Some integral inequalities for functions with \((n - 1)\)st derivatives of bounded variation

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Abstract

In this paper, we generalize Cerone’s results, and a unified treatment of error estimates for a general inequality satisfying \(f^{(n-1)}\) being of bounded variation is presented. We derive the estimates for the remainder terms of the mid-point, trapezoid, and Simpson formulas. All constants of the errors are sharp. Applications in numerical integration are also given.

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1. Introduction

In 2000, Cerone, Dragomir and Pearce [1] proved the following trapezoid type inequalities.

**Theorem 1.** Let \(f : [a, b] \rightarrow \mathbb{R}\) be a function of bounded variation. Then we have the inequality

\[
\left| \int_a^b f(t) \, dt - \left[ (x - a)f(a) + (b - x)f(b) \right] \right| \leq \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \int_a^b (f) \tag{1.1}
\]

for all \(x \in [a, b]\), where \(\int_a^b (f)\) denotes the total variation of \(f\) on the interval \([a, b]\).

The inequality (1.1) is a perturbed generalization of the trapezoidal inequality for mapping of bounded variation. Using (1.1), Cerone et al. further obtained the following error estimate for the composite quadrature rule.

**Theorem 2.** Let \(f\) be defined as in Theorem 1; then we have

\[
\int_a^b f(t) \, dt = \sum_{i=0}^{n-1} \left( (\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) \right) + R(f). \tag{1.2}
\]
The remainder term $R(f)$ satisfies the estimate

$$|R(f)| \leq \left[ \frac{v(l)}{2} + \max_{i=0,1,\ldots,n-1} \left| \xi_i - \frac{x_i + x_{i+1}}{2} \right| \right] \int_a^b f(t) \, dt \leq v(l) \int_a^b f(t) \, dt,$$

(1.3)

where $v(l) := \max\{|l_i| : i = 0, 1, \ldots, n - 1\}$, $l_i = x_{i+1} - x_i$ and $\xi_i \in [x_i, x_{i+1}]$.

In this paper, following the main ideas of Vinogradov [2], we give a unified treatment of error estimates for a general quadrature rule satisfying $f^{(n-1)}$ being of bounded variation. Using the perturbed inequality, we obtain the error bounds for the mid-point, trapezoid and Simpson quadrature formulas. We also generalize Euler trapezoid formulas [3].

2. The main results

A sequence of polynomials $\{u_k\}^\infty_0$ is called a sequence of Appell type polynomials if $u_0 = 1, u'_k = u_{k-1} (k \in \mathbb{Z}_+)$.

**Lemma 1.** Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $f^{(n-1)}$ is a function of bounded variation on $[a, b]$ for some $n \geq 1$, $n \in \mathbb{Z}_+$. Moreover, if $n = 1$, $f(t)$ is continuous at $x, x \in [a, b]$. Suppose that $\{r_k\}, \{s_k\}$ are sequences of Appell type polynomials on $[a, x]$ and $[u_k], \{v_k\}$ are sequences of Appell type polynomials on $(x, b)$. Let $m \in \mathbb{N}, m \leq n,$

$$k_n(x, t) = \begin{cases} p_n(t) = r_{n-m}(t)s_m(t), & t \in [a, x); \\ q_n(t) = u_{n-m}(t)v_m(t), & t \in (x, b]. \end{cases}$$

Then we have the following equality:

$$\int_a^b f(t) \, dt - \frac{(-1)^n}{C_n^m} \int_a^b k_n(x, t) \, dt f^{(n-1)}(t) = \begin{cases} \frac{1}{C_n^m} \sum_{k=0}^{n-1} (-1)^{n-k-1} \left[ q_n^{(k)}(b) f^{(n-1-k)}(b) \\ - q_n^{(k)}(a) f^{(n-1-k)}(a) \right], & x = a; \\ \frac{1}{C_n^m} \sum_{k=0}^{n-1} (-1)^{n-k-1} \left[ (p_n^{(k)}(x) - q_n^{(k)}(x)) f^{(n-1-k)}(x) \\ + q_n^{(k)}(b) f^{(n-1-k)}(b) - p_n^{(k)}(a) f^{(n-1-k)}(a) \right], & x \in (a, b); \\ \frac{1}{C_n^m} \sum_{k=0}^{n-1} (-1)^{n-k-1} \left[ p_n^{(k)}(b) f^{(n-1-k)}(b) \\ - p_n^{(k)}(a) f^{(n-1-k)}(a) \right], & x = b, \end{cases}$$

