beta u^a_{12} u^2}{H^e_2(1+n)} \right] - (1+n) < 0$.

The assumption $u^a_{12}(c,d) \leq 0$ means that an increase of consumption today leads to a decrease of marginal utility tomorrow. This provides some incentive for the altruistic agent to increase his future consumption. The consumption levels over the life-cycle are therefore similar to complementary goods. Theorem 1 proves that under this complementarity assumption:

\textsuperscript{10}See Michel and Venditti [16].
i) the determinacy property does not depend on the size $p$ of the infinitely lived consumer;

ii) this results still holds when income effects are as large as substitution effects;

iii) this result still holds if the Diamond economy with heterogeneous agents and without bequest motive has globally indeterminate perfect foresight equilibria.\(^\text{11}\)

In the particular case of an additively separable utility function for altruistic agents, provided bequests are positive, our model is similar to that studied by Muller and Woodford [17]. In this regard, their paper contains two crucial contributions:

i) Proposition 4 (p.276) which shows that if the infinite lived agent is “small”, there exists some open set of economies in which the steady state is locally indeterminate. The concept of “small” agent is defined by Muller and Woodford as follows: the infinite lived consumers own a large fraction of total wealth but their consumption is not too great a part of total consumption. Their proof consists in showing that an indeterminate steady state of an OLG economy is preserved by the mere adjunction of a “small” infinite lived consumer.

ii) Proposition 8 (p.282) which shows that if for finite lived agents income effects following some price changes are in a neighbourhood of zero, the steady state cannot be locally indeterminate.

Considering simultaneously these propositions allows to provide the following assertion: for any given substitution effects greater than income effects, there exist some economies with locally indeterminate steady state provided infinite lived agents have small enough consumptions.

In our model, the determinacy property does not depend on the consumption levels of altruistic agents. But small altruistic dynasties are obviously associated with a small proportion $p$. Theorem 1 proves that provided $p > 0$, if substitution effects are greater than income effects, the steady state is determinate for any arbitrarily low value of $p$. This result is quite different from the conclusions of Muller and Woodford.

\(^{11}\)Following Galor and Ryder [10], a robust indeterminacy of equilibria may appear in the Diamond model as soon as the saving function of non altruistic agents is a decreasing function of the interest factor, i.e. $s^e_2(w, R) < 0$. 

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We may actually provide two types of explanations for such a difference. Firstly, as mentioned in the introduction, local indeterminacy cannot occur in a one-dimensional Diamond model. However, in a formulation with exogenous proportions of overlapping generations of finite lived agents and infinite lived households, Nourry and Venditti [18] have exhibited an example in which the equilibrium is globally determinate when \( p = 0 \), but locally indeterminate for some positive values of the proportion \( p \) when the stationary wealth of infinitely lived agents is negative and their stationary consumption is low enough. The constraint of non negative bequests appears to prevent the existence of locally indeterminate perfect foresight equilibria.\(^{12}\)

Secondly, if \( p \) is very close to zero, the necessary and sufficient condition for positive stationary bequest becomes

\[
 s^{De}(k^*) < (1 + n)k^*
\]

Using the budget constraint of the first period of life for altruistic agents, and the capital accumulation condition (14) considered at the steady state gives

\[
 p(x^* - \bar{c}^a) = (1 + n)k^* - (1 - p)s^e(w^*, R^*) - pw^*
\]

It follows easily that

\[
 \lim_{p \to 0} p(x^* - \bar{c}^a) = (1 + n)k^* - (1 - p)s^e(w^*, R^*) \equiv \ell > 0
\]

and thus

\[
 \lim_{p \to 0} px^* \geq \ell > 0
\]

Considering now the discounted life-cycle stationary consumption of altruistic agents, we obtain:

\[
 p\left[\bar{c}^a + \frac{\bar{d}^a}{R^*}\right] = pw^* + \frac{(r^* - n)}{R^*}px^*
\]

and finally

\[
 \lim_{p \to 0} p\left[\bar{c}^a + \frac{\bar{d}^a}{R^*}\right] \geq \frac{(r^* - n)}{R^*} \ell > 0
\]

Stationary bequests and discounted life-cycle stationary consumption of altruistic agents therefore converge to \(+\infty\) as \( p \) goes toward zero.\(^{13}\) Note also that these properties explain

\(^{12}\)See also the examples in Section 5.

