Factoring N-Cycles and Counting Maps of Given Genus

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We present an explicit expression for the number of decompositions of an \( n \)–cycle as a product of any two permutations of cycle types given by partitions \( \lambda \) and \( \mu \). The same expression is also counting the number of unicellular rooted bicolor maps on an orientable surface of genus \( g \) with vertex degree distribution given by \( \lambda \) and \( \mu \). The relation between the genus and the partitions \( \lambda \) and \( \mu \) is given by \( \ell(\lambda) + \ell(\mu) = n + 1 - 2g \) where \( \ell(\lambda) \) is the number of parts of \( \lambda \). We use character theory and the group algebra of the symmetric group to develop our expression. The key argument is the construction of a bijection involving the character formula at one end and our final expression at the other end.

1. Introduction

A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) is a finite non-increasing sequence of positive integers \( \lambda_i \) such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \). The non-zero terms of \( \lambda \) are called the parts of \( \lambda \) and the number \( k \) of parts is the length of \( \lambda \), denoted \( \ell(\lambda) \). We also write \( \lambda = 1^{a_1} \cdot 2^{a_2} \cdot \cdots \cdot n^{a_n} \) when \( a_i \) parts of \( \lambda \) are equal to \( i \) (\( i = 1, \ldots, n \)). When the sum \( \lambda_1 + \lambda_2 + \cdots + \lambda_k = n \), we call \( n \) the weight of \( \lambda \) and we write \( \lambda \vdash n \) or \( |\lambda| = n \).

The conjugacy classes \( C_\lambda \) of the symmetric group \( S_n \) are indexed by partitions of \( n \) which are called the cycle types of the permutations \( \sigma \in C_\lambda \). Let \( \mathbb{Q}[S_n] \) be the group algebra of the symmetric group over the rational numbers \( \mathbb{Q} \) and let \( C[S_n] \) be the center of this group algebra. The formal sum of the permutations in a conjugacy class \( C_\lambda \) belongs to \( C[S_n] \) and we denote it \( K_\lambda \). We abuse notation and also call the elements \( K_\lambda \) conjugacy classes. The set \( \{K_\lambda\}_{\lambda \vdash n} \) of these formal sums forms a linear basis for the center \( C[S_n] \). Thus a product of two conjugacy classes \( K_\lambda \cdot K_\mu \) in \( C[S_n] \) can be decomposed in the basis \( \{K_\lambda\}_{\lambda \vdash n} \) and the coefficients of the decomposition are non-negative integers called connection coefficients or structure constants of \( C[S_n] \):

\[
K_\lambda \cdot K_\mu = \sum_{\gamma \vdash n} c_{\lambda,\mu}^{\gamma} K_\gamma.
\]

Equivalently, the number \( c_{\lambda,\mu}^{\gamma} \) counts the number of solutions \((\sigma, \rho) \in C_\lambda \times C_\mu \) of the equation \( \sigma \rho = \pi \) where \( \pi \) is any fixed permutation of \( C_\gamma \).

Efforts for computing them have been made by several authors, beginning in the late 1970s with Walkup [18], Bertram and Wei [2], Boccara [3] and also Stanley [17], Jackson [10].

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Katriel [13], and Jones [12]. All these authors compute special values of \( c_{\gamma}^{\lambda,\mu} \) and most of their explicit results concern the case \( \gamma = n \), i.e., at least one permutation is an \( n \)-cycle, and restricted values of \( \lambda \) and \( \mu \). Our main result (Theorem 2.1) is an explicit expression for \( c_{\lambda,\mu}^{n} \), valid for any partitions \( \lambda \) and \( \mu \) of \( n \).

From now on, we shall be interested only with this case \( \gamma = n \) and thus we can define the genus \( g(\lambda,\mu) \) of a pair of partitions of weight \( n \) by the equation

\[
\ell(\lambda) + \ell(\mu) = n + 1 - 2g(\lambda, \mu).
\]

We shall assume that \( g(\lambda, \mu) \) is a non-negative integer, as a straightforward decomposition into transpositions proves that otherwise \( c_{\lambda,\mu}^{n} = 0 \).

Our main result (Theorem 2.1) is the explicit computation of a family \((S_k, g(X))_{k \geq 0}\) of symmetric polynomials of degree \( 2g \) such that, for any partitions \( \lambda \), \( \mu \) of \( n \) of genus \( g = g(\lambda, \mu) \),

\[
c_{\lambda,\mu}^{n} = \prod_{i} a_{i}! \prod_{i} b_{i}! 2^{2g} \sum_{g_1 + g_2 = g} S_{\ell(\lambda), g_1}(\lambda)S_{\ell(\mu), g_2}(\mu).
\]

In a previous result [8, Theorem 3.1], the simple expression for the coefficients \( c_{\lambda,\mu}^{n} \), when \( g(\lambda, \mu) = 0 \) was derived. This coefficient was later interpreted combinatorially by Goulden and Jackson [6] as the number of unicellular rooted bicolored maps with \( n \) edges on a surface of genus zero, the vertices of each color having degree distribution given by \( \lambda \) and \( \mu \), respectively. Our main result (Theorem 2.1) is the extension of this result to maps on a surface of arbitrary genus i.e., to pairs of partitions \((\lambda, \mu)\) with arbitrary \( g(\lambda, \mu) \). Except for much more restricted values of \((\lambda, \mu, \gamma)\), this is, as far as we know, the first general expression for connection coefficients which do not belong to the so-called minimal transitive case \( g = 0 \).

In Section 2, we state our main result (Theorem 2.1) and outline the successive steps of the proof. Unlike Goulden and Jackson’s proof of the special case \( g = 0 \), ours follows from the general character theoretic expression and is not constructive. However, crucial use is made of a new bijective interpretation during the course of the proof. In Section 3, we give the proof of Theorem 2.1.

In Section 4, we give some special cases of Theorem 2.1 and we recall the relation between pairs of permutations and maps on oriented surfaces and restate our result in this context. From this point of view, our result generalizes a formula of Walsh and Lehman [19, Formula 9] for some monochromatic maps to the corresponding bicolored maps.

From their formula, Walsh and Lehman were able to sum over all types of monochromatic unicellular maps with given genus and number of vertices to obtain a formula for the number of maps with given number of vertices. Several equivalent expressions for this coarser enumeration were obtained by Harer and Zagier [9] in connection with the Euler characteristic of the moduli spaces of algebraic curves. Independently, Jackson [10] used the character theoretic approach to obtain the same result. His approach was simplified by Zagier [20] and extended by Jackson and Visentin [11]. Following Zagier, Adrianov [1] obtained recursive expressions which are analogue to Harer–Zagier Formulae for bicolored unicellular maps. In Section 4, using Theorem 2.1, we are able to deduce explicit expressions for the number of these maps. Although equivalent expressions were already obtained for these coarser enumerations (in [1] and [11]), our expressions are of a different nature: they involve summations of positive contributions, with number of terms bounded by a function of \( g \) and independent of \( n \). In particular for fixed genus, asymptotic results are derived when \( n \) goes to infinity.
2. The Main Result: A General Expression for $c_{\lambda, \mu}^n$

A composition $p$ of $n$ is a finite sequence of non-negative integers summing up to $n$ and we denote it by $p \models n$. Following [15], given a partition $\lambda$, let $z_{\lambda} = \prod \alpha_i! \alpha_i^i$ for a partition $\lambda = \alpha_1 \ldots \alpha_n$.

