EXPERIMENTAL VERIFICATION OF THE BUTTERFLY ATTRACTOR IN A MODIFIED LORENZ SYSTEM

S. ÖZOĞUZ
Istanbul Technical University, Faculty of Electrical–Electronics Engineering, 80626, Maslak, Istanbul, Turkey

A. S. ELWAKIL*
Department of Electrical & Electronic Engineering, University of Sharjah, P.O. Box 27272, Emirates

M. P. KENNEDY
Department of Microelectronic Engineering, University College Cork, Ireland

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An electronic circuit realization of a modified Lorenz system, which is multiplier-free, is described. The well-known butterfly chaotic attractor is experimentally observed verifying that the proposed modified system does capture the essential dynamics of the original Lorenz system. Furthermore, we clarify that the butterfly attractor is a compound structure obtained by merging together two simple attractors after performing one mirror operation.

Keywords: Lorenz equations; butterfly chaos; chaotic oscillators.

1. Introduction

The Lorenz system is one of the most famous sets of differential equations which can generate chaos [Lorenz, 1963]. It is an autonomous third-order system which relies on two multipliers to introduce the nonlinearity necessary for folding trajectories. The chaotic attractor observed from this system is the well-known butterfly attractor. Recently, some novel systems which are not topologically equivalent to the Lorenz system, but also exhibit butterfly chaos, have been proposed [Chen & Ueta, 1999; Ueta & Chen, 2000]. However, nonlinearity in these systems is also achieved via multipliers. The utilization of multipliers renders circuit implementations of these systems particularly difficult as can be seen clearly in [Tokunaga et al., 1989] and most recently in [Gonzales et al., 2000].

In this work, we describe a novel electronic circuit which can generate butterfly chaos without need for multipliers. The circuit is a direct implementation of a modified Lorenz system where one of the multipliers is replaced by an absolute value function \( f(X) = |X| \) while the other multiplier is replaced with a bipolar switching constant. An additional parameter is used to verify the compound nature of the butterfly. In particular, it is possible to confine the chaotic dynamics to one or the other of the butterfly wings, forming two simple attractors which when merged together form the butterfly.

*Permanent address: Reactor Department, Nuclear Research Centre, INSHAS Egypt.
Fig. 1. Chaotic attractors observed from the modified Lorenz system: (a) Compound butterfly ($m = 0$). (b) Left-wing half attractor ($m = -1$).
2. Modified Lorenz System

The multiplier-free modified Lorenz system is given by:

\[ \begin{align*}
\dot{X} &= a(Y - X) \\
\dot{Y} &= K(b - Z) + m \\
\dot{Z} &= |X| - cZ
\end{align*} \tag{1} \]

where \( a, b, c \) and \( m \) are constants. \( K \) is a bipolar switching constant given by:

\[ K = \begin{cases} 
1 & X \geq 0 \\
-1 & X < 0 
\end{cases} \tag{2} \]

The constant \( a \) controls the condition for oscillation while \( b \) acts as an amplitude threshold. Parameter \( c \) is a damping constant which if large results in smooth transitions between the two butterfly wings while if sufficiently small results in nonsmooth transitions. \( m \) is a displacement constant; when set to +1, a simple chaotic attractor corresponding to the right-hand wing is obtained and when set to \(-1\), the mirror image of this attractor, i.e. the left-hand wing is obtained. The even-symmetrical nature of the nonlinearity \(|X|\) dictates that only a mirror operation without flipping is performed when these two attractors merge together for \( m = 0 \) to form the butterfly [Elwakil & Soliman, 1998; Gilmore, 1998].

Numerical integration results of the above model using a fourth-order Runge–Kutta algorithm with 0.001 step size are shown in Figs. 1(a) and 1(b) for the two cases \( m = 0 \) and \( m = -1 \), respectively. Here, \( a = 0.9 \), \( b = 2 \) and \( c = 0.1 \). Note that \( Z \) lies only in the positive half-space.

