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A GENERALIZED MANDELBROT SET OF POLYNOMIALS OF TYPE E_d FOR BICOMPLEX NUMBERS

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Abstract. We use a commutative generalization of complex numbers called bicomplex numbers to introduce the bicomplex dynamics of *polynomials of type* E_d, $f_c(w) = w(w + c)^d$. Rochon [9] proved that the Mandelbrot set of quadratic polynomials in bicomplex numbers of the form $w^2 + c$ is connected. We prove that our generalized Mandelbrot set of polynomials of type E_d, $f_c(w) = w(w + c)^d$, is connected.

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1. Introduction

In 1982, A. Norton [8] gave some straightforward algorithms for generation and display fractal shapes in 3-D. For the first time, iteration by quaternions appeared. Subsequently, theoretical results were obtained in [5] for the quaternionic Mandelbrot set defined by quadratic polynomial in quaternions of the form $q^2 + c$. Rochon [9] used a commutative generalization of complex numbers called bicomplex numbers to give a new version of the Mandelbrot set in dimensions three and four. Moreover, he proved that the generalization in dimention four is connected.

In this article, we introduce a one-parameter family of polynomials of type E_d, $f_c(z) = z(z + c)^d$. First, in the complex dynamics of f_c, we consider the connectedness locus or, what is the same, the *Mandelbrot set*,

$$\mathcal{C}_d = \{c \in \mathbb{C} : \text{the filled Julia set} \quad K(f_c) \quad \text{is connected}\},$$

and prove that the connectedness locus $\mathcal{C}_d$ of polynomials of type E_d is connected:

Theorem 1. *The Mandelbrot set* $\mathcal{C}_d$ *is connected.*

Finally, by using the concept of filled Julia set for bicomplex numbers, we give a version of the connectedness locus for a bicomplex number and prove that our generalization of the Mandelbrot set for bicomplex numbers is connected:

Theorem 2. *The generalized Mandelbrot set* $\mathcal{C}_d^2$ *is connected.*

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2. Polynomials of Type E_d

Let us recall (see [2]) the terminology and definitions in the monic family of higher degree polynomials. Consider the monic family of complex polynomials $f_c(z) = z(z+c)^d$, where $c \in \mathbb{C}$ and $d \geq 2$. Each f_c has degree $d+1$ and has exactly two critical points: $-c$ with multiplicity $d-1$, and $c_0 = \frac{-c}{d+1}$ with multiplicity 1. Moreover, $f_c(-c) = 0$ and 0 is fixed. It is proved (see [2]) that polynomials with these features can always be expressed in the form f_c.

Definition 1 ([2])**.** A monic polynomial f of degree $d \geq 2$ is of type E_d if it satisfies the following properties:

1. f has two critical points: $-c$ of multiplicity $d-1$ and c_0 of multiplicity 1.
2. f has a fixed point at $z = 0$.
3. $f(-c) = 0$.

Proposition 1. *Any monic polynomial $f(z)$ of degree $d+1$ which is of type E_d is of the form*

$$f_c(z) = z(z+c)^d.$$

Moreover, if f_c and $f_{c'}$ of type E_d are affine conjugate with $d \geq 2$, then $c = c'$.

The proof is straightforward and is omitted. □

Notation. Denote $\mathbb{D}$ as the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

3. Connectedness

As we know, the Mandelbrot set of quadratic polynomials is connected. In this section we are going to prove that the Mandelbrot set $\mathcal{C}_d$ of polynomials of type E_d is connected. For this purpose, we need slight modifications of some well known results and methods.

If we apply the Böttcher theorem([3]) to the polynomial f_c, for a sufficiently large positive number R_c, we get a *Böttcher function* $\phi_c(z)$ on $D_c = \{z \in \mathbb{C} : |z| \geq R_c\}$, such that $\phi_c(f_c(z)) = (\phi_c(z))^{d+1}$.

Let us recall (see [3]) that, in general, a Böttcher function $\phi_c(z)$ cannot be continued analytically to the whole of $A_c(\infty)$. However, since $A_c(\infty) = \bigcup_{n=1}^{\infty}(f_c^n)^{-1}(D_c)$, we can extend the harmonic function $G_c(z) = \log|\phi_c(z)|$ to the whole of $A_c(\infty)$ by setting

$$G_c(z) = \frac{1}{(d+1)^{n+1}} G_c(f_c^n(z)) \quad \text{if} \quad z \in (f_c^n)^{-1}(D_c).$$

This extension is well defined and $G_c(z)$ is clearly harmonic on $A_c(\infty)$. Moreover, we set $G_c(z) = 0$ for every $z \in K_c$. Then $G_c(z)$ is a continuous subharmonic function on $\mathbb{C}$, which is called the *Green function* associated to $K(f_c)$.