Theorem 2.1. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be any two partitions of $n$ with $g(\lambda, \mu) = g$. Then

$$c_{\lambda, \mu}^n = \frac{n}{z_{\lambda} z_{\mu} 2^{2g}} \sum_{g_1 + g_2 = g} (\ell + 2g_1 - 1)! (m + 2g_2 - 1)! \sum_{(i_1, \ldots, i_\ell) \models g_1} \prod_k \left( \begin{array}{c} \lambda_k \\ 2i_k + 1 \end{array} \right) \sum_{(j_1, \ldots, j_m) \models g_2} \prod_k \left( \begin{array}{c} \mu_k \\ 2j_k + 1 \end{array} \right),$$

or equivalently, with $\lambda = 1^{a_1} \ldots n^{a_n}$ and $\mu = 1^{b_1} \ldots n^{b_m}$,

$$c_{\lambda, \mu}^n = \prod \alpha_i! \prod \beta_i! 2^{2g} \sum_{g_1 + g_2 = g} S_{g_1}(\lambda) S_{g_2}(\mu),$$

where

$$S_k(x_1, \ldots, x_k) = (k + 2g - 1)! \sum_{(p_1, \ldots, p_k) \models g} \prod_i \frac{1}{2p_i + 1} \left( \begin{array}{c} x_i - 1 \\ 2p_i \end{array} \right)$$

is a symmetric polynomial of degree $2g$ in the $x_i$.

This formula has several interesting properties:

1. The coefficient $c_{\lambda, \mu}^n$ is expressed as a sum of positive contributions, thus easily giving an idea of its order of magnitude. This is used in Section 4 to obtain asymptotic results for maps.
2. The product form of the formula is rather surprising: it suggests in particular that the two partitions contribute independently to the genus.
3. The number of terms in the summation is $n(\ell, m, g) = \sum_{g_1 + g_2 = g} (\ell + g_1 - 1)(m + g_2 - 1)$ and hence is a polynomial in $\ell$ and $m$ of total degree $g$.

These properties are meant to be compared with those of the known expression which is readily obtained from the character theoretic point of view (Eqns (4) and (5) in Section 3.1). Indeed, this latter expression involves the evaluations $\chi_{\lambda}^{(n-r)}$ of some characters of the symmetric group which can be exponential with respect to the number of parts of $\lambda$. Moreover, in this expression, an asymptotic evaluation of $c_{\lambda, \mu}^n$ is made very difficult by the occurrence of alternating signs.

2.1. Survey of the proof. In order to help the reader we provide a short overview of the successive steps in the proof.

The first step is based on the character theoretic approach which is exposed in [7] or [10] (among many other references). Using explicit expressions for characters of the symmetric group, a first formula (Formula (4)) is given in Section 3.1.

Then we produce a sequence of bijections establishing a correspondence between two sets that are respectively the combinatorial models of Formula (4) and of our final expression: In Section 3.2 our first combinatorial model, ‘quasi-painted diagram’, is introduced and the
evaluation of some characters are given as weighted summations over the corresponding set of objects. In Section 3.3 we use a bijection to replace quasi-painted diagrams by properly ‘painted diagrams’ and we rewrite Formula (4) as a weighted summation over some ‘painted diagram matchings’ (Formula (8)). The introduction in Section 3.4 of ‘connected components’ of diagram matchings allows us to set apart the diagram matching from its painting and to show that the weight depends only on the painting (Formula (9)). This is used in Section 3.5 to apply a sign-reversing involution. As expected, the fix-points yield positive contributions (Formula (10)). These contributions count ‘colorings’ of the diagram matchings as explained in Section 3.6. In Section 3.7, we show that colored diagrams are enumerated by the formula of Theorem 2.1, thus completing the proof.

3. Proof of the Main Theorem

3.1. The character theoretic expression. Detailed proofs for this section can be found in [7, 10, 12, 15, 17], so we only give all ingredients for the sake of completeness.

Characters $\chi^\lambda$ of irreducible representations of $S_n$ are class functions and can therefore be seen as elements of the center $C[S_n]$:

$$\chi^\lambda = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \sigma \in C[S_n].$$

The set $\{\chi^\lambda\}_{\lambda \vdash n}$ forms a linear basis and a set of fundamental idempotents (up to a constant) for $C[S_n]$. This observation and the fact that the transition matrix between the bases $\{K^\lambda\}_{\lambda \vdash n}$ and $\{\chi^\lambda\}_{\lambda \vdash n}$ is the character table, allows us to develop the Frobenius formula:

$$c^{\lambda \mu} = \frac{|C_\lambda| |C_\mu|}{n!} \sum_{\nu \vdash n} \frac{1}{f^\nu} \chi^\nu_\lambda \chi^\nu_\mu$$

where $\chi^\nu_\lambda$ is the character $\chi^\nu$ evaluated at the conjugacy class $C_\lambda$ and $f^\nu$ is the dimension of the representation indexed by $\nu$. The Murnaghan–Nakayama rule recursively computes the values of the characters $\chi^\nu_\lambda$. We obtain from this rule the value

$$\chi^{(n-r)}_\lambda = \begin{cases} (-1)^r \text{ if } \nu = 1^r(n-r), \\ 0 \text{ otherwise.} \end{cases}$$

of a character $\chi^\nu$ evaluated at a full cycle, i.e., at an element of the conjugacy class $C(n)$. From the hook formula, we also have $f^{1^r(n-r)} = \binom{n-1}{r}$. Using these two observations, we rewrite Formula (3) as

$$c^{\lambda \mu} = \frac{n}{z_\lambda z_\mu} \sum_{r=0}^{n-1} (-1)^r r!(n-r)! \chi^{1^r(n-r)}_\lambda \chi^{1^r(n-r)}_\mu.$$

The following formula is well known (see Goupil [7], Stanley [17] or Littlewood [14, p. 139]) and can be derived from the Murnaghan–Nakayama rule: let $\lambda = 1^{a_1} \ldots n^{a_n}$ and $\ell$ be the smallest index such that $a_\ell \neq 0$, then

$$\chi^{1^r(n-r)}_\lambda = (-1)^r [T^{r+}] (1 + T + \ldots + T^{\ell-1})(1 - T^{\ell})^{a_{\ell+1}} \prod_{i \geq \ell+1} (1 - T^i)^{a_i}. \quad (5)$$

Our first goal is to give a combinatorial interpretation of this formula.
3.2. Painted diagrams: a combinatorial model. Let \( \lambda \) be a partition of \( n \) and write \( \lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)}) \) or \( \lambda = \ell a_\ell \cdots n a_n \) with \( a_\ell \neq 0 \). The Ferrers’ diagram of shape \( \lambda \) is a planar representation of \( \lambda \) made of rows positioned on top of each other with the \( i \)th row, starting from the bottom, having \( \lambda_i \) cells. In the sequel \( \lambda \) shall indistinctly denote a partition or its Ferrers’ diagram, hence we shall refer to lines and cells of \( \lambda \), meaning lines and cells of the Ferrers’ diagram of shape \( \lambda \). As the top line \( \lambda_{\ell(\lambda)} = \ell \) and its cells are treated differently in our combinatorial model, we shall refer to them as the top line and cells of \( \lambda \). We define \( \lambda^* = (\ell - 1) a_{\ell - 1} \cdots n a_n \) and \( \lambda^* = \ell a_{\ell - 1} \cdots n a_n \). In particular, when \( \ell = 1, \lambda^* = \lambda^* \).

A dotted diagram of shape \( \lambda \) is a diagram of shape \( \lambda \) in which the rightmost cell of the top line contains a bullet. In particular, if \( \lambda \) is dotted then the empty cells of \( \lambda \) form a diagram of shape \( \lambda^* \).

