Packing Designs with Block Size 5 and Indexes 8, 12, 16

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A $(v, K, 1)$ packing design of order $v$, block size $K$, and index $1$ is a collection of $K$-element subsets, called blocks, of a set $V$ such that every 2-subset of $V$ occurs in at most $1$ blocks. The packing problem is to determine the maximum number of blocks in a packing design. In this paper we solve the packing problem with $K = 5$, $1 = 8, 12, 16$, and all positive integers $v$ with the possible exceptions of $(v, 1) = (19, 16) (22, 16) (24, 16) (27, 16) (28, 12)$. © 1992 Academic Press, Inc.

1. INTRODUCTION

A $(v, K, 1)$ packing design of order $v$, block size $K$, and index $1$ is a collection $B$ of $K$-element subsets, called blocks, of a $v$-set $V$ such that every 2-subset of $V$ occurs in at most $1$ blocks. If every 2-subset of $V$ occurs in at least $1$ blocks, then it is called a covering design.

Let $\sigma(v, K, 1)$ denote the maximum number of blocks in a $(v, K, 1)$ packing design. A $(v, K, 1)$ packing design with $|B| = \sigma(v, K, 1)$ will be called a maximum packing design.

Schoenheim [8] has shown that

\[ \sigma(v, K, 1) \leq \left[ \frac{v}{K} \left[ \frac{v - 1}{K - 1} \right] \right] = \psi(v, K, 1), \]

where $[x]$ is the largest integer satisfying $[x] \leq x$. Hanani [6] has sharpened this bound in certain cases by proving the following result.
THEOREM 1.1. If \( \lambda(v-1) \equiv 0 \pmod{\kappa-1} \) and \( \lambda v(v-1)/(\kappa-1) \equiv 1 \pmod{\kappa} \) then \( \sigma(v, \kappa, \lambda) \leq \psi(v, \kappa, \lambda) - 1 \).

The value of \( \sigma(v, 3, \lambda) \) for all \( v \) and \( \lambda \) has been determined by Hanani [6]. The value of \( \sigma(v, 4, 1) \) for all \( v \) has been determined by Brouwer [5]. The value \( \sigma(v, 4, \lambda) \) for all \( v \) and \( \lambda > 1 \) has been determined by Assaf [1] and by Hartman [7].

In the case \( \kappa = 5 \) and any \( \lambda \) the only result known is that of Assaf and Hartman [2] in which they have determined \( \sigma(v, 5, 4) \) for all \( v \).

A balanced incomplete block design, \( B[v, \kappa, \lambda] \), is a \((v, \kappa, 2)\) packing design where every 2-subset of points is contained in precisely \( \lambda \) blocks. If a \( B[v, \kappa, \lambda] \) exists, then it is clear that \( \sigma(v, \kappa, \lambda) = \psi(v, \kappa, \lambda) = \lambda v(v-1)/\kappa(\kappa-1) \) and Hanani [6] has proved the following existence theorem for \( K = 5 \).

THEOREM 1.2. Necessary and sufficient conditions for the existence of a \( B[v, 5, \lambda] \) are that \( \lambda(v-1) \equiv 0 \pmod{4} \) and \( \lambda v(v-1) \equiv 0 \pmod{20} \) and \( (4, \lambda) \neq (15, 2) \).

This theorem implies that \( \alpha(v, 5, \lambda) = \psi(v, 5, \lambda) \) for \( \lambda = 8, 12, 16 \) and \( v \equiv 0 \) or 1 (mod 5).

In this paper we are interested in determining the values of \( \sigma(v, 5, \lambda) \), where \( \lambda = 8, 12, 16 \). Our goal is to prove that \( \sigma(v, 5, \lambda) = \psi(v, 5, \lambda) \) for all \( v \geq 5 \), with some few possible exceptions. Specifically we prove the following.

THEOREM 1.3. Let \( v \geq 5 \) be a positive integer. Then

1. \( \sigma(v, 5, 12) = \psi(v, 5, 12) \), where \( v \equiv 3 \pmod{5} \) with the possible exception of \( v = 28 \).
2. \( \sigma(v, 5, 12) = \psi(v, 5, 12) - 1 \), where \( v \equiv 2 \) or 4 (mod 5).
3. \( \sigma(v, 5, \lambda) = \psi(v, 5, \lambda) \) for all \( v \geq 5 \) and \( \lambda = 8, 16 \) with the possible exceptions of \( (v, \lambda) = (19, 16) \) (22, 16) (24, 16) (27, 16).