\(^{13}\)Note that under Assumption 2 we necessarily have \( s^a\left(w(k^*) + x^*, -(1 + n)x^*, R(k^*)\right) > s^{De}(k^*) \). The result is thus coherent with Proposition 3.
why the stationary capital labor ratio is a discontinuous function of \( p \) for \( p = 0 \). Our concept of “small” infinite lived dynasty thus differs drastically from the one used by Muller and Woodford. Actually, the only configuration in which consumptions converge toward zero with \( p \) is the converse inequality \( s^{De}(k^*) > (1 + n)k^* \). In this case however, it may be easily shown that there exists a proportion \( \tilde{p} \in [0, 1] \) such that stationary bequests equal zero for all \( p \in [0, \tilde{p}] \), and in the long run the economy is only composed of non altruistic agents.

Assume now that \( \psi_4 > 0 \). We thus consider the cases in which the consumption levels of altruistic agents are substitutable goods in terms of marginal utility, i.e. \( u^1_{12}(c, d) > 0 \), and / or income effects are greater than substitution effects, i.e. \( s^e_w(w^*, R^*) < 0 \). We introduce the following Assumption which will be substituted to Assumption 5.

**Assumption 6.** The saving function of non altruistic agents satisfies at the steady state

\[
s^e_2(w^*, R^*) + k^* s^e_1(w^*, R^*) - (1 + n)[f''(k^*)]^{-1} > 0
\]

Our model is defined with consumers having distinct utility functions. If the non altruistic consumers’ utility function only depends on second period consumption, their marginal propensity to save out of the first period income is equal to one. Thus, if preferences of altruistic households are oppositely characterized, the saving of the former may be greater than the saving of the latter. In this case, the saddle point condition of Proposition 4 prevents indeterminacy.

**Theorem 2.** Under Assumptions 1-4,6, let \( 0 < p \leq 1 \) and \( \psi_4 > 0 \). If in the long run, the non altruistic agents save more than the altruistic ones, i.e. \( s^e_w(w^*, R^*) \geq s^a(w^* + x^*, -(1 + n)x^*, R^*) \), and if

\[
2 \left( \frac{1 + \beta}{\beta} \right) H_u^a(1 + n) + f''(k^*)u^2_2 \left( u^a_{22} + \frac{1 + \beta}{1 + n}u^a_{12} + \frac{\beta}{(1 + n)^2}u^a_{11} \right) \geq 0, \quad (21)
\]

then the steady state \((k^*, q^*, x^*)\) cannot be locally indeterminate. A sufficient condition for equation (21) to hold is that the utility function of altruistic agents satisfies

\[
\left( u^a_{22} + \frac{1 + \beta}{1 + n}u^a_{12} + \frac{\beta}{(1 + n)^2}u^a_{11} \right) \leq 0
\]
Assume finally that the altruistic agents save more in the long run than the non altruistic ones. Under Assumption 2, this is the standard configuration when both type of agents have similar utility functions. It follows that a strengthening of the saddle point condition of Proposition 4 rules out the possibility of indeterminate equilibria.

**Theorem 3.** Under Assumptions 1-4,6, let \( 0 < p \leq 1 \) and \( \psi_4^* > 0 \). If in the long run, the altruistic agents save more than the non altruistic ones, i.e. \( s^a(w^* + x^* - (1+n)x^*, R^*) \geq s^e(w^*, R^*) \), and if the utility function of altruistic agents satisfies

\[
2 \left( \frac{1 + \beta}{\beta} \right) \tilde{H}_u^a(1+n) + f''(k^*) u_2^a \left( u_{22}^a + \frac{1 + \beta}{1+n} u_{12}^a + \frac{\beta}{(1+n)^2} u_{11}^a \right) > \\
2 \frac{f''(k^*) u_2^a}{1+n} \left( -u_{12}^a + \frac{\beta u_{11}^a}{1+n} \right) > 0
\]

then the steady state \((k^*, q^*, x^*)\) cannot be locally indeterminate.

**Remark.** We may also consider altruistic agents with some heterogeneity on their level of altruism. Let \( p_h \) be the proportion of altruistic agents with degree of altruism \( \delta_h \), for \( h = 1, \ldots, H \), and \( \sum_{h=1}^H p_h = p \), and let \( q = 1 - p \) be the proportion of non altruistic agents. For simplicity, let \( \delta_1 \leq \delta_2 \leq \cdots \leq \delta_H \). As in Vidal [22], there exists a finite date \( T > 0 \) such that for all \( t \geq T \), the optimal bequests \( x_t^h \) equal zero for \( h = 1, \ldots, H - 1 \), i.e. the stationary capital labor ratio only depends on the degree of altruism of the most altruistic agents. In a neighborhood of the steady state, since each altruistic agent \( h = 1, \ldots, H - 1 \) behaves as non altruistic ones, we can thus consider a model with a proportion \( p_H \) of altruistic dynasties, and a proportion \( 1 - p_H \) of overlapping generations of finite lived consumers. All our results therefore apply to this configuration.