where $C_n^m = \frac{n!}{m!(n-m)!}$.

**Proof.** Integrating by parts in the sense of Riemann and Stieltjes, we can easily obtain Lemma 1. □

**Remark 1.** Actually, $f^{(n-1)}$ is continuous if it is of bounded variation when $n > 1$. If $k_1(x, t)$ is continuous at $x$, we can weaken the conditions of Lemma 1. In this case, it is not necessary that $f(t)$ is continuous at $x$.

**Theorem 3.** Let $f$ be defined as in Lemma 1. Suppose that $m \in \mathbb{N}, n \in \mathbb{Z}_+, m \leq n$ and $\lambda \in [0, 1]$. Then we have

$$\int_a^b f(t) \, dt - \frac{1}{C_n^m} \sum_{j=0}^{n-1} \left[ \sum_{i=L}^{U} \sum_{j=0}^{n-m-1} C_j^i C_{n-j}^{n-m-i} (1 - \lambda)^{m-j+i} \right] \frac{(b - x)^{n-j} - (a - x)^{n-j}}{(n-j)!} f^{(n-1-j)}(x)$$

$$- \frac{1}{C_n^m} \sum_{j=n-m}^{n-1} C_j^{n-m} \lambda^{-j} (x - a)^{n-j} f^{(n-1-j)}(a) - (x - b)^{n-j} f^{(n-1-j)}(b) \right|$$
\[
\sup_{0 < \tau < 1} \frac{\tau^{n-m}|\tau - \lambda|^m}{n!} \int_a^b (f^{(n-1)}) \max\{|x - a|^n, |x - b|^n\},
\]

(2.1)

where \( U = \min\{j, n - m\} \), \( L = \max\{0, j - m\} \).

**Proof.** Let

\[
k_n(x, t) = \begin{cases} p_n(t) = \frac{(t - a)^{n-m}(t - \alpha)^m}{m!(n - m)!}, & t \in [a, x); \\ q_n(t) = \frac{(t - b)^{n-m}(t - \beta)^m}{m!(n - m)!}, & t \in (x, b], \end{cases}
\]

and \( \alpha = \lambda x + (1 - \lambda)a \), \( \beta = \lambda x + (1 - \lambda)b \).

Thus, it follows from a straightforward calculation that

\[
p_n^{(j)}(x+) = \sum_{i=L}^U C_j^i C_{n-j}^{m-i} \frac{(1 - \lambda)^{m-j+i}(x - a)^{n-j}}{(n - j)!},
\]

\[
q_n^{(j)}(x-) = \sum_{i=L}^U C_j^i C_{n-j}^{m-i} \frac{(1 - \lambda)^{m-j+i}(x - b)^{n-j}}{(n - j)!}.
\]

On the other hand, we have

\[
\left| \int_a^b k_n(x, t) \, d f^{(n-1)}(t) \right| = \left| \int_a^x \frac{(t - a)^{n-m}(t - \alpha)^m}{m!(n - m)!} \, d f^{(n-1)}(t) + \int_x^b \frac{(t - b)^{n-m}(t - \beta)^m}{m!(n - m)!} \, d f^{(n-1)}(t) \right|
\]

\[
\leq \sup_{a < t < x} \frac{|(t - a)^{n-m}(t - \alpha)^m|}{m!(n - m)!} \int_a^x |d f^{(n-1)}(t)|
\]