2. CONSTRUCTIONS OF PACKING DESIGNS

To prove our theorem, we do not need any recursive construction. The constructions are direct constructions, using the method used in [1], that is, using designs on a different number of points and then identifying points to obtain optimal packing. This method is very useful in the case \( \lambda > 1 \) and it will play an important role in determining the covering and packing designs with \( \kappa = 5 \) and \( \lambda > 1 \). To describe our construction we need the notion of designs with a hole.
Let \((V, \beta)\) be a \((v, \kappa, \lambda)\) packing design, and let \(H\) be a subset of \(V\) of cardinality \(h\). We shall say that \((V, \beta)\) is an exact packing design with a hole of size \(h\) if no 2-subset of \(H\) appears in any block, and every other 2-subset of \(V\) appears in precisely \(\lambda\) blocks.

The following results are most essential to prove our theorem.

**Lemma 2.1 (Assaf and Hartman [2]).**

(i) Let \(v \equiv 2 \text{ or } 4 \pmod{5}\). An exact \((v, 5, 4)\) packing design with a hole of size 2 exists for all \(v \neq 7\).

(ii) Let \(v \equiv 3 \pmod{5}\). An exact packing design with a hole of size 3 and \(\lambda = 4\) exists for all \(v \neq 8\) and possible exceptions of \(v = 43, 68\).

**Lemma 2.2 (Assaf and Shalaby [3]).** For all positive integers \(v\) there is a \((v, 5, 4)\) covering design with the possible exceptions of \(v = 17, 18, 19, 22, 24, 27, 28, 78, 98\). Furthermore, the repeated pairs in the case \(v \equiv 2 \text{ or } 4 \pmod{5}\) are the pairs of a triple; each appears exactly six times.

In the following two lemmas we improve the results of Lemmas 2.1 and 2.2.

**Lemma 2.3.** Let \(v \equiv 3 \pmod{5}\). An exact packing design with a hole of size 3 and \(\lambda = 4\) exists for all \(v \neq 8\), with the possible exception of \(v = 68\).

**Proof.** By Lemma 2.1 we only need to handle the case \(v = 43\). A \((43, 5, 4)\) packing design with a hole of size 3 can be constructed as follows:

(i) Take the blocks of a \(B[41, 5, 2]\) on \(X = \{1, \ldots, 41\}\).

(ii) Take the blocks of a \(B[45, 5, 2]\) on \(X = \{1, \ldots, 45\}\) and assume we have two copies of the block \(\langle 41, 42, 43, 44, 45 \rangle\) (we may assume this since a \(B[45, 5, 1]\) exists). Drop these two blocks and in the remaining blocks of this design change 45 to 43 and 44 to 42. The resultant blocks of (i) and (ii) are the blocks of a \((43, 5, 4)\) packing design with a hole of size 3.

**Lemma 2.4.** There exists a \((v, 5, 4)\) covering design for \(v = 17, 18\).

**Proof.** For \(v = 17\) let \(V = \mathbb{Z}_{17}\). Then the required blocks are

\[
\langle 1 2 4 10 13 \rangle \quad \langle 1 2 8 14 17 \rangle \quad \langle 1 2 6 11 15 \rangle \\
\langle 1 2 7 10 12 \rangle \quad \langle 1 3 8 11 17 \rangle \quad \langle 1 3 9 14 16 \rangle \\
\langle 1 3 8 10 13 \rangle \quad \langle 1 3 9 11 12 \rangle \quad \langle 1 4 12 14 17 \rangle \\
\langle 1 4 5 6 16 \rangle \quad \langle 1 4 9 15 17 \rangle \quad \langle 1 5 7 14 15 \rangle \\
\langle 1 5 6 7 9 \rangle \quad \langle 1 5 7 11 13 \rangle \quad \langle 1 6 10 13 16 \rangle \\
\langle 2 3 7 16 17 \rangle \quad \langle 2 3 5 8 12 \rangle \quad \langle 2 3 4 5 6 \rangle \\
\langle 2 3 7 11 16 \rangle \quad \langle 2 4 7 9 14 \rangle \quad \langle 2 4 8 13 15 \rangle \\
\langle 2 5 13 14 16 \rangle \quad \langle 2 5 9 14 15 \rangle \quad \langle 2 6 10 11 12 \rangle \\
\langle 2 6 8 9 11 \rangle \quad \langle 2 9 12 13 17 \rangle \quad \langle 2 10 15 16 17 \rangle
g
For $u = 18$ let $V = \mathbb{Z}_{18}$. Then the required blocks are