### 5 Examples

Theorems 1, 2 and 3 give some sufficient conditions to prevent the existence of multiple equilibrium paths which depend on separability and curvature properties of preferences and technology. One might then wonder about the robustness of this determinacy result when the above conditions do not hold. More precisely, when the sufficient conditions
fail are not satisfied, is it possible to exhibit a continuum of perfect foresight equilibria, starting from one initial position, which converge to a steady state equilibrium with positive bequest?

To answer this question, we have carefully analyzed some examples with CES utility functions such that

$$u^i(c^i, d^i) = A_i \left( (c^i)^{\rho_i} + \gamma_i(d^i)^{\rho_i} \right)^{\frac{\alpha_i}{\rho_i}}$$

with $A_i > 0$, $\rho_i > 0$, $\gamma_i \in [0, 1]$, $\alpha_i \leq 1$, $i = a, e$. The saving functions are the following:

$$s^e(w_t, R_{t+1}) = \frac{w_t}{1 + \gamma^e R_{t+1}^{1-\rho_e}}$$

and

$$s^a\left(w_t + x_t, -(1 + n)x_{t+1}, R_{t+1}\right) = \frac{w_t + x_t + (\gamma_a R_{t+1})^{-\rho_a}(1 + n)x_{t+1}}{1 + \gamma_a^{-\rho_a} R_{t+1}^{1-\rho_a}}$$

When the intertemporal elasticity of substitution $\rho_e$ is less than 1, the saving function of non altruistic agents $s^e(w_t, R_{t+1})$ is decreasing with respect to the interest factor. For altruistic households, the sign of this derivative also depends on bequests. By experimenting with different values of the parameters, we consider respectively the three configurations of Theorems 1, 2 and 3.  

In the Diamond model with $p = 0$, the capital accumulation equation $(1 + n)k_{t+1} = s^e(w_t, R_{t+1})$ may be written under Assumption 2 as a forward-looking equation $k_t = \psi(k_{t+1})$. The global determinacy properties of equilibrium paths are given by the characteristics of the function $\psi(.)$ which depend on the intertemporal elasticity of substitution in consumption $\rho_e$ and the elasticity of substitution in capital characterizing the technology. We have considered Cobb-Douglas or CES production functions, respectively such that $f(k) = Bk^\varphi$, or $f(k) = C(\theta k^{-\sigma} + 1 - \theta)^{-1/\sigma}$, with $B, C > 0$, $\varphi \in [0, 1]$, $\theta \in [0, 1]$, and $\sigma > -1$. With a Cobb-Douglas technology, the function $\psi(.)$ is monotone increasing and thus globally invertible for any value of the intertemporal elasticity of substitution $\rho_e$ and the elasticity of substitution in capital $\varphi$.  

14The Assumptions of Theorem 1 are satisfied when $\alpha_a = -\rho_a (w^a_{t+2} = 0)$, or $\alpha_a < 0$ with $A_a = 1/\alpha_a (w^a_{t+2} < 0)$. The configuration of Theorem 2 is obtained when $\gamma_a \simeq 0$ and $\gamma_e \simeq 1 (s^e \geq s^a)$, while the Assumptions of Theorem 3 hold when $\gamma_a$ and $\gamma_e$ are close ($s^a \geq s^e$).

15See Nourry and Venditti [18].
is therefore globally determinate. On the contrary, with a CES technology, when \( \rho_c \) and the elasticity of substitution in capital \( (1 + \sigma)^{-1} \) are close enough to zero, the function \( \psi(.) \) becomes non monotone and uniqueness of equilibrium path no longer holds.\(^{16}\)

In each case, we have chosen the values of the parameters for which the curvature conditions on preferences and technology are not satisfied.\(^{17}\) Even if the stationary total consumption of altruistic agents is kept close to zero, it clearly appears that either the stationary bequest is positive and the steady state is determinate, or the bequest constraint is binding while the sufficient conditions still hold. These results differ drastically from the example given in Nourry and Venditti [18] in which local indeterminacy of equilibria appears when the saving of finite lived households decreases with the interest factor, and infinite lived agents have small consumption and negative wealth at the steady state. Thus, taking into account the constraint for operative bequest introduces some strong restrictions which make that indeterminate perfect foresight equilibria are not likely.