\[
+ \frac{|(t - b)^{n-m}(t - \beta)^m|}{m!(n - m)!} \int_x^b |d f^{(n-1)}(t)|
\]

\[
= \sup_{0 < \tau < 1} \frac{\tau^{n-m}|\tau - \lambda|^m}{m!(n - m)!} \left[ \int_a^x (f^{(n-1)})|x - a|^n + \int_x^b (f^{(n-1)})|x - b|^n \right]
\]

\[
\leq \sup_{0 < \tau < 1} \frac{\tau^{n-m}|\tau - \lambda|^m}{(n - m)!} \int_a^b (f^{(n-1)}) \max\{|x - a|^n, |x - b|^n\}.
\]

According to Lemma 1, we derive (2.1) and the proof is completed. \( \square \)

**Remark 2.** Theorem 3 contains many classical formulas. The advantage of this theorem is that we have three parameters \( \lambda, x \) and \( m \) to choose.

**Corollary 1.** Let \( f \) be defined as in Theorem 3. Suppose that \( n \in \mathbb{Z}_+, \lambda \in [0, 1] \). Then we have

\[
\left| \int_a^b f(t) \, dt - \frac{1}{n} \left\{ \lambda(x - a) f(a) + \lambda(b - x) f(b) + \sum_{j=0}^{n-1} \left( \sum_{i=\max\{0,j-1\}}^{i} C_j^i C_{n-j}^{l-i} (1 - \lambda)^{i+1-j} \right) \right\} \frac{(b - x)^{n-j} - (a - x)^{n-j}}{(n - j)!} f^{(n-1)}(x) \right| \leq C_n \int_a^b (f^{(n-1)}) \max\{|x - a|^n, |x - b|^n\}
\]

(2.2)

where

\[
C_n = \begin{cases} \max\{\lambda, 1 - \lambda\}, & n = 1; \\ \frac{1}{n!} \max\left\{ \frac{n-1}{n^n} \lambda^n, 1 - \lambda \right\}, & n > 1. \end{cases}
\]
Proof. Let $g(\tau) = \tau^{n-1}(\tau - \lambda)$, $\tau \in [0, 1]$. Hence
\[
g'(\tau) = \begin{cases} 
1, & n = 1; \\
n^{n-2}(n\tau - \lambda), & n > 1.
\end{cases}
\]
Clearly, we can obtain the following inequality:
\[
|g(\tau)| \leq \max \{ |g(0)|, |g(1)| \} = \max \{ n\lambda, 1 - \lambda \},
\]
\[
\quad n = 1; \quad \quad \max \left\{ \frac{1}{n^n} \left( \frac{\lambda}{n} \right)^n, |g(1)| \right\} = \max \left\{ \frac{n-1}{n^{n-1}}, 1 - \lambda \right\}, \quad n > 1.
\]
Therefore, setting $m = 1$ in Theorem 3 we have (2.2) and the proof is completed. \hfill \Box

Corollary 2. Let $f$ be defined as in Theorem 3. Suppose that $n \in \mathbb{Z}^+$ and $0^0 = 1$. Then we have
\[
\int_a^b f(t) \, dt = - \frac{1}{n} \sum_{j=1}^{n-1} \frac{n - j}{j!} \left( \lambda \int_a^1 f^{(j-1)}(t) \, dt \right) + \frac{1}{n} \sum_{j=0}^{n-1} \frac{\lambda^{n-j}}{(n-j)!} \left( \int_a^b f^{(n-j)}(t) \, dt \right)
\]
\[
\leq \frac{(n-1)^{n-1}}{n^{n-1}n!} \int_a^b (f^{(n-1)}) \max \{|x-a|^n, |x-b|^n\}.
\]
(2.3)