A painted diagram \( \lambda_{RG} \) of shape \( \lambda \) is a Ferrers’ diagram of shape \( \lambda \) in which to each cell is assigned one of two colors, say red and green, in such a way that:

1. the subdiagram \( \lambda^* \) is painted,
2. in the top line of \( \lambda \) (which contains the dotted cell), red cells, if any, appear on the left of green cells, if any.

Let \( P_r(\lambda) \) denote the set of painted diagrams of shape \( \lambda \) with \( r \) red cells and \( P(\lambda) = \bigcup_{0 \leq r \leq |\lambda|} P_r(\lambda) \).

A quasi-painted diagram \( \lambda_{RG} \) of shape \( \lambda \) is a dotted Ferrers’ diagram of shape \( \lambda \) in which each cell, except the dotted one, is assigned one of two colors, say red and green, in such a way that:

1. the subdiagram \( \lambda^* \) is painted,
2. in the top line of \( \lambda \) (which contains the dotted cell), red cells, if any, appear on the left of green cells, if any.

Let \( QP_r(\lambda) \) denote the set of quasi-painted diagrams of shape \( \lambda \) with \( r \) red cells.

Given a quasi-painted diagram \( \lambda_{RG} \), we shall denote by \( \lambda_R \) the induced red diagram, \( \lambda_G \) the induced green diagram and \( \lambda^*_{RG} \) the induced painted diagram of shape \( \lambda^* \). Then \( |\lambda_R| \) is the number of red cells of \( \lambda_{RG} \). We shall also denote by \( \lambda^*_R \) and \( \lambda^*_G \) the diagrams induced by the coloring of \( \lambda^*_{RG} \). The following equalities are straightforward:

\[
|\lambda| = |\lambda^*| + 1 = |\lambda_R| + |\lambda_G| + 1,
\]

\[
\ell(\lambda) = \ell(\lambda^*) + 1 = \ell(\lambda^*_R) + \ell(\lambda^*_G) + 1.
\]

Example. Let \( \lambda = (6, 5, 4, 4) \). Then,

\[
\lambda = \begin{array}{cccccc}
\otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\
\end{array}
, \quad \lambda^* = \begin{array}{cccc}
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\end{array}
, \quad \lambda^* = \begin{array}{cccc}
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes & \otimes \\
\end{array}
.
\]

Now let \( \lambda_{RG} \) be the following quasi-painted diagram, and \( \lambda^*_{RG} \) the induced painting of \( \lambda^* \):

\[
\lambda_{RG} = \begin{array}{cccccc}
R & R & G & \cdot \\
G & G & G & G \\
R & R & R & R \\
R & R & R & R \\
R & R & R & R \\
R & R & R & R \\
\end{array}, \quad \lambda^*_{RG} = \begin{array}{cccc}
G & G & G & G \\
R & R & R & R \\
R & R & R & R \\
R & R & R & R \\
R & R & R & R \\
\end{array}.
\]
We also have the following induced diagrams:

\[
\lambda_R = \begin{array}{ccc}
R & R \\
R & R & R & R \\
R & R & R & R & R
\end{array}, \quad \lambda_G = \begin{array}{ccc}
G \\
G & G & G & G
\end{array}
\]

With these notations and standard arguments on generating functions, Formula (5) is rewritten as

\[
f_{\lambda}^{(n-r)} = (-1)^r \sum_{\lambda_{RG} \in \mathcal{QP}_r(\lambda)} (-1)^{\ell(\lambda_{RG})}.
\]

3.3. From quasi-painted diagrams to painted diagram matchings. For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) and any non-negative integer \( \ell \) such that \( 1 \leq \ell \leq \lambda_k - 1 \), define

\[
\lambda^{\ell} = (\lambda_1, \ldots, \lambda_{k-1}, \ell, 1, \ldots, 1)
\]

\( \lambda_k - 1 \) times

and

\[
\lambda^0 = (\lambda_1, \ldots, \lambda_{k-1}, 1, \ldots, 1)
\]

\( \lambda_k - 1 \) times

In particular, if \( \lambda_k = 1 \) then \( \lambda^0 = \lambda^* \).

Let \( \{R, G\}^k \) denote the set of words of length \( k \) on the alphabet \( \{R, G\} \) and \( S(R^k G^h) \) the set of words of \( k \) letters \( R \) and \( h \) letters \( G \). Also let \( R\{R, V\}^l \) denote the set of words of length \( i + 1 \) starting with an \( R \).

**Lemma 3.1.** Let \( \lambda \) be a partition of \( n \), \( \ell_0 = \lambda_{c(\lambda)} \). There exists a bijection

\[
\mathcal{QP}_r(\lambda) \times \{R, G\}^{\ell_0} \simeq \bigcup_{\ell=0}^{\ell_0-1} (\mathcal{P}_r(\lambda^\ell) \times \{R, G\}^{|\ell| h_0})
\]

**Proof.** We shall use the following fact: let \( k \) and \( h \) be non-negative integers. Then the application \( f_{k,h}: w \mapsto (u, v) \) where \( u \) is the maximal left factor of \( w \) such that \( uv = w \), \( |u|_R \leq k \) and \( |u|_G \leq h \) defines a bijection

\[
\{R, G\}^{k+h+1} \simeq \left( \bigcup_{i=0}^{k} S(R^i G^h) \times \{R, G\}^{k-i} \right) \cup \left( \bigcup_{j=0}^{h} S(R^k G^j) \times \{R, G\}^{h-j} \right)
\]

Indeed the application \( g: (u, v) \mapsto uv \) is clearly the inverse of \( f_{k,h} \). Then we check that the image of \( \{R, G\}^{k+h+1} \) is as described.

**Example.** Let \( k = 5, h = 4 \), then the image of the word \( GGRGRGGRGGRG \) is the factorization \( (GGRGRG, GGRG) \), because the left factor would be extended by a letter \( G \), whereas it already contains \( q = 4 \) of them. Similarly \( f_{5,4}(RRRGRGGRG) = (RRRGRGGRG, G) \).

Now let \( \lambda \) be a partition of \( n \), \( \ell_0 = \lambda_{c(\lambda)} \). Let also \( \lambda_{RG} \in \mathcal{QP}(\lambda) \) and \( w \in \{R, G\}^{\ell_0} \). The image of \( \lambda_{RG}, w \) is defined as follows: let \( k \) and \( h \) be, respectively, the number of letters \( R \) and \( G \) in the top line of \( \lambda_{RG} \) and let \( (u, v) = f_{k,h}(w) \) and \( \ell = |v| - 1 \). If \( \ell = 0 \) let \( v' = v \) otherwise let \( v = xv \) where \( x \) is the first letter of \( v \) (so that \( u \) contains as many letters \( x \) as the top line of \( \lambda_{RG} \)). Finally, let \( \lambda'_{RG} \) be the painted diagram of shape \( \lambda^\ell \) in which:
(1) Lines of $\lambda^*$ are painted according to $\lambda_{RG}$.
(2) The line of length $\ell$ of the hook of $\lambda^\ell$ has the opposite color than $x$.
(3) The $\ell_0 - \ell$ lines of length 1 of the hook of $\lambda^\ell$ are painted according to the word $u$
(which has length $\ell_0 - \ell$) written from top to bottom.

The image of $(\lambda_{RG}, u)$ is then defined to be $(\lambda^\ell_{RG}, u')$, which satisfies $|u'| = \ell + \delta_{\ell,0}$ and this
is clearly a bijection preserving the respective numbers of red and green cells in the diagrams.