### 6 Concluding comments

We have considered an OLG model with production and both altruistic and non altruistic agents having distinct utility functions. A necessary and sufficient condition for positive stationary bequests is given. It is thus shown that the stationary capital labor ratio is a discontinuous function of the proportion \( p \) of altruistic consumers when \( p = 0 \). We first prove that under a complementarity assumption for consumptions of altruistic agents, the equilibrium is determinate for any “size” of the infinite lived altruistic dynasty as long as income effects are not too strong with respect to substitution effects. Some conditions based on a trade-off between the curvatures of preferences and technology extend the same result to the general case. Indeterminate perfect foresight equilibria appear thus to be unlikely when the constraint for operative bequest is considered.

The conventional Barro’s formulation of altruism postulates that each generation con-

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\(^{16}\)This is the case for example when \( A_c = 10, \rho_c = 0.4, \gamma_c = 0.9, \alpha_c = 1 \) and \( \sigma = 4 \). It is worth noticing that with these parameter’s values, there exist two stationary equilibria.

\(^{17}\)The numerical computations are available upon request.
considers the welfare of its immediate offspring by simply adding their utility function to its own. Such an assumption leads to the consideration of the Bellman equation \( V_t = \max \{ u^a(c_t^a, d_{t+1}^a) + \beta V_{t+1} \} \). We have stressed the important role of separability assumptions for the life-cycle utility function on the local dynamic properties of equilibrium paths. It should therefore be interesting to consider a non linear recursive formulation such that \( V_t = \max \{ u(c_t, d_{t+1}, V_{t+1}) \} \). The existence and uniqueness of solutions for this type of non linear altruism have been analyzed by Hori and Kanaya [11]. A further work would consist in studying the influence on the determinacy properties of equilibria of this additional non linearity in an OLG model with heterogeneous agents.

7 Appendix

**Proof of Proposition 1:**
The proof, available upon request, is a straightforward extension to the case with altruistic and non altruistic agents of the results recently provided by Thibault [21].

**Proof of Proposition 2:**
Under Assumption 3, \( f'(k) \) is a monotone decreasing function which satisfies \( \lim_{k \to 0} f'(k) = \infty \), and \( \lim_{k \to \infty} f'(k) = 1 - \mu \). The existence and uniqueness of \( k^* \) is thus ensured. Let us now consider the stationary values for bequest and consumption. At the steady state, the optimization program (1) of non altruistic agents can be reformulated as follows:

\[
s^e = \arg\max_s u^e (w^* - s, R^*s)
\]

with \( s \in [0, w^*] \), \( w^* = f(k^*) - k^*f'(k^*) \), and \( R^* = f'(k^*) \). Under Assumption 1, such a \( s^e \) exists and is unique. Substituting this value in the budget constraints give some unique stationary consumption levels \( \bar{c}^e, \bar{d}^e \).

In the same way, the optimization program (3) of the altruistic agents can be reformulated at the steady state as:

\[
s^a = \arg\max_s u^a (\omega^1 - s, R^*s + \omega^2)
\]

with \( s \in [0, \omega^1] \), and \( \omega^1, \omega^2 \) the income of the first and second period of life respectively. Under Assumption 1, such a \( s^a \) exists and is unique for any given \( (\omega^1, \omega^2, R^*) \). Substituting
s^e and s^a in the capital accumulation equation (14) gives the following function of bequest:

\[ G(x) = (1 - p)s^e + ps^a \left( w^* + x, -(1 + n)x^*, R^* \right) - (1 + n)k^* \]

The stationary bequest is thus solution of \( G(x) = 0 \). Note that under Assumption 2, \( G(x) \) is a strictly monotone increasing function. Moreover, under Assumption 4, \( G(0) = (1 - p)s^e(w^*, R^*) + ps^a(w^*, 0, R^*) - (1 + n)k^* < 0 \). Note finally that we necessarily have \( R^* s^a \geq (1 + n)x \), so that:

\[ \lim_{x \to +\infty} G(x) = +\infty \]