Proof. We consider the case $m = n - 1, \alpha = \beta = x$ in Theorem 3. In this case, we can get $\lambda = 1$. Let $g(\tau) = \tau^{n-1}(\tau - 1)$, $\tau \in [0, 1]$. Hence
\[
g'(\tau) = \begin{cases} 
1, & n = 1; \\
(\tau-1)^{n-2}(n\tau - 1), & n > 1.
\end{cases}
\]
We can obtain the following inequality:
\[
|g(\tau)| \leq \max \{ 1, g \left( \frac{1}{n} \right) \} = \frac{(n-1)^{n-1}}{n^n}, \quad n = 1; \quad \quad \frac{1}{n^n} \left( \frac{\lambda}{n} \right)^n \leq \frac{1}{n!} \max \{ \lambda^n, (1-\lambda)^n \} \int_a^b (f^{(n-1)}) \max \{|x-a|^n, |x-b|^n\}.
\]
(2.4)

Proof. We take $m = n$ in Theorem 3, and the corollary is proved. \hfill \Box

Corollary 3. Let $f$ be defined as in Theorem 3, $n \in \mathbb{Z}^+$, $\lambda \in [0, 1]$. Then
\[
\int_a^b f(t) \, dt = - \frac{1}{n} \sum_{j=0}^{n-1} \frac{(1-\lambda)^{n-j}(b-x)^{n-j} - (a-x)^{n-j}}{(n-j)!} f^{(n-1-j)}(x)
\]
\[
- \frac{1}{n} \sum_{j=0}^{n-1} \frac{\lambda^{n-j}}{(n-j)!} \left[ (x-a)^{n-j} f^{(n-1-j)}(a) - (x-b)^{n-j} f^{(n-1-j)}(b) \right]
\]
\[
\leq \frac{1}{n!} \max \{ \lambda^n, (1-\lambda)^n \} \int_a^b (f^{(n-1)}) \max \{|x-a|^n, |x-b|^n\}.
\]
(2.4)

Proof. We take $m = n$ in Theorem 3, and the corollary is proved. \hfill \Box

Remark 3. For $n = 1$, we have
\[
\int_a^b f(t) \, dt = (1-\lambda)(b-a) f(x) - (x-a) f(a) - (b-x) f(b)
\]
\[
\leq \max \{ \lambda, 1-\lambda \} \int_a^b (f) \left[ \frac{b-a}{2} + \frac{x-a+b}{2} \right].
\]
(2.5)

Choosing $\lambda = 1$, we can obtain (1.1). Furthermore, when $x = (a+b)/2$, for $\lambda = 0$, $\lambda = 1$ and $\lambda = \lambda = 1/3$ we obtain the estimates for the errors of the mid-point rule, trapezoid rule and Simpson rule respectively.
Theorem 4. Suppose that $X := \{x_i \mid i = 0, 1, \ldots, k - 1, k \in \mathbb{Z}_+\}$ is a set of $k$ points satisfying $a \leq x_0 < x_1 < \cdots < x_{k-1} \leq b$. Let $p_i \geq 0$, $\sum_{i=0}^{k-1} p_i = 1$, and $f^{(n-1)}$ be a function of bounded variation. Moreover, when $n = 1$, $f(t)$ is continuous at $x_i$, $i = 0, 1, \ldots, k - 1$, then we have

$$
\int_a^b f(t) \, dt - (b - a) \sum_{i=0}^{k-1} p_i \frac{x_i - t}{b - a} + \sum_{j=1}^{n-1} \frac{(b - a)^j}{j!} \left[ f^{(j-1)}(b) - f^{(j-1)}(a) \right] \sum_{i=0}^{k-1} p_i B_j \left( \frac{x_i - a}{b - a} \right) 
\leq K_n (b - a)^n \int_a^b (f^{(n-1)}),
$$

(2.6)

where

$$
K_n = \frac{1}{n!} \sup_{a < t < b} \left[ \sum_{i=0}^{k-1} p_i \left[ B_n^* \left( \frac{x_i - t}{b - a} \right) - B_n \left( \frac{x_i - a}{b - a} \right) \right] \right],
$$

and $B_n^*$ is a 1-periodic function that coincides with the Bernoulli polynomial $B_n$ on $[0, 1]$.