According to Lemma 3.1, Formula (6) can be rewritten as

$$\ell_0 \sum_{\ell=0}^{\ell_0-1} 2^{\ell_0 - \ell + \delta_{\ell,0}} \sum_{\lambda_{RG} \in P_{\ell}^{\ell}(\ell)} (-1)^{\ell(\lambda_{RG})}$$

where $\ell_0 = \ell(\lambda)$.

We now consider the equation of Eqn (4):

$$n ! = \sum_{\ell=0}^{n-1} (-1)^\ell r!(n - 1 - r)! H^{(n-r)} \lambda \mu (n-r),$$

$$= \sum_{\ell=0}^{n} z_{\lambda,\mu} 2^{\ell_0 + m_0} \sum_{0 < l < m_0} 2^{\ell + \delta_{\ell,0} + m + \delta_{m,0}} \sum_{\phi(RG) \in P_{\ell}^{\ell}(\ell)} (-1)^{\phi(RG)} r!(n - 1 - r)!.$$

As $r!$ is the number of bijections between red cells of $\lambda^\ell_{RG}$ and red cells of $\mu^m_{RG}$, and $(n - 1 - r)!$
is the number of bijections between green cells of those painted diagrams, we are led to consider
bijections $\phi: \lambda^\ell_{RG} \rightarrow \mu^m_{RG}$ which preserve colors.

A Ferrers’ diagram matching of type $(\lambda, \mu)$ is a bijection from cells of $\lambda$ onto cells of $\mu$.

Let $M(\lambda, \mu)$ denote the set of Ferrers’ diagram matchings of type $(\lambda, \mu)$.

A painted diagram matching $\phi_{RG}$ of type $(\lambda, \mu)$ is a triple $(\lambda_{RG}, \mu_{RG}, \phi)$ where $\lambda_{RG} \in P(\lambda), \mu_{RG} \in P(\mu)$ and $\phi \in M(\lambda, \mu)$ such that $\phi(\lambda_{RG}) = \mu_{RG}$. Let $PM(\lambda, \mu)$ denote the set
of painted diagram matchings of type $(\lambda, \mu)$.

For $\phi_{RG}$ in $PM(\lambda^\ell, \mu^m)$, let $c(\phi_{RG}) = |\lambda^\ell_{RG}| - \ell(\lambda^\ell) - \ell(\mu^m)$. Then, using these definitions, we obtain

$$c^n_{\lambda,\mu} = \frac{n}{z_{\lambda,\mu}} 2^{\ell_0 + m_0} \sum_{0 < l < m_0} 2^{\ell + \delta_{\ell,0} + m + \delta_{m,0}} \sum_{\phi(RG) \in PM(\lambda^\ell, \mu^m)} (-1)^{\phi(RG)}.$$

For any pair $(\lambda, \mu)$ of partitions of the same weight with $g(\lambda, \mu) = g$, let $h(\lambda, \mu) = |\lambda| - \ell(\lambda) - \ell(\mu)$. Then, by definition of $\ell^\ell$ and $\mu^m$ and as $h(\lambda, \mu) = 2g - 1$, we obtain $h(\lambda^\ell, \mu^m) = 2g - (\ell_0 + m_0) + (\ell + \delta_{\ell,0} + m + \delta_{m,0})$ and

$$c^n_{\lambda,\mu} = \frac{n}{z_{\lambda,\mu}} 2^{2g} \sum_{0 < l < m_0} 2^{h(\lambda^\ell, \mu^m)} \sum_{\phi(RG) \in PM(\lambda^\ell, \mu^m)} (-1)^{\phi(RG)}.$$ 

3.4. Connected components of painted diagram matchings. Let $\phi: \lambda^\ell \rightarrow \mu^m$ be a diagram matchings. Let $x \sim_p y$ be the following relation on cells of $\lambda^\ell \cup \mu^m$:

- $x \sim_p y$ if $x \in \lambda^\ell, y \in \mu^m$ and $y = \phi(x)$,
- $x \in \mu^m, y \in \lambda^\ell$ and $x = \phi(y)$,
- $x \in \lambda^\ell, y \in \lambda^\ell$ with $x$ and $y$ in the same line,
- $x \in \mu^m, y \in \mu^m$ with $x$ and $y$ in the same line.
The transitive closure $\sim_p$ of $\sim$ is an equivalence relation. Each equivalence class contains an equal number of cells of $\lambda^\ell$ and $\mu^m$ because $\phi$ is stable on them. Hence, equivalence classes induce decompositions of $\lambda^\ell$, $\mu^m$ and $\phi$ and we refer to the factors as the connected components of $\lambda^\ell$, $\mu^m$ and $\phi$. Let $c(\phi)$ denote the number of connected components of type $\lambda$, $\mu$ and $\phi$ and let $\phi = \phi^{(1)} + \phi^{(2)} + \ldots + \phi^{(c(\phi))}$ where $\phi^{(i)} : \lambda^{(i)} \rightarrow \mu^{(i)}$ is the restriction of $\phi$ to the $i$th equivalence class of $\sim$.

Our interest in connected components lies in the fact that all cells in a connected component have the same color in a painting compatible with $\phi$. Hence colors can be given to components rather than to cells. Indeed as soon as colors are given to components, the bijection $\phi_{RG}$ induced from $\phi$ is stable on colors and $\lambda_{RG}$ is painted as well as $\mu_{RG}$. For a given diagram matching $\phi$, the associated painted diagram matchings $\phi_{RG}$ are therefore in one-to-one correspondence with subsets of $[c(\phi)] = \{1, \ldots, c(\phi)\}$, giving

**Lemma 3.2.** There exists a bijection

$$PM(\lambda^\ell, \mu^m) \simeq \{ (\phi, R) \mid \phi \in M(\lambda^\ell, \mu^m), R \subseteq [c(\phi)] \}.$$  

Let $\phi$ be in $M(\lambda^\ell, \mu^m)$ and for all $i \in [c(\phi)]$, let

$$h^{(i)} = |\lambda^{(i)}| - \ell(\lambda^{(i)}) - \ell(\mu^{(i)}),$$

where $\ell(\lambda^{(i)})$ denotes the number of lines in the component $\phi^{(i)}$ which also belong to $\lambda^*$ (i.e., cells of the top hook of $\lambda^\ell$ are ignored).

If $(\phi, R)$ is the image of a painted diagram matching $\phi_{RG}$ of type $(\lambda^\ell, \mu^m)$ via Lemma 3.2, then we have

$$\sum_{i \in R} h^{(i)} = |\lambda_{RG}| - \ell(\lambda_{RG}) - \ell(\mu_{RG}) = \varepsilon(\phi_{RG}).$$

Hence summation (8) becomes

$$c_{\lambda^{\ell}, \mu^{m}} = \sum_{i_0, c, t_{0, c}, a_{0, c}, m_{0, c}} 2^{h(\lambda^\ell, \mu^m)} \sum_{\phi \in M(\lambda^\ell, \mu^m)} \sum_{R \subseteq [c(\phi)]} (-1)^{\sum_{i \in R} h^{(i)}}. \quad (9)$$

**3.5. Sign reversing involution.**

**Lemma 3.3.** Let $\phi$ be a diagram matching of type $(\lambda^\ell, \mu^m)$, then

$$\sum_{R \subseteq [c(\phi)]} (-1)^{\sum_{i \in R} h^{(i)}} \begin{cases} 0 & \text{if } \exists i \mid h^{(i)} \equiv 1 \pmod{2}, \\ 2^{c(\phi)} & \text{otherwise}. \end{cases}$$

**Proof.** Suppose there exists $i$ such that $h^{(i)}$ is odd and take $i_0$ the least such index. Then the involution

$$R \mapsto \begin{cases} R \setminus \{i_0\} & \text{if } i_0 \in R, \\ R \cup \{i_0\} & \text{if } i_0 \notin R \end{cases}$$

on subsets of $[c(\phi)]$ changes the sign in the summation. Therefore all terms cancel. Otherwise all $h^{(i)}$ are even, hence $\sum_{i \in R} h^{(i)}$ is even and the sum is equal to the number of subsets of $[c(\phi)]$ which is $2^{c(\phi)}$. \qed

According to Lemma 3.3, negative terms cancel in the summation (9), and we have
**Proposition 3.1.**

\[ C^\pi_{\lambda, \mu} = \frac{n}{z_{\lambda, \mu}^{2g}} \sum_{0 \leq i \leq \ell(\mu)} \sum_{0 \leq n \leq m(\mu)} 2^{h(\lambda, \mu) + c(\phi)}, \]  

(10)

where the second summation ranges over all \( \phi \in \mathcal{M}(\lambda, \mu) \) such that each connected component \( \phi^{(i)} \) has an even \( h^{(i)} \).