Therefore there exists a unique stationary bequest \( x^* \) such that \( G(x^*) = 0 \). Substituting \( k^* \) and \( x^* \) in the budget constraints of altruistic agents give unique \( c^a \) and \( d^a \). A stationary value for the implicit price \( q^* \) is finally given by \( q^* = u^a_1 \left( c^a, d^a \right) = u^a_2 \left( c^a, d^a \right) / \delta \).

\[ \square \]

**Proof of Proposition 3:**

Consider the capital accumulation condition

\[ (1 - p)s^e(w^*, R^*) + ps^a \left( w^* + x^*, -(1 + n)x^*, R^* \right) - (1 + n)k^* = 0 \]

From Proposition 2, we know that \( \partial k^*/\partial p = 0 \). It follows easily that

\[ \frac{\partial x^*}{\partial p} = - \frac{s^a \left( w^* + x^*, -(1 + n)x^*, R^* \right) - s^e \left( w^*, R^* \right)}{p \left[ s^a_1 \left( w^* + x^*, -(1 + n)x^*, R^* \right) - (1 + n)s^a_2 \left( w^* + x^*, -(1 + n)x^*, R^* \right) \right]} \]

Under Assumption 2, \( \partial x^*/\partial p \leq 0 \) if and only if

\[ s^a \left( w^* + x^*, -(1 + n)x^*, R^* \right) \geq s^e \left( w^*, R^* \right) \]

Considering the stationary utility level \( \bar{u}^a \equiv u^a(\bar{c}^a, \bar{d}^a) \), the result then follows from the first order conditions (6-7) and the budget constraints in (3). Finally, if both type of agents have the same utility function, then under Assumption 2, \( s^a(, , ,) \geq s^e(, ,) \).

\[ \square \]

**Proof of Lemma 1:**

Substituting equation (8) into equations (6) and (7) gives:

\[ u^a_1 \left( c^a_t, d^a_{t+1} \right) - q_t = 0 \]  \hspace{1cm} (22)

\[ u^a_2 \left( c^a_t, d^a_{t+1} \right) - \frac{q_t}{R_{t+1}} = 0 \]  \hspace{1cm} (23)
Under Assumption 1, along an interior optimal path, for each \((q_t, R_{t+1})\) there exist some unique consumption levels \((c_t^a, d_{t+1}^a)\) satisfying equations (22) and (23). The Jacobian matrix of the left-hand side with respect to \((c_t^a, d_{t+1}^a)\) has a determinant equal to \(\tilde{H}_a^a \left( c_t^a, d_{t+1}^a \right) > 0\). The differentiability follows from the implicit function theorem:

$$
\begin{pmatrix}
\frac{\partial c_t^a}{\partial q_t} & \frac{\partial c_t^a}{\partial R_{t+1}} \\
\frac{\partial d_{t+1}^a}{\partial q_t} & \frac{\partial d_{t+1}^a}{\partial R_{t+1}}
\end{pmatrix} = \frac{1}{H_a^a} \begin{pmatrix}
u_{22}^a - \frac{u_{12}^a}{R_{t+1}} & \frac{u_{12}^a q_t}{(R_{t+1})^2} \\
-\nu_{12}^a + \frac{u_{11}^a}{R_{t+1}} & -\frac{u_{11}^a q_t}{(R_{t+1})^2}
\end{pmatrix}
$$

Proof of Lemma 2:

Using the competitive equilibrium prices, the capital accumulation equation (17) becomes:

$$
\psi(k_t, q_t, x_t, k_{t+1}) = (1-p)s^e \left( w(k_t), R(k_{t+1}) \right) + p \left( w(k_t) + x_t - c^a (q_t, R(k_{t+1})) \right) - (1+n)k_{t+1}
$$

The derivative with respect to \(k_{t+1}\) is

$$
\psi_4(k_t, q_t, x_t, k_{t+1}) = f''(k_{t+1}) \left[ (1-p)s_2^e (w_t, R_{t+1}) - p \frac{u_{12}^a q_t}{H_a^a (R_{t+1})^2} \right] - (1+n)
$$

Assume now that \(\psi_4(k^*, q^*, x^*, k^*) \neq 0\). The implicit function theorem implies that in the neighborhood of \((k^*, x^*, q^*)\), \(k_{t+1} = \phi(k_t, q_t, x_t)\), with \(\phi\) a differentiable function such that:

$$
\begin{align*}
\phi_1 &= \frac{k_t f''(k_t)}{\psi_4} \left[ (1-p)s_1^e + p \right] \\
\phi_2 &= \frac{p (\partial c_t^a / \partial q_t)}{\psi_4} \\
\phi_3 &= \frac{p}{\psi_4}
\end{align*}
$$

and \(\partial c_t^a / \partial q_t\) given in the proof of Lemma 1.