\begin{align*}
\langle 1 & 2 8 15 17 \rangle & \langle 1 & 2 5 14 17 \rangle & \langle 1 & 2 6 10 13 \rangle \\
\langle 1 & 2 4 12 16 \rangle & \langle 1 & 3 9 10 12 \rangle & \langle 1 & 3 11 13 16 \rangle \\
\langle 1 & 3 5 6 15 \rangle & \langle 1 & 3 10 11 14 \rangle & \langle 1 & 4 12 16 18 \rangle \\
\langle 1 & 4 5 7 13 \rangle & \langle 1 & 4 9 13 15 \rangle & \langle 1 & 5 8 11 15 \rangle \\
\langle 1 & 6 7 8 10 \rangle & \langle 1 & 6 7 16 17 \rangle & \langle 1 & 7 9 14 18 \rangle \\
\langle 1 & 8 9 14 18 \rangle & \langle 1 & 11 12 17 18 \rangle & \langle 2 & 3 4 10 13 \rangle \\
\langle 2 & 3 8 9 12 \rangle & \langle 2 & 3 6 11 18 \rangle & \langle 2 & 4 6 9 15 \rangle \\
\langle 2 & 5 14 16 18 \rangle & \langle 2 & 6 12 14 17 \rangle & \langle 2 & 7 9 12 18 \rangle \\
\langle 2 & 7 10 15 18 \rangle & \langle 2 & 7 8 13 14 \rangle & \langle 2 & 7 11 16 17 \rangle \\
\langle 2 & 8 11 13 15 \rangle & \langle 2 & 9 10 11 16 \rangle & \langle 3 & 4 14 15 16 \rangle \\
\langle 3 & 4 8 14 17 \rangle & \langle 3 & 4 7 17 18 \rangle & \langle 3 & 5 9 11 18 \rangle \\
\langle 3 & 5 10 14 17 \rangle & \langle 3 & 6 7 9 15 \rangle & \langle 3 & 6 13 16 18 \rangle \\
\langle 3 & 7 12 15 17 \rangle & \langle 3 & 8 12 13 16 \rangle & \langle 4 & 5 6 9 14 \rangle \\
\langle 4 & 5 7 10 16 \rangle & \langle 4 & 6 10 11 17 \rangle & \langle 4 & 6 8 11 18 \rangle \\
\langle 4 & 8 9 11 17 \rangle & \langle 4 & 10 12 15 18 \rangle & \langle 4 & 11 12 13 14 \rangle \\
\langle 5 & 6 8 16 18 \rangle & \langle 5 & 6 8 10 12 \rangle & \langle 5 & 7 10 11 12 \rangle \\
\langle 5 & 7 9 11 13 \rangle & \langle 5 & 8 12 15 16 \rangle & \langle 5 & 9 12 13 17 \rangle \\
\langle 5 & 13 15 17 18 \rangle & \langle 6 & 7 12 13 14 \rangle & \langle 6 & 9 13 16 17 \rangle \\
\langle 6 & 11 12 14 15 \rangle & \langle 7 & 11 14 15 16 \rangle & \langle 8 & 9 10 16 17 \rangle \\
\langle 8 & 10 13 14 18 \rangle & \langle 9 & 10 14 15 16 \rangle & \langle 10 & 13 15 17 18 \rangle \\
\langle 2 & 5 7 8 3 \rangle & \langle 2 & 5 7 8 4 \rangle
\end{align*}

With Lemmas 2.1–2.4 at our hands we can prove the following lemmas.