**3.6. Colored diagram matchings.** We introduce another type of coloring:

A diagram is colored if one of two colors, say black and white, is given to each cell. A parity function \( C \) on a couple \( (\lambda, \mu) \) of partitions is a map from the set of lines of \( \lambda \) and \( \mu \) into \( \{0, 1\} \). Let even \( (C) \) be the number of even lines in \( \lambda \) and \( \mu \) according to \( C \) (i.e., a line \( L \) is even if \( C(L) = 0 \); even \( (C) = |C^{-1}\{0\}) | \).

A diagram matching \( \phi \) of \( \mathcal{M}(\lambda, \mu) \) is colored if \( \lambda \) and \( \mu \) are colored and \( \phi \) respects the colors (i.e., maps black cells onto black cells).

A colored diagram matching \( \phi \) of \( \mathcal{M}(\lambda, \mu) \) is \( C \)-correct if:

1. The number of black cells in each line \( L \) of \( \lambda \) has same parity as \( C(L) \),
2. The number of white cells in each line \( L \) of \( \mu \) has same parity as \( C(L) \).

**Example.** In order to draw a diagram matching we put numbers in cells so that matched cells share the same number. The following example is a \( C \)-correct colored diagram matching with respect to be the parity function \( C \) whose values are written in front of each line.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 13_B & 12_B & 11_W & 10_W & 9_W & 8_W \\
1 & 6_W & 7_W & 8_B & 9_B & 10_B & 11_B \\
0 & 1_B & 2_B & 3_W & 4_B & 5_B & 6_B \\
0 & 12_B & 13_B & 12_W & 12_B & 11_B & 12_W \\
0 & 5_B & 3_W & 1_W & 1_B & 2_W & 4_B \\
\hline
\end{array}
\]

**Theorem 3.1.** The number of \( C \)-correct colorings of a diagram matching \( \phi \) of type \( (\lambda, \mu) \) with exactly one connected component is

\[
2^{h(\lambda, \mu) + 1} \quad \text{if } h(\lambda, \mu) \equiv \text{even}(C) \text{ (mod 2)},
\]

\[
0 \quad \text{otherwise},
\]

where \( h(\lambda, \mu) = |\lambda| - \ell(\lambda) - \ell(\mu) \).

**Proof.** The proof is by induction on \( |\lambda| \).

First suppose that \( |\lambda| = 1 \). There is only one possible diagram matching \( \phi \) which maps the only cell of \( \lambda \) onto the only cell of \( \mu \). In this case, \( C \) completely determines the colors of a \( C \)-correct colored matching and as the two cells must have the same color because of \( \phi \), there is a solution only if \( \text{even}(C) = 1 \). This is in accordance with \( h(\lambda, \mu) = -1 \) and the result holds.

Now suppose that the theorem is proved for all shapes \( \lambda \) and \( \mu \) with \( |\lambda| < n \). Let \( \lambda, \mu, \phi \) and \( C \) be given with \( |\lambda| = |\mu| = n \). Consider the multigraph \( G(\phi) \) whose vertices are the lines and whose edges are the fibers of \( \phi \) (i.e., for each \( x \) in \( \lambda \) and \( y \) in \( \mu \) such that \( y = \phi(x) \) add an edge between corresponding lines). This graph is connected as its connected components are exactly those of \( \phi \). There are two cases:

1. **1st case:** there exists at least one line of length 1 in \( \lambda \) or \( \mu \).  
   Suppose without loss of generality that there is such a line in \( \lambda \) and let \( i_0 \) be the lowest such line \( (i_0 \) denote here indifferently the
line or the cell). As $\phi$ is connected, $j_0 = \phi(i_0)$ is not alone in its line $L_0$ (except if $|\lambda| = 1$ but this case has already been treated). The color of $i_0$ is imposed by $C(i_0)$ and $j_0$ has the same color of $i_0$ because of $\phi$.

Let $\lambda' = \lambda \setminus \{i_0\}$, $\mu' = \mu \setminus \{j_0\}$, $\phi'$ the corresponding restriction of $\phi$ and let $C'$ be the parity function defined by:

$$C' : L \mapsto \begin{cases} C(L) & \text{if } L \not= L_0, \\ C(L_0) & \text{if } L = L_0 \text{ and } C(i_0) = 1, \\ C(L_0) + 1 \mod 2 & \text{if } L = L_0 \text{ and } C(i_0) = 0. \end{cases}$$

With these definitions, a $C$-correct coloring of $\phi$ induces a $C'$-correct coloring of $\phi'$ and, conversely, there is a unique way to extend a $C'$-correct coloring of $\phi'$ into a $C$-correct coloring of $\phi$.

Now the diagram matching $\phi'$ has exactly one connected component because we have only suppressed a leaf of the graph $G(\phi)$ to obtain $G(\phi')$. As $|\lambda'| = n - 1 < n$, by induction hypothesis, the number of $C'$-correct colorings of $\phi'$ is $2^{h(\lambda', \mu') + 1}$ if $h(\lambda', \mu') \equiv \text{even}(C')$ (mod 2), and 0 otherwise. But $h(\lambda', \mu') = h(\lambda, \mu)$ and even($C'$) $\equiv$ even($C$) (mod 2) by construction. The result is therefore proved in this case.

2nd case: all lines in $\lambda$ and $\mu$ have more than one cell. Hence the graph $G(\phi)$ has no vertex of degree 1 (no leaf). Such a graph contains at least a simple cycle and therefore at least an edge which is not a bridge (i.e., if this edge is suppressed, the induced subgraph is still connected). Let $(x_0, y_0) \in \lambda \times \mu$ be the unique edge of such type whose cell $x_0$ is the lowest rightmost possible. (Recall that by definition of edges, $y_0 = \phi(x_0)$.)

The cells $x_0$ and $y_0$ are either both black or both white in any $C$-correct coloring of $\phi$. For any choice of color we now construct restrictions $\phi'$ and $C'$ such that $C'$-correct colorings of $\phi$ with $x_0$ of the chosen color are in one-to-one correspondence with $C'$-correct colorings of $\phi'$ and we count the latter:

Let $\lambda' = \lambda \setminus \{x_0\}$, $\mu' = \mu \setminus \{y_0\}$ and $\phi'$ be the corresponding restriction of $\phi$, which is still connected because the edge $(x_0, y_0)$ is not a bridge of $G(\phi)$. If $(x_0, y_0)$ is black, the parity of the line $L_{x_0}$ containing $x_0$ must be adjusted. In this case, we define:

$$C' : L \mapsto \begin{cases} C(L) & \text{if } L \not= L_{x_0}, \\ C(L_{x_0}) & \text{if } L = L_{x_0}. \end{cases}$$

Otherwise $(x_0, y_0)$ is white and we define $C'$ by adjusting instead the parity of the line $L_{y_0}$ containing $y_0$.