Proof of Proposition 4:

The characteristic polynomial is obtained from the Jacobian matrix \(J\) of the three-dimensional dynamical system evaluated at the steady state as follows:
\[
\text{Det } [J - \lambda I_3] \equiv \begin{vmatrix}
\phi_1^* - \lambda & \phi_2^* & \phi_3^* \\
\frac{\beta}{1 + n} f''(k^*) q^* \phi_1^* & 1 - \frac{\beta}{1 + n} f''(k^*) q^* \phi_2^* - \lambda & - \frac{\beta}{1 + n} f''(k^*) q^* \phi_3^* \\
A_1 & A_2 & A_3 - \lambda \\
1 + n & 1 + n & 1 + n
\end{vmatrix} = 0
\]

with \( \partial c_t^a / \partial q_t, \partial c_t^a / \partial R_{t+1}, \partial d_t^a / \partial q_t, \partial d_t^a / \partial R_{t+1} \) given in the proof of Lemma 1, \( \phi_1^*, \phi_2^*, \phi_3^*, \psi_4^* \) given in the proof of Lemma 2, and

\[
\begin{align*}
A_1 &= f''(k^*) \left( \phi_1^* \left( s^a - (1 + n)\beta^{-1} \frac{\partial c_t^a}{\partial R_{t+1}} - \frac{\partial d_t^a}{\partial R_{t+1}} \right) - (1 + n)\beta^{-1} k^* \right) \\
A_2 &= f''(k^*) \phi_2^* \left( s^a - (1 + n)\beta^{-1} \frac{\partial c_t^a}{\partial R_{t+1}} - \frac{\partial d_t^a}{\partial R_{t+1}} \right) - (1 + n)\beta^{-1} \frac{\partial c_t^a}{\partial q_t} - \frac{\partial d_t^a}{\partial q_t} \\
A_3 &= f''(k^*) \phi_3^* \left( s^a - (1 + n)\beta^{-1} \frac{\partial c_t^a}{\partial R_{t+1}} - \frac{\partial d_t^a}{\partial R_{t+1}} \right) + (1 + n)\beta^{-1}
\end{align*}
\]

Considering the expression \( \beta f''(k^*) q^* / (1 + n) \) as a common factor of the second line of the matrix \((J - \lambda I_3)\), and substituting line 1 by the sum of lines 1 and 2, some tedious computations finally give:

\[
P(\lambda) = \lambda^3 - \lambda^2 T + \lambda \Im - D = 0 \tag{24}
\]

with

\[
\begin{align*}
T &= 1 + \beta^{-1} + \frac{k^* f''(k^*)}{\psi_4^*}(1 - p)s_1^a(w^*, R^*) + \frac{f''(k^*)p}{\psi_4^* (1 + n)}[(1 + n)k^* - s^a] \\
&\quad - \frac{\beta f''(k^*) pq^*}{\psi_4^* (1 + n)} \left[ \frac{\partial c_t^a}{\partial q_t} + \frac{1}{(1 + n)} \frac{\partial d_t^a}{\partial q_t} \right] \\
\Im &= \frac{f''(k^*)p}{\psi_4^* (1 + n)}[(1 + n)k^* - s^a] + \beta^{-1} + \frac{k^* f''(k^*)}{\psi_4^*}(1 - p)s_1^a(w^*, R^*)(1 + \beta^{-1}) \\
D &= \beta^{-1} \frac{k^* f''(k^*)}{\psi_4^*}(1 - p)s_1^a(w^*, R^*)
\end{align*}
\]

Let the proportion \( p \) of altruistic agents be equal to one. First note that since \( P(0) = -D = 0 \), one eigenvalue is equal to zero. Moreover we have \( (1 + n) k^* = s^a \). Using the fact that \( q^* = u_2^a(1 + n)/\beta \), it remains thus to solve the following degree-2 polynomial:

\[
Q(\lambda) = \lambda^2 - \lambda T + \Im = 0
\]

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with
\[ T = 1 + \beta^{-1} - \frac{f''(k^*)u_2^a}{\psi_4^s} \left[ \frac{\partial c_t^a}{\partial q_t} + \frac{1}{(1 + n)} \frac{\partial d_{t+1}^a}{\partial q_t} \right] \]
\[ \mathbb{G} = \beta^{-1} \]