**Lemma 2.5.** For all positive integers $v \geq 5$, we have $\sigma(v, 5, 8) = \psi(v, 5, 8)$.

**Proof.** For $v \equiv 2$ or $4 \pmod{5}$, $v \neq 7$, the blocks of a $(v, 5, 8)$ packing design are those of a $(v, 5, 4)$ packing design, each block taken twice.

For $v \equiv 3 \pmod{5}$ the blocks of a $(v, 5, 8)$ packing design $v \neq 8, 68$ can be constructed as follows:

(i) Take the blocks of a $(v - 1, 5, 4)$ packing design.

(ii) Take the blocks of a $(v + 1, 5, 4)$ packing design. Since $v + 1 \equiv 4 \pmod{5}$, then by Lemma 2.1 there is one pair, say $(v + 1, v)$, that does not appear at all, so change the point $v + 1$ to $v$ in all the blocks of the $(v + 1, 5, 4)$ packing design.
The above construction does not work for $v = 7, 8$. For $v = 7$ let $X = \mathbb{Z}_7$, then the required blocks are

\[
\begin{align*}
\langle 0, 1, 2, 4, 5 \rangle & \quad \langle 0, 1, 2, 4, 6 \rangle & \quad \langle 0, 1, 2, 5, 6 \rangle & \quad \langle 0, 1, 4, 5, 6 \rangle \\
\langle 0, 1, 3, 4, 5 \rangle & \quad \langle 0, 1, 3, 4, 6 \rangle & \quad \langle 0, 1, 3, 5, 6 \rangle & \quad \langle 1, 2, 4, 5, 6 \rangle \\
\langle 0, 2, 3, 4, 5 \rangle & \quad \langle 0, 2, 3, 4, 6 \rangle & \quad \langle 0, 2, 3, 5, 6 \rangle & \quad \langle 2, 3, 4, 5, 6 \rangle \\
\langle 1, 2, 3, 4, 5 \rangle & \quad \langle 1, 2, 3, 4, 6 \rangle & \quad \langle 1, 2, 3, 5, 6 \rangle & \quad \langle 0, 3, 4, 5, 6 \rangle
\end{align*}
\]

A nice construction for $v = 8$ is given below. Let $V = \mathbb{Z}_8 \cup \{a, b\}$. Then the required blocks are

\[
\begin{align*}
\langle a, b, 0, 2, 4 \rangle + i, & \quad i \in \mathbb{Z}_4 \\
\langle 0, 1, 2, 3, a \rangle & \quad \pmod{6} \\
\langle 0, 1, 2, 3, b \rangle & \quad \pmod{6} \\
\langle k, k + 1, k + 3, k + 4, f(k) \rangle & \quad \pmod{6},
\end{align*}
\]

where $f(k) = a$ if $k$ is even and $f(k) = b$ if $k$ is odd.

**Lemma 2.6.** For all positive integers $v \geq 5$, we have $\sigma(v, 5, 12) = \Psi(v, 5, 12) - 1$ if $v \equiv 2$ or $4 \pmod{5}$ and $\sigma(v, 5, 12) = \Psi(v, 5, 12)$ if $v \equiv 3 \pmod{5}$, $v \neq 28$.

**Proof.** We distinguish the following two cases:

**Case 1.** $v \equiv 2$ or $4 \pmod{5}$. In this case the blocks of a $(v, 5, 12)$ packing design, $v \neq 7$, are those of a $(v, 5, 4)$ and $(v, 5, 8)$ packing design. Note that we applied Theorem 1.1. For $v = 7$ let $X = \mathbb{Z}_7 \cup \{\infty\}$ then the blocks are

\[
\begin{align*}
\langle 0, 1, 2, 3, 4 \rangle & \quad \pmod{6} \\
\langle 0, 1, 2, 4, \infty \rangle & \quad \pmod{6} \\
\langle 0, 1, 2, 5, \infty \rangle & \quad \pmod{6} \\
\langle 0, 1, 3, 4, \infty \rangle & \quad \pmod{6}
\end{align*}
\]

**Case 2.** $v \equiv 3 \pmod{5}$. A $(v, 5, 12)$ packing design, $v \neq 8, 18, 23, 28, 68$ can be constructed as follows:

(i) Take the blocks of a $(v - 1, 5, 4)$ covering design. By Lemmas 2.2 and 2.4 such design exists for all positive integers $v$ with the possible exceptions of $v = 20, 23, 25, 28, 29, 79, 99$. Also by Lemma 2.2 there is a triple, say $\{a, b, c\}$, such that each pair of this triple occurs exactly six times in the blocks of a $(v - 1, 5, 4)$ covering design, and all other pairs occur exactly four times.