In both cases, $|\lambda'| = n - 1 < n$ so that induction hypothesis can be applied. Moreover, in both case the number of $C'$-correct colorings of $\phi'$ is $2^{h(\lambda', \mu') + 1}$ or 0. The number of $C$-correct colorings of $\phi$ is therefore $2 \cdot 2^{h(\lambda', \mu') + 1}$ if $h(\lambda', \mu') \equiv \text{even}(C')$ (mod 2) and 0 otherwise. As $h(\lambda, \mu) = h(\lambda', \mu') + 1$ and the parity of exactly one line has changed, the theorem is proved. $\square$

### 3.7. Direct enumeration of colored diagram matchings

Let $C_{\ell,m}$ be the parity function defined on the couple $(\lambda^\ell, \mu^m)$ which is odd on every line of $\lambda^\ast$ and $\mu^\ast$ and even on the lines of the top hooks of $\lambda^\ell$ and $\mu^m$.

Let $\phi$ be a diagram matching of type $(\lambda^\ell, \mu^m)$, then all $C_{\ell,m}$-correct colorings of $\phi$ may be constructed component by component. Hence according to Theorem 3.1, there exists some $C_{\ell,m}$-correct colorings of $\phi$ if and only if each connected component $(\lambda^{(i)}, \mu^{(i)})$ of $(\lambda^\ell, \mu^m)$ satisfies $h(\lambda^{(i)}, \mu^{(i)}) \equiv \text{even}(C^{(i)})$ (mod 2) where $C^{(i)}$ is the restriction of $C_{\ell,m}$ to the connected component $(\lambda^{(i)}, \mu^{(i)})$. But as odd lines are precisely lines of $\lambda^\ast$ and $\mu^\ast$, $\text{even}(C^{(i)}) = \ell(\lambda^{(i)}) - \ell(\lambda^\ast) + \ell(\mu^{(i)}) - \ell(\mu^\ast) + h(\lambda^{(i)}, \mu^{(i)}) - \text{even}(C^{(i)}) = h^{(i)}$...
(mod 2). Hence the diagram matchings having $C_{ℓ,m}$-correct colorings are precisely those which are involved in summation (10). Moreover, the number of $C_{ℓ,m}$-correct colorings of these $φ$ is, according to Theorem 3.1, $\prod_{i} 2^{h(λ^{i},µ^{i})+1} = 2^{h(λ^{i},µ^{i})+c(φ)}$ and corresponds to the weight of each $φ$ in summation (10).

Let $CM(λ^{i}, µ^{m})$ denote the set of $C_{ℓ,m}$-correct colored diagram matchings of type $(λ^{i}, µ^{m})$. According to the previous discussion, summation (10) reads

$$c_{k,µ}^{n} = \frac{n}{z_{k}z_{µ}2^{g}} \sum_{0_{c1}, 0_{c2}=g} card(CM(λ^{i}, µ^{m})).$$

Now all $C_{ℓ,m}$-correct diagrams of type $(λ^{i}, µ^{m})$ can be constructed as follows: choose the numbers $p$ of black cells and $q$ of white cells $(p+q = n-1)$. As, according to $C_{ℓ,m}$, there are exactly $ℓ(λ) − 1$ odd lines in $λ$, we have $p ≥ ℓ(λ) − 1$ and similarly $q ≥ ℓ(µ) − 1$. Then choose the odd number $2k_{1} + 1$ of black cells in each line $λ_{k}$, $k ∈ [1, ℓ(λ) − 1]$ of $λ^{*}$, and the even number $2ℓ(λ)$ of black cells in the horizontal part (of length $ℓ$) of the top hook of $λ$. We have $(2l_{1} + 1) + \cdots + (2l_{ℓ(λ)}) = p + 1$ hence $p = 2(l_{1} + \cdots + l_{ℓ(λ)}) + ℓ(λ) − 1$ and similarly $q = 2(j_{1} + \cdots + j_{ℓ(µ)}) + ℓ(µ) − 1$. Let $g_{1} = (p - ℓ(λ) + 1)/2$ and $g_{2} = (q - ℓ(µ) + 1)/2$ then $g_{1} + g_{2} = g(λ, µ)$ and $g_{1}$ and $g_{2}$ are non-negative integers. Choose similarly $j_{1} + \cdots + j_{ℓ(µ)} = g_{2}$. Then in each line $λ_{k}$ of $λ^{*}$ the black cells can be placed in $2^{λ_{k}+1}$ ways for $k ∈ [1, ℓ(λ) − 1]$ and in $2^{g_{2}}$ ways in the horizontal line of length $ℓ$ of the hook. The same holds for white cells in $µ^{m}$. To conclude we need to define the matching $φ$. This has to be done independently on the two colors, hence in $p!q!$ ways, that is in $(ℓ(λ) − 1 + 2g_{1})!(ℓ(µ) − 1 + 2g_{2})!$ ways. Therefore card $(CM(λ^{i}, µ^{m}))$ is equal to

$$\sum_{g_{1} + g_{2} = g(λ, µ)} (ℓ(λ) − 1 + 2g_{1})!(ℓ(µ) − 1 + 2g_{2})! \times \sum_{j_{1} + \cdots + j_{ℓ(µ)} = g_{2}} (\ell_{2)}(2l_{ℓ(λ)}) m \prod_{k=1}^{ℓ(µ)} (\lambda_{k})^{ℓ(µ)} \prod_{k=1}^{ℓ(λ)} (\mu_{k})^{ℓ(λ)}.$$  

As $ℓ_{0} = ℓ_{ℓ(λ)}$ and $\sum_{ℓ_{0}=0}^{ℓ(λ)} \binom{ℓ(λ)}{ℓ_{0}}$, summation of the cardinals on $ℓ$ and $m$ yields,

$$c_{k,µ}^{n} = \frac{n}{z_{k}z_{µ}2^{g}} \sum_{g_{1} + g_{2} = g} (ℓ(λ) − 1 + 2g_{1})!(ℓ(µ) − 1 + 2g_{2})! \times \sum_{j_{1} + \cdots + j_{ℓ(µ)} = g_{2}} \prod_{k=1}^{ℓ(µ)} (\lambda_{k})^{ℓ(µ)} \prod_{k=1}^{ℓ(λ)} (\mu_{k})^{ℓ(λ)}.$$  

and Theorem 2.1 is proved.

After this proof, we kindly encourage the reader to pause and enjoy a lighter activity such as bird-watching or gardening before continuing to read.

4. APPLICATIONS

4.1. Special cases

It was observed by several authors that in some cases, for a given diagram $λ$, distinct diagrams $µ$ yields the same value of $c_{k,µ}^{n}$. For a given diagram $λ$ we introduce the set of diagrams $µ$ that force $λ$ to connect, i.e., such that all diagram matchings $φ$ in all $M(λ^{i}, µ^{m})$ have exactly one connected component. Then
We recover here the result of [2] that expression does not depend on the particular shape $\mu$:

$$\text{Case } k = 1 \text{ obtains a combinatorial interpretation for the coefficients. A constructive proof was given for the well-known formula (see [16] for an elementary constructive proof).}$$

Component is 2 do not contribute to the summation. The contribution of diagrams with only one connected component is still the same: for each $i = \lambda(n - k)$, forces connectivity and we recover the well-known formula (see [16] for an elementary constructive proof)

$$c_{1(n-1),n}^\mu = 2(n - 2)!, \quad \forall \mu \text{ odd.}$$

More generally, for a given shape $\lambda$, one can classify all essentially distinct diagram matchings that can be constructed on the diagrams $\lambda^\ell$ and compute the contribution of each of these essential configurations. The previous case is the simplest case because there is only one essential configuration, which consists of a unique connected component.