Some tedious but straightforward algebra give:
\[ \psi_4^s = -\frac{f''(k^*)u_{12}^a u_2^a + f'(k^*)H_u^a(1 + n)}{f'(k^*)H_u^a} \]
\[ T = -\frac{1}{\psi_4^s H_u^a} \left[ (1 + \beta)f'(k^*)H_u^a + f''(k^*)u_2^a \left( u_{22}^a + \frac{\beta u_{11}^a}{(1 + n)^2} \right) \right] \]
\[ Q(-1) = \frac{1}{\psi_4^s H_u^a} \left[ 2 \left(1 + \frac{\beta}{\beta} \right) H_u^a(1 + n) + f''(k^*)u_2^a \left( u_{22}^a + \frac{1 + \beta}{1 + n} u_{12}^a + \frac{\beta u_{11}^a}{(1 + n)^2} \right) \right] \]

Since \( Q(0) = \mathbb{G} \) is greater than one, indeterminacy is impossible, and the sign of the eigenvalues is given by the sign of the trace \( T \). Under Assumptions 1, 2 and 3, we have \( T > 0 \) if and only if \( f''(k^*)u_{12}^a u_2^a + f'(k^*)H_u^a(1 + n) > 0 \).

First consider the case \( T > 0 \) with two positive eigenvalues. Since we have \( \psi_4^s < 0 \), it follows that \( Q(1) < 0 \). The steady state is thus a regular saddle point.

Assume now that \( T < 0 \). The two eigenvalues are negative, and the steady state is a regular saddle point if and only if \( Q(-1) < 0 \).

\[ \square \]

**Proof of Theorem 1:**

Consider the characteristic polynomial (24) given in the proof of Proposition 4. When \( \lambda = 1 \), we have:
\[ P(1) = \frac{\beta f''(k^*)pq^*}{\psi_4^s(1 + n)} \left( \frac{\partial c_t^a}{\partial q_t} + \frac{1}{(1 + n)} \frac{\partial d_{t+1}^a}{\partial q_t} \right) \]

Under Assumptions 3 and 5, if \( u_{12}^a(c, d) \leq 0 \), then \( \psi_4^s < 0 \). Moreover, under Assumptions 1 and 2, the term between parenthesis is strictly negative. \( P(1) \) is thus strictly negative. Note also that
\[ \lim_{\lambda \to +\infty} P(\lambda) = +\infty \]
Therefore, there exists \( \lambda^* > 1 \) such that \( P(\lambda^*) = 0 \).

\[ \square \]
Proof of Theorem 2:
When \( \lambda = -1 \), the characteristic polynomial (24) becomes:
\[
P(-1) = -2 \left\{ \left(1 + \beta^{-1}\right) \left[1 + \frac{k^* f''(k^*)}{\psi_4^*} (1 - p) s_1^e\right] + \frac{f''(k^*) p}{\psi_4^*} \left(k^* - \frac{s^a}{1 + n}\right)\right\} \\
+ \frac{\beta f''(k^*) p q^*}{\psi_4^*(1 + n)} \left(\frac{\partial c^a_i}{\partial q_t} + \frac{1}{(1 + n)} \frac{\partial d^a_{i+1}}{\partial q_t}\right)
\]
Considering the expressions of \( \psi_4^*, q^*, \partial c^a_i / \partial q_t, \partial d^a_{i+1} / \partial q_t \), and rearranging terms give:
\[
P(-1) = -2 \frac{f''(k^*)}{\psi_4^*} \left\{ \left(1 + \frac{\beta}{\beta}\right) (1 - p) \left[s_2^e + k^* s_1^e - \frac{1 + n}{f''(k^*)}\right] + p \left(k^* - \frac{s^a}{1 + n}\right)\right\} \\
+ \frac{p}{\psi_4^* H_a^a} \left\{2 \left(1 + \frac{\beta}{\beta}\right) H_a^a(1 + n) + f''(k^*) u_2^a \left(u_{22}^a + \frac{1 + \beta}{1 + n} u_{12}^a + \frac{\beta u_{11}^a}{(1 + n)^2}\right)\right\}
\]
Under Assumptions 1-4,6, let \( u_{12}^a > 0 \) be such that \( \psi_4^* > 0 \), and let \( s^e(w^*, R^*) \geq s^a(w^* + x^*, -(1 + n)x^*, R^*) \). It follows easily that \( k^* - s^a/(1 + n) \geq 0 \). \( P(-1) \) is thus strictly positive if
\[
2 \left(1 + \frac{\beta}{\beta}\right) H_a^a(1 + n) + f''(k^*) u_2^a \left(u_{22}^a + \frac{1 + \beta}{1 + n} u_{12}^a + \frac{\beta u_{11}^a}{(1 + n)^2}\right) \geq 0
\]
Note also that
\[
\lim_{\lambda \to -\infty} P(\lambda) = -\infty
\]
Therefore, there exists \( \lambda^* < -1 \) such that \( P(\lambda^*) = 0. \)