3. Circuit Realization

Consider the circuit shown in Fig. 2 which contains three capacitors \((C_x, C_y, C_z)\); the voltage across
each corresponds to the marked state variable of the system. The two resistors labeled \( R_a \) and \( R_c \) each controls the corresponding parameter in (1) indicated by its subscript. \( V_b \) and \( V_m \) are DC voltages which control the constants \( b \) and \( m \) in (1), respectively. The remaining resistors \((R_1, R_2, R_m)\) and the associated op amps are utilized for necessary voltage to current conversion operations.

It can be verified that for the choice of \( C_X = C_Y = C_Z = C \), \( R_1 = R_2 = R_m = R \), \( R_a = R/a \), \( R_c = R/c \), \( V_m = mV_{\text{ref}} \), \( V_b = bV_{\text{ref}} \) (\( V_{\text{ref}} \) is an arbitrary normalization voltage) and by defining the quantities \( X = V_X/V_{\text{ref}} \), \( Y = V_Y/V_{\text{ref}} \), \( Z = V_Z/V_{\text{ref}} \), and \( t_n = t/RC \), the circuit in Fig. 2 realizes equation sets (1) and (2). The absolute value function \(|X|\) is implemented by a full-wave rectifier circuit [Toumazon et al., 1994] composed of the four diodes \((D_1 – D_4)\) while the switching constant \( K \) is realized via the MOS transistor switches \((N_{1,2} & P_{1,2})\) controlled via the output of a comparator controlled in turn by \( V_X \).

An experimental setup was constructed using an LM311 comparator chip, a 4007 MOS transistor array and general purpose signal diodes. The rest of the op amps in the circuit are all AD844 current feedback op amps which offer a constant bandwidth almost independent of the gain [Analog Devices, 1990]. It is clear from Fig. 2 that only the current output terminal \((C)\) of these op amps was used. All elements were biased with \( \pm 9\)V supplies.

Figure 3(a) represents the observed butterfly attractor when \( R = 5.1 \) k\( \Omega \), \( C = 1 \) nF, \( R_a = 5.54 \) k\( \Omega \), \( R_c = 35 \) k\( \Omega \), and \( V_b = 2 \) V corresponding to \( a = 0.92 \), \( b = 2 \) and \( c = 0.15 \), respectively. Since \( m = 0 \) in this case, the biasing voltage \( V_m \) and the related op amp were simply removed from the circuit. We note that as the damping constant \( c \) is increased, transition between the two wings becomes smoother, as shown in Fig. 3(b) for \( c = 0.36 \) (\( R_c = 14 \) k\( \Omega \)).

Finally, in order to verify the compound nature of the attractor, \( V_m \) was set to \( \pm 1 \) V respectively. The left-hand half attractor \((V_m = -1)\) is shown in Fig. 3(c). Here also \( c = 0.15 \) and a similar right-hand mirror image results when \( V_m = +1 \). These experimental observations confirm the third

Fig. 3. Experimental observations from a constructed circuit (a) Butterfly attractor with nonsmooth transition \((c = 0.15)\) \((x\text{-axis: } 0.2 \) V/div, \(y\text{-axis: } 0.2 \) V/div). (b) Butterfly with smooth transition \((c = 0.36)\) \((x\text{-axis: } 0.45 \) V/div, \(y\text{-axis: } 0.3 \) V/div). (c) Left-wing half attractor \((m = -1, c = 0.15)\) \((x\text{-axis: } 0.25 \) V/div, \(y\text{-axis: } 0.2 \) V/div).
Fig. 3. (Continued)
conjecture of [Elwakil & Kennedy, 2001] which states that a chaotic attractor observed from a symmetrical nonlinear function (odd or even) is a compound attractor formed by merging together two chaotic attractors corresponding to the anti-symmetrical halves of this nonlinearity after performing one mirror operation and an extra flip operation if the symmetrical function was odd.

It is worth noting that the full-wave rectifier subcircuit can also be replaced by four MOS transistors all controlled by $V_X$. However, in this case severe distortion was noticed due to the mismatches between the PMOS and NMOS transistors.

4. Conclusion

We have verified experimentally the observation of the butterfly attractor from a modified Lorenz system without need for multipliers. The key elements in the circuit realization are an absolute value circuit and ganged voltage-controlled switches.

References