Proof. To prove this theorem, we set $m = 0$ in Lemma 1 and take

$$
k_n(x, t) = (-1)^n \frac{(b - a)^n}{n!} B_n^* \left( \frac{x - t}{b - a} \right).
$$

Hence we have

$$
\int_a^b f(t) \, dt = f(x)(b - a) - \sum_{j=1}^{n-1} \frac{(b - a)^j}{j!} B_j \left( \frac{x - a}{b - a} \right) \left[ f^{(j-1)}(b) - f^{(j-1)}(a) \right] 
+ \frac{(b - a)^n}{n!} \int_a^b \left[ B_n^* \left( \frac{x - t}{b - a} \right) - B_n \left( \frac{x - a}{b - a} \right) \right] df^{(n-1)}(t).
$$

Making the change of variables $x = x_i$, $i = 0, 1, \ldots, k - 1$, and using $\sum_{i=0}^{k-1} p_i = 1$, we obtain

$$
\int_a^b f(t) \, dt = (b - a) \sum_{i=0}^{k-1} p_i f(x_i) + \sum_{j=1}^{n-1} \frac{(b - a)^j}{j!} \left[ f^{(j-1)}(b) - f^{(j-1)}(a) \right] \sum_{i=0}^{k-1} p_i B_j \left( \frac{x_i - a}{b - a} \right) 
+ \frac{(b - a)^n}{n!} \int_a^b \left[ B_n^* \left( \frac{x_i - t}{b - a} \right) - B_n \left( \frac{x_i - a}{b - a} \right) \right] df^{(n-1)}(t).
$$

Since

$$
\int_a^b \sum_{i=0}^{k-1} p_i \left[ B_n^* \left( \frac{x_i - t}{b - a} \right) - B_n \left( \frac{x_i - a}{b - a} \right) \right] \, df^{(n-1)}(t)
\leq \sup_{a < t < b} \left| \sum_{i=0}^{k-1} p_i \left[ B_n^* \left( \frac{x_i - t}{b - a} \right) - B_n \left( \frac{x_i - a}{b - a} \right) \right] \right| \int_a^b | df^{(n-1)}(t) |
= \sup_{a < t < b} \left| \sum_{i=0}^{k-1} p_i \left[ B_n^* \left( \frac{x_i - t}{b - a} \right) - B_n \left( \frac{x_i - a}{b - a} \right) \right] \right| \int_a^b (f^{(n-1)})
$$

We can easily derive (2.6) and the proof is completed. □

We define $h = (b - a) / k$. Setting $p_i = 1/k$, $x_i = a + (i + x) h$, $i = 0, 1, \ldots, k - 1$, in Theorem 4, we obtain the Euler–Maclaurin formula;
Trapezoid rule:

\[ \left| \int_a^b f(t) \, dt - h \sum_{i=0}^{k-1} f(a + (i + \frac{1}{2})h) \right| \leq \frac{(b-a)^n}{n!k^n} \sup_{0 < t < 1} |B_n(t) - B_n(x)| \frac{b}{a} \left( f^{(n-1)} \right) . \]

As regards applications of the Euler–Maclaurin formula, one can see [4]. Now, we consider the general quadrature

\[ \int_a^b f(t) \, dt = (b-a) \sum_{i=0}^{k-1} p_i f(x_i) + R_n(f) \] (2.7)

and obtain the following corollary.

**Corollary 4.** Let \( x_i \in [a, b] \) and \( p_i \geq 0 \) be such that

\[ \sum_{i=0}^{k-1} p_i x_i^j = \frac{b^{j+1} - a^{j+1}}{(j+1)(b-a)}, \quad j \in \{0, 1, \ldots, n-1\}, \] (2.8)
i.e. (2.7) is exact for any polynomial of degree less than \( n \); then we have

\[ \left| \int_a^b f(t) \, dt - (b-a) \sum_{i=0}^{k-1} p_i f(x_i) \right| \leq K_n (b-a)^n \frac{b}{a} \left( f^{(n-1)} \right), \] (2.9)

where \( K_n = \frac{1}{n!} \sup_{a < t < b} \sum_{i=0}^{k-1} B_n^\ast \left( \frac{x_i-a}{b-a} \right) - B_n \left( \frac{x_i-a}{b-a} \right) \).