(ii) Take a $(v, 5, 4)$ packing design. By Lemma 2.3 such a design has a hole of size 3 with the exception of $v = 8$ and possible exception of $v = 68$. Assume the hole is $\{a, b, c\}$. 


(iii) Take the blocks of a \((v+1, 5, 4)\) packing design. By Lemma 2.1 such design exists for all \(v \neq 6\); furthermore, there is exactly one pair say \((v, v+1)\) that does not occur in any block. So in these blocks change \(v+1\) to \(v\).

Since a \((v, 5, 4)\) covering design, \(v = 19, 22, 24, 27\), is still unknown and since \((8, 5, 4)\) and \((68, 5, 4)\) packing designs have no holes of size 3, the above construction fails in the cases of \(v = 8, 18, 23, 28, 68\).

For \(v = 8\) let \(V = \mathbb{Z}_7 \cup \{a, b, c\}\). Then the required blocks are

\[
\begin{align*}
&\langle 0, 1, 2, a, b \rangle \pmod{5} \\
&\langle 0, 2, 3, a, c \rangle \pmod{5} \\
&\langle 0, 1, 2, b, c \rangle \pmod{5} \\
&\langle 0, 2, 3, a, b \rangle \pmod{5} \\
&\langle 0, 2, 3, a, c \rangle \pmod{5} \\
&\langle 0, 1, 2, b, c \rangle \pmod{5} \\
&\langle 0, 1, 2, 3, 4 \rangle \text{ three times.}
\end{align*}
\]

For \(v = 18\) let \(X = \mathbb{Z}_{15} \cup \{a, b, c\}\). Then the required blocks are

\[
\begin{align*}
&\langle 0, 3, 6, 9, 12 \rangle + i, i \in \mathbb{Z}_3 \\
&\langle 0, 5, 10, a, b \rangle + i, i \in \mathbb{Z}_5 \text{ twice} \\
&\langle 0, 5, 10, a, c \rangle + i, i \in \mathbb{Z}_5 \text{ twice} \\
&\langle 0, 5, 10, b, c \rangle + i, i \in \mathbb{Z}_5 \text{ twice} \\
&\langle 0, 1, 3, 4, 11 \rangle \pmod{15} \\
&\langle 0, 2, 3, 4, 6 \rangle \pmod{15} \\
&\langle 0, 1, 7, 8, 14 \rangle \pmod{15} \\
&\langle 0, 2, 4, 8, 11 \rangle \pmod{15} \\
&\langle 0, 1, 3, 6, a \rangle \pmod{15} \\
&\langle 0, 1, 4, 9, a \rangle \pmod{15} \\
&\langle 0, 2, 6, 9, b \rangle \pmod{15} \\
&\langle 0, 1, 3, 5, b \rangle \pmod{15} \\
&\langle 0, 1, 5, 7, c \rangle \pmod{15} \\
&\langle 0, 1, 4, 9, c \rangle \pmod{15} \\
\end{align*}
\]

For \(v = 23\) take the blocks of a \(B_{[23, 5, 10]}\) and the blocks of a \((23, 5, 2)\) packing design. See [4] for the construction of the last design. A \((68, 5, 12)\) packing design may be constructed as follows. Take a \(GD[6, 2, 5, 35]\) (see [6, p. 262] for a definition of group divisible designs, and [6, p. 286] for the existence of this design) and delete one point from last group. The resultant design is a \(GD[\{5, 6\}, 2, \{4^*, 5\}, 34]\), that is, a group divisible design with blocks of size 5 and 6 and groups of size 5 and exactly one group of size 4. Now inflate this design by 2 and \(\lambda = 6\): that is, replace each point \(x\) of \(GD[\{5, 6\}, 2, \{4^*, 5\}, 34]\) by two points \(x_0, x_1\),
and replace each block of size 5 and each block of size 6 by the blocks of a GD[5, 6, 2, 10] and GD[5, 6, 2, 12]. These two designs are actually three copies of GD[5, 2, 2, 10] and GD[5, 2, 2, 12], respectively, and they may be found in [3].