Proof of Theorem 3:
Consider the expression \( P(-1) \) in the proof of Theorem 2. Rearranging terms gives:
\[
P(-1) = -2 \frac{f''(k^*)}{\psi_4^*} \left\{ \left(1 + \frac{\beta}{\beta}\right) (1 - p) \left[s_2^e + k^* s_1^e - \frac{1 + n}{f''(k^*)}\right] + p k^*\right\} \\
+ \frac{p}{\psi_4^* H_a^a} \left\{2 \left(1 + \frac{\beta}{\beta}\right) H_a^a(1 + n) + f''(k^*) u_2^a \left(u_{22}^a + \frac{1 + \beta}{1 + n} u_{12}^a + \frac{\beta u_{11}^a}{(1 + n)^2}\right)\right\} \\
+ \frac{p}{\psi_4^* H_a^a} \left\{2 f''(k^*) H_a^a \frac{s^a}{1 + n}\right\}
\]
Consider also the following functions
\[
\psi(k_t, q_t, x_t, k_{t+1}) = (1 - p) s^e(w_t, R_{t+1}) + p \left(w_t + x_t - c^a(q_t, R_{t+1})\right) - (1 + n) k_{t+1}
\]
\[
\theta(k_t, x_t, x_{t+1}, k_{t+1}) = (1 - p) s^e(w_t, R_{t+1}) + p s^a(w_t + x_t, (1 + n) x_{t+1}, R_{t+1}) - (1 + n) k_{t+1}
\]

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Taking into account the fact that

\[ x_{t+1} = \frac{1}{1+n} [R_{t+1} (x_t + w_t - c^a(q_t, R_{t+1})) - d^a(q_t, R_{t+1})] \equiv \varphi(k_t, q_t, x_t, k_{t+1}) \]

it follows

\[ \psi(k_t, q_t, x_t, k_{t+1}) = \theta (k_t, x_t, \varphi(k_t, q_t, x_t, k_{t+1}), k_{t+1}) \]

Therefore, we have

\[ \psi_4^* = \theta_3^* \varphi_4^* + \theta_4^* \]

\[ = f''(k^*)[(1-p)s_2^{e} + p s_3^{e}] - (1+n) - ps_2^e f''(k^*) \left( s^a - R^* \frac{\partial c^a}{\partial R_{t+1}} - \frac{\partial d^a}{\partial R_{t+1}} \right) \]

\[ = f''(k^*)[(1-p)s_2^{e} + p s_3^{e}] - (1+n) - ps_2^e f''(k^*) \left( s^a + u_2^a \frac{\partial d^a}{\partial q_t} \right) \]

Under Assumptions 1-4, if \( u_{12}^a > 0 \) is such that \( \psi_4^* > 0 \), then it follows

\[ s^a + u_2^a \frac{\partial d^a}{\partial q_t} = s^a - \frac{u_2^a}{H_u} \left( -u_{12}^a + \frac{\beta u_{11}^a}{1+n} \right) < 0 \]

Substituting this expression in equation (25) gives

\[ P(-1) = - \frac{2f''(k^*)}{\psi_4^*} \left\{ \left( \frac{1+\beta}{\beta} \right) (1-p) \left[ s_2^{e} + k^* s_1^{e} - \frac{1+n}{f''(k^*)} \right] + pk^* \right\} \]

\[ + \frac{p}{\psi_4^* H_u} \left\{ 2 \left( \frac{1+\beta}{\beta} \right) H_u^a (1+n) + f''(k^*) u_2^a \left( u_{22}^a + \frac{1+\beta}{1+n} u_{12}^a + \frac{\beta u_{11}^a}{(1+n)^2} \right) \right\} \]

\[ - \frac{p}{\psi_4^* H_u} \left\{ 2f''(k^*) \frac{u_2^a}{1+n} \left( -u_{12}^a + \frac{\beta u_{11}^a}{1+n} \right) \right\} \]

The rest of the proof is obvious.

\[ \square \]

**References**