**Proof.** We first note that \( B_j((t-a)/(b-a)) \) is a polynomial of degree \( j \). By (2.8), we obtain

\[ \sum_{i=0}^{k-1} p_i B_j \left( \frac{x_i-a}{b-a} \right) = \frac{1}{b-a} \int_a^b B_j \left( \frac{t-a}{b-a} \right) \, dt = 0, \quad j \in \{1, 2, \ldots, n-1\}. \]

According to (2.6), we derive (2.9) and complete the proof. \( \square \)

**Remark 4.** It is worth mentioning that the result was derived by Wang [5] in 1978. Further, if \( f \) is discontinuous at \( x_i \) when \( n = 1 \), Corollary 4 also holds. We can generalize it to the functions of bounded \( p \)-variation [6]. Theorem 4 generalizes the classical Euler–Maclaurin formula as can be found in [7,8]. It is also a generalization of Euler trapezoid formulas [3]. In particular, we can evaluate the error constants for some quadrature formulas which are well known.

1. Mid-point rule: \( k = 1, p_0 = 1, x_0 = \frac{b+a}{2}, K_1 = \frac{1}{2}, K_2 = \frac{1}{8} \).
2. Trapezoid rule: \( k = 2, p_0 = p_1 = \frac{1}{2}, x_0 = a, x_1 = b, K_1 = \frac{1}{2}, K_2 = \frac{1}{8} \).
3. Simpson rule: \( k = 3, p_0 = p_2 = \frac{1}{6}, p_1 = \frac{1}{2}, x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b, K_1 = \frac{1}{2}, K_2 = \frac{1}{24}, K_3 = \frac{1}{324}, K_4 = \frac{1}{1152} \).

All constants of the errors are sharp. It is obvious that \( f(t) \) is of bounded variation if \( |f'(t)| < \infty \) or \( f(t) \) is Lipschitz continuous. For further investigation of these cases, one can refer to Ostrowski’s inequality and its extensions ([9–13]).

We define \( \hat{h} = (b-a)/r, a_j = a + j\hat{h}, \quad (j = 0, 1, \ldots, r) \). We apply Corollary 4 on the interval \([a_j, a_{j+1}]\) and we have the following corollary.

**Corollary 5 (Cf. [5]).** Let \( 0 \leq t_0 < t_1 < \cdots < t_{k-1} \leq 1 \). Suppose that the following quadrature rule:

\[ \int_0^1 f(t) \, dt = \sum_{i=0}^{k-1} p_i f(t_i) \]
is exact for any polynomial of degree less than \( n \). Let \( f : [a, b] \rightarrow \mathbb{R} \) be such that \( f^{(n-1)} \) is a function of bounded variation. Then we have

\[
\int_a^b f(t) \, dt = \hat{h} \sum_{j=0}^{r-1} \sum_{i=0}^{k-1} p_i f(a_j + t_i \hat{h}) + R(f),
\]

(2.10)

where

\[
R(f) = \hat{h}^n \int_a^b G_n \left( \frac{t - a}{b - a} \right) \, df^{(n-1)}(t)
\]

and

\[
G_n(t) = \frac{1}{n!} \sum_{i=0}^{k-1} p_i (B^n_n(t_i - t) - B_n(t_i)).
\]

Moreover,

\[
|R(f)| \leq \frac{1}{n!} \left( \frac{b - a}{r} \right)^n \max_{a \leq t \leq b} |f^{(n-1)}(t)| \sup_{0 < \xi < 1} \left| \sum_{i=0}^{k-1} p_i (B^n_n(t_i - t) - B_n(t_i)) \right|.
\]

References