Suppose now $\lambda = 1^k(n - k)$ with $k > 0$ and $\mu$ contains a part of length 1. Then $\lambda^0 = 1^{k-1}(n - k)$ and the essential configuration with a unique connected component is still the only relevant one. Indeed in other configurations, except for the connected component which contains the part $(n - k)$ of $\lambda$, all connected components contain only parts of length 1 in $\lambda$, and therefore only one part in $\mu$. But for such a component, $h^{(i)} = -1$ and these diagrams do not contribute to the summation. The contribution of diagrams with only one connected component is $2^{h(\lambda^\ell, \mu^n)+1} = 2^{2\ell+1}$ and the number of such diagrams is readily obtained by an exclusion/inclusion argument. If $\mu$ has no part of length 1 the proof is more complicated but the final formula is the same: for $k > 0$ and $\mu = 1^{\beta_1} \ldots n^{\beta_k}$ a partition of $n$ of the same parity as $k$, and with $(x)_i = x(x - 1) \ldots (x - i + 1)$ the descending factorial,

$$c_{1^k(n-k),n}^\mu = \frac{2(n - k - 1)!}{k!} \sum_{h=0}^{k-1} (k - 1)h(n - \beta_1 - h)_{k-1-h} \left( \prod_{i=2}^{\ell} \beta_i \right).$$

We recover here the result of [2] that $c_{1^k(n-k),n}^\mu$ depends only on $\beta_1, \ldots, \beta_{k-1}, n$ and $k$ and obtain a combinatorial interpretation for the coefficients. A constructive proof was given for case $k = 2$ in [4] and this explicit formula was known for case $k = 1, 2, 3, 4$ (see [2] and [12]):

$$c_{1(n-1),n}^\mu = 2(n - 2)!,$$

$$c_{1^2(n-2),n}^\mu = (n - 3)!(n - \beta_1),$$

$$c_{1^3(n-3),n}^\mu = \frac{(n - 4)!}{3} [(n - \beta_1)(n - \beta_1 - 1) - 2\beta_2],$$

$$c_{1^4(n-4),n}^\mu = \frac{(n - 5)!}{4 \cdot 3} [6(n - \beta_1) - 1 - 2\beta_2 - 6\beta_3],$$

$$c_{1^5(n-5),n}^\mu = \frac{(n - 6)!}{5 \cdot 4 \cdot 3} [(n - \beta_1) + 12\beta_2(\beta_2 - 1) - 12(n - \beta_1 - 2)(n - \beta_1 - 3)\beta_2 - 24(n - \beta_1 - 3)\beta_3 - 24\beta_4].$$
4.2. Applications to the enumeration of maps. An account of the relations between pairs of permutations and maps on oriented surfaces can be found e.g., in [5]. But in order to have a self-contained exposition, we shall briefly sketch the definitions of maps: a map \((S, \mathcal{G})\) on a compact oriented surface \(S\) without boundary is a graph \(\mathcal{G}\) together with an embedding of \(\mathcal{G}\) into \(S\) such that connected components of the complement \(S\setminus \mathcal{G}\) of the embedding of \(\mathcal{G}\) in \(S\), called the faces of the map, are homeomorphic to discs. Multiple edges are allowed and our maps are rooted, i.e., one edge of \(\mathcal{G}\) is distinguished. Two maps \((S, \mathcal{G})\) and \((S', \mathcal{G}')\) are isomorphic if there exists an orientation-preserving homeomorphism \(f : S \to S'\) such that \(f(\mathcal{G}) = \mathcal{G}'\). A map is bicolored if its vertices are colored in black or white so that each edge is incident to one vertex of each color. A map is unicellular if it has one face. The type of a bicolored unicellular map \(M\) with \(n\) edges is a pair of partitions \((\lambda, \mu)\) whose parts give the respective degrees of black and white vertices of \(M\).

A relation between products of conjugacy classes and enumeration of maps is presented in the following proposition (see [5] for more details):

**Proposition 4.1.** Bicolored unicellular maps of type \((\lambda, \mu)\) are maps on a compact orientable surface of genus \(g\) which satisfy

\[
g = g(\lambda, \mu)
\]

Moreover, the number \(\text{Bi}(\lambda, \mu)\) of bicolored unicellular maps of type \((\lambda, \mu)\) with \(n\) edges is the number of pairs \((\sigma, \tau)\) \(\in C_\lambda \times C_\mu\) such that \(\sigma \tau = (1, 2, \ldots, n)\), which is also the coefficient \(c_{\lambda, \mu}^n\):

\[
\text{Bi}(\lambda, \mu) = c_{\lambda, \mu}^n.
\]

In particular, when \(g(\lambda, \mu) = 0\), we are interested in unicellular maps on the sphere which are plane trees as remarked by Goulden and Jackson [6].

We observe that if we fix a partition \(\lambda\) of \(n\) and the parameter \(g\) and we sum over all conjugacy classes indexed by partitions \(\mu\) satisfying \(\ell(\mu) = m = n + 1 - 2g - \ell(\lambda)\), then the coefficient

\[
\text{Bi}(\lambda, m, n) = \sum_{\mu \vdash n \atop \ell(\mu) = m} \text{Bi}(\lambda, \mu, n) = \sum_{\mu \vdash n \atop \ell(\mu) = m} c_{\lambda, \mu}^n
\]

counts the number of bicolored unicellular rooted maps on a surface of genus \(g\) with black degree distribution given by \(\lambda\) and no restriction on white degree distribution other than the number \(m\) of white vertices. Similarly, if we sum over all conjugacy classes with partition \(\lambda\) of a fixed length (or rank), then the coefficient

\[
\text{Bi}(\ell, m, n) = \sum_{\lambda \vdash m \atop \ell(\lambda) = \ell} \text{Bi}(\lambda, m, n) = \sum_{\lambda \vdash m, \ell(\lambda) = \ell \atop \mu \vdash n, \ell(\mu) = m} c_{\lambda, \mu}^n
\]

counts the number \(\text{Bi}(\ell, m, n)\) of unicellular bicolored rooted maps of genus \(g\) with \(\ell\) black vertices and \(m = n + 1 - 2g - \ell\) white vertices. These maps were recently considered by Adrianov [1]. Finally, if we sum the product in (12) over all possible lengths of \(\lambda\), we obtain

\[
\text{Bi}(n, g) = \sum_{\ell, \lambda \vdash n \atop \ell + \text{area} + 1 - 2g = \ell(\lambda)} \text{Bi}(\ell, m, n)
\]

which is the number \(\text{Bi}(n, g)\) of unicellular bicolored rooted maps of genus \(g\) with \(n\) edges. The purpose of this section is to give explicit expressions for the sums of coefficients in (11), (12) and (13). These explicit expressions are derived from Theorem 2.1 using the following lemma, which already appears in [19], and performing usual algebraic manipulations on summations. Therefore we state the results without giving the rather cumbersome calculations.
where $c_g \eta$.

Moreover, if $g$ is fixed,

$$
\sum_{\ell(\mu) \leq m} \frac{1}{\ell \mu} \sum_{\ell(\eta) = \ell(\mu)} \prod_{k=1}^{m} \left( \frac{\mu_k}{2q_k + 1} \right) = \frac{1}{m!} \left( \frac{n - 1}{m + 2g - 1} \right) \prod_{k=1}^{\ell(\eta)} \frac{1}{2q_k + 1} \sum_{\ell(\eta) = \ell(\mu)} 1.
$$

A straightforward consequence of this lemma and Theorem 2.1 is the following result.