Finally, on the groups of size 10 construct a B[10, 5, 12] and on the group of size 8 construct a (8, 5, 12) packing design.

**Lemma 2.7.** For all positive integers \( v \geq 5, \) \( v \neq 19, 22, 24, 27, \) we have \( \sigma(v, 5, 16) = \psi(v, 5, 16). \)

**Proof.** We distinguish the following two cases:

**Case 1.** \( v \equiv 3 \pmod{5}. \) In this case the blocks of a \((v, 5, 16)\) packing design are those of a \((v, 5, 8)\) packing design each block taken twice.

**Case 2.** \( v = 2 \) or \( 4 \pmod{5}. \) In this case the construction of a \((v, 5, 16)\) packing design for all \( v \neq 7, 19, 22, 24, 27 \) can be done as follows:

(i) Take the blocks of a \((v, 5, 4)\) covering design \( v \neq 19, 22, 24, 27. \) In this design there is a triple, say \( \{a, b, c\} \), such that each pair of this triple appears six times in the blocks of the \((v, 5, 4)\) covering design.

(ii) Take three copies of a \((v, 5, 4)\) packing design \( v \neq 7. \) By Lemma 2.1 a \((v, 5, 4)\) packing design, \( v \neq 7, \) has a hole of size 2. In the first copy assume the hole is \((a, b)\); in the second the hole is \((a, c)\); and in the third the hole is \((b, c)\). Then the blocks from (i) and (ii) are the blocks of a \((v, 5, 16)\) packing design. Since there is no \((7, 5, 4)\) packing design; the previous method does not work for \( v = 7. \)

For \( v = 7 \) let \( X = Z_2 \times Z_3 \cup \{\infty\}. \) Then the required blocks are

\[
\begin{align*}
\langle(0, 0) (0, 1) (1, 0) (1, 2) \rangle &\mod(-, 3), \text{ twice} \\
\langle(0, 0) (0, 1) (1, 1) (1, 2) \rangle &\mod(-, 3), \text{ twice} \\
\langle(0, 0) (0, 1) (1, 0) (1, 1) \rangle &\mod(-, 3), \text{ twice} \\
\langle(0, 0) (0, 1) (0, 2) (1, 2) \rangle &\mod(-, 3) \\
\langle(0, 0) (1, 0) (1, 1) (1, 2) \rangle &\mod(-, 3) \\
\langle(0, 0) (0, 1) (0, 2) (1, 0) (1, 1) \rangle &\mod(-, 3), \text{ twice} \\
\langle(0, 0) (0, 1) (1, 0) (1, 1) (1, 2) \rangle &\mod(-, 3)
\end{align*}
\]

We now prove our main theorem which is restated below for the reader's convenience.

**Theorem 1.3.** Let \( v \geq 5 \) be a positive integer. Then

1. \( \sigma(v, 5, 12) = \psi(v, 5, 12), \) where \( v \equiv 3 \pmod{5} \) with the possible exception of \( v = 28. \)

2. \( \sigma(v, 5, 12) = \psi(v, 5, 12) - 1, \) where \( v \equiv 2 \) or \( 4 \pmod{5}. \)
\(\sigma(v, 5, \lambda) = \psi(v, 5, \lambda)\) for all \(v \geq 5\) and \(\lambda = 8, 16\) with the possible exceptions of \((v, \lambda) = (19, 16)\) \((22, 16)\) \((24, 16)\) \((27, 16)\).

**Proof.** We distinguish three cases:

**Case 1.** \(\lambda = 8\), the proof of this case follows from Lemmas 2.5 and 2.8.

**Case 2.** \(\lambda = 12\), the proof of this case follows from Lemma 2.6.

**Case 3.** \(\lambda = 16\), the proof of this case follows from Lemma 2.7.

**References**