**Theorem 4.1.** Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ and $m, g$ two non-negative integers such that $\ell + m = n + 1 - 2g$. Then

$$
\text{Bi}(\lambda, m, n) = \frac{n!}{2^g (n + 1 - 2g)!} \sum_{\gamma_1 + \cdots + \gamma_\ell = g} \prod_{i=1}^{\ell} \gamma_i! (2\ell + 1)^{\gamma_i},
$$

where $P_{\ell}(x)$ is a polynomial in $x$ of degree $g$ defined by:

$$
P_{\ell}(x) = \sum_{y \geq x + 1} (y)_\ell (y)_{\ell(\gamma)} \prod_{i=1}^{\ell} \gamma_i! (2\ell + 1)^{\gamma_i},
$$

with $(y)_\ell = y(y - 1)(y - 2) \cdots (y - \ell + 1)$ the descending factorial.

If we set $\mu = 2^{\ell/2}$, all white vertices have degree two and can be seen as centers of edges linking black vertices to black vertices so that the white vertices can be forgotten and we obtain the monochromatic unicellular maps counted by Walsh and Lehman [19]. Therefore, let us call $\text{Mono}(n, g)$ the number of monochromatic maps of genus $g$ with $n$ edges and use Theorem 4.1 to obtain

**Corollary 4.2 ([19]).** Let $n$ be a positive integer and $g$ be non-negative integer, then

$$
\text{Mono}(n, g) = \text{Bi}((2^n), n + 1 - 2g, 2n) = \frac{(2n)!}{2^g n! (n + 1 - 2g)!} P_k(n + 1 - 2g).
$$

Moreover, if $g$ is fixed,

$$
\text{Mono}(n, g) \sim_{n \to \infty} \frac{\mu^{g/2} 14n}{\sqrt{\pi} g^{1/2} 128^n}.
$$

Using once again Lemma 4.1 to sum over all $\lambda$ in Theorem 4.1, we obtain:

**Theorem 4.2.** Let $\ell, m, n, g$ be non-negative integers such that $\ell + m = n + 1 - 2g$. Then

$$
\text{Bi}(\ell, m, n) = \frac{(n + 1)2^g}{(n + 1)2^g} \sum_{\ell(\gamma_1 + \cdots + \gamma_\ell = g)} \prod_{i=1}^{\ell} \gamma_i! (2\ell + 1)^{\gamma_i} P_{\ell}(\ell) P_k(m).
$$

Eventually, summing over all $\ell$ and $m$ with $\ell + m = n$, we obtain:

**Theorem 4.3.** Let $n, g$ be non-negative integers. Then

$$
\text{Bi}(n, g) = \frac{(n + 1)2^g}{(n + 1)2^g} \sum_{\ell(\gamma_1 + \cdots + \gamma_\ell = g)} \prod_{i=1}^{\ell} \gamma_i! (2\ell + 1)^{\gamma_i} \sum_{c_{\ell_1, \ell_2}} \sum_{\ell_1 + \ell_2 = \ell} c_{\ell_1, \ell_2} (n + 1 - 2g)_{\ell_1 + \ell_2} \left( \frac{2n - 2g - \ell_1 - \ell_2}{n - 2g - \ell_1} \right),
$$

where the $c_{\ell, \gamma}$ are constants given by:

$$
c_{\ell, \gamma} = \sum_{\ell(\gamma_1 + \cdots + \gamma_\ell = \ell)} \frac{1}{\prod_{i=1}^{\ell} \gamma_i! (2\ell + 1)^{\gamma_i}}.
$$
In particular, we recover Adrianov’s result [1, Corollary 5] that for fixed $g$,

$$
B_i(\lambda, \mu, n) = \frac{n(\lambda - 1)(m - 1)!}{l_1! l_2! m_1! m_2!} \left[ \binom{\lambda + 1}{2} \cdot \sum_i \binom{\lambda_i - 1}{2} + \binom{m + 1}{2} \cdot \sum_i \binom{n_i - 1}{2} \right]
$$

$P_\mathcal{A}(x)$

<table>
<thead>
<tr>
<th>Genus</th>
<th>$B_i(\lambda, \mu, n)$</th>
<th>$P_\mathcal{A}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{n(\lambda - 1)(m - 1)!}{l_1! l_2! m_1! m_2!}$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{n(\lambda - 1)(m - 1)!}{l_1! l_2! m_1! m_2!} \left[ \binom{\lambda + 1}{2} \cdot \sum_i \binom{\lambda_i - 1}{2} + \binom{m + 1}{2} \cdot \sum_i \binom{n_i - 1}{2} \right]$</td>
<td>$\frac{x^5}{5}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{n(\lambda - 1)(m - 1)!}{l_1! l_2! m_1! m_2!} \left[ \binom{\lambda + 3}{4} \cdot \sum_i \binom{\lambda_i - 1}{2} + 20 \sum_{i \neq j} \binom{\lambda_i - 1}{2} \binom{\lambda_j - 1}{2} \right]$</td>
<td>$\frac{x^{5(5+13)}}{90}$</td>
</tr>
</tbody>
</table>

Table 1.

The number of rooted bicolored unicellular maps of genus $g$ on $n$ edges with vertex degree distributions given by $\lambda$ and $\mu$.

<table>
<thead>
<tr>
<th>Genus</th>
<th>$B_i(\ell, m, n)$</th>
<th>$B(n, g)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{1}{\ell} \binom{n}{\ell-1} \binom{n}{m-1}$</td>
<td>$\frac{1}{2n+1} \binom{2n+1}{n}$ Catalan numbers</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{\ell} \binom{n}{\ell-1} \binom{n}{m-1}$</td>
<td>$\frac{1}{2n+1} \binom{2n+1}{n}$ Catalan numbers</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{\ell} \binom{n}{\ell-1} \binom{n}{m-1}$</td>
<td>$\frac{1}{2n+1} \binom{2n+1}{n}$ Catalan numbers</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{\ell} \binom{n}{\ell-1} \binom{n}{m-1}$</td>
<td>$\frac{1}{2n+1} \binom{2n+1}{n}$ Catalan numbers</td>
</tr>
</tbody>
</table>

Table 2.

The number of rooted bicolored unicellular maps of genus $g$ on $n$ edges.

COROLLARY 4.3. Let $g$ be a given non-negative integer, then

$$
B_i(n, g) \sim_{n \to \infty} n^3 \left( \frac{1}{4} \right)^g
$$

In particular, we recover Adrianov’s result [1, Corollary 5] that for fixed $g$,

$$
\frac{B(n, g)}{\text{Mono}(n, g)} \sim_{n \to \infty} \left( \frac{1}{4} \right)^g
$$

In Tables 1 and 2, we give the expressions of low genera for the polynomials $B_i(\lambda, \mu, n)$, $P_\mathcal{A}(x)$, $B_i(\ell, m, n)$ and $B(n, g)$ where

$q_2(x, y) = 13(x^2 + y^2) + 5xy(x + y) + 86(x + y) + 47xy + 129,$

$q_3(x, y) = 502(x^4 + y^4) + 35x^2y^2(x^2 + y^2) + 273xy(x^3 + y^3) + 70x^3y^3 + 9978(x^3 + y^3) + 1260x^2y^2(x + y) + 6512xy(x^2 + y^2) + 71842(x^2 + y^2) + 13410x^2y^2 + 54123xy(x + y) + 185554xy + 219918(x + y) + 238480.$

REFERENCES


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