Proposition 2 (Sibony's Theorem)**.** *For every $A \geq 0$ there exists $\alpha = \alpha(A) \geq 0$ such that the restriction of the Green function G_c to the disk $|c| \leq A$ is $\alpha-$ Hölder .*

Proof. We repeat the arguments given in the proof of Theorem 3.2, page 138 of [3], by making necessary modifications. Assume that $A \geq 10$. Let $z \in \mathbb{C} - K(f_c)$ and let $\delta(z) = dist(z, K(f_c))$. Let us take the closest point $z_0 \in K_c$ to z, and let $S = \{z_0 + t(z - z_0) : \quad 0 \leq t \leq 1\}$. Take $N = N(z)$ to satisfy $|f_c^n(w)| < A^2$ for all $w \in S$ and all $n < N$, while $|f_c^N(z_1)| \geq A^2$ for some $z_1 \in S$. Using $f_c'(z) = (z+c)^{d-1}((d+1)z + c)$ and the chain rule, we see that $|(f_c^n)'(w)| \leq ((A^2 + A)^{d-1}((d+1)A^2 + A))^n$ for all $n < N$ and $w \in S$. Also, $|f_c^n(z_0)| < 2A$ for all n, or else the iterates

$$|f_c^{n+1}(z_0)| \geq |f_c^n(z_0)|(|f_c^n(z_0)| - |c|)^d \geq |f_c^n(z_0)|A^d > |f_c^n(z_0)|$$

would tend to ∞. The mean value theorem then implies that

$$|f_c^N(z_1) - f_c^N(z_0)| \leq ((A^2 + A)^{d-1}((d+1)A^2 + A))^N \delta(z_1).$$

Then we have

$$|f_c^N(z_1)| \leq 2A + ((A^2 + A)^{d-1}((d+1)A^2 + A))^N \delta(z_1).$$

But $|f_c^N(z_1)| \geq A^2$, hence $((A^2 + A)^{d-1}((d+1)A^2 + A))^N \delta(z_1) \geq 1$, and $((A^2 + A)^{d-1}((d+1)A^2 + A))^N \delta(z) \geq 1$.

For $\alpha = \frac{\log(d+1)}{\log((A^2+A)^{d-1}((d+1)A^2+A))}$ we have $\delta(z)^\alpha > (d+1)^{-N}$ so that

$$G_c(z) = G_c(f_c^N(z))(d+1)^{-N} \leq M\delta(z)^\alpha,$$

where M depends only on A. Consider two points z_1, z_2 and suppose $\delta(z_1) \geq \delta(z_2)$. We would like to prove

$$|G_c(z_1) - G_c(z_2)| \leq C|z_1 - z_2|^\alpha.$$

If $|z_1 - z_2| > \frac{1}{2}\delta(z_1)$, this follows from the above estimate. If $|z_1 - z_2| \leq \frac{1}{2}\delta(z_1)$, we use the Harnack inequality for the positive harmonic function $G_c(z)$ in the disk $D(z_1, \delta(z_1))$ to conclude that

$$|G_c(z_1) - G_c(z_2)| \leq C_0 M\delta(z_1)^\alpha \frac{|z_1 - z_2|}{\delta(z_1)} \leq C|z_1 - z_2|^\alpha. \qquad \square$$

Proposition 3. *If $c_n \to c$, then the corresponding Green functions $G_{c_n}(z)$ converge uniformly on $\mathbb{C}$ to G_c. Thus $G_c(z)$ is jointly continuous in c and z.*

Proof. Proposition 2 guarantees the equicontinuity of the sequence $G_c(z)$ and the arguments of [3], page 139, can be reproduced. $\square$

Finally we have

Theorem 1. *The Mandelbrot set $\mathcal{C}_d$ is connected.*

Proof. As $\hat{\mathbb{C}} - \overline{\mathbb{D}}$ is simply connected, it is enough to find a homeomorphism from $\hat{\mathbb{C}} - \mathcal{C}_d$ onto $\hat{\mathbb{C}} - \overline{\mathbb{D}}$. We will need to find a conformal map

$$\Phi : \mathbb{C} - \mathcal{C}_d \to \mathbb{C} - \overline{\mathbb{D}}.$$

Let $\Omega = \{(z, c) \in \mathbb{C} \times \mathbb{C} : c \in \mathbb{C} - \mathcal{C}_d, G_c(z) > G_c(c_0)\}$. For every $n \geq 0$ the pre-critical points, as $(z, c) \in \mathbb{C}^2$ such that $f_c^n(z) = c_0$, and the pre-images of the origin, as $(z, c) \in \mathbb{C}^2$ such that $f_c^n(z) = 0$, do not belong to Ω. Hence the

Böttcher function $\phi(z,c) := \phi_c(z)$ is well defined and analytic on Ω. It has the following representation

$$\phi(z,c) = \lim_{n\to\infty} (f_c^n(z))^{\frac{1}{(d+1)^n}} = z\prod_{n=1}^{\infty} \frac{(f_c^n(z))^{\frac{1}{(d+1)^n}}}{(f_c^{n-1}(z))^{\frac{1}{(d+1)^{n-1}}}}$$
$$= z\prod_{n=1}^{\infty}\left(\frac{f_c^{n-1}(z)(f_c^{n-1}(z)+c)^d}{(f_c^{n-1}(z))^{d+1}}\right)^{\frac{1}{(d+1)^n}} = z\prod_{n=0}^{\infty}\left(1+\frac{c}{f_c^n(z)}\right)^{\frac{d}{(d+1)^{n+1}}}.$$

We obviously have $(f_c(c_0), c) \in \Omega$. Now we define

$$\Phi : \mathbb{C} - \mathcal{C}_d \to \mathbb{C} - \overline{\mathbb{D}}, \quad \Phi(c) = \phi(f_c(c_0), c),$$

and prove that the function Φ is onto and conformal. By the representation

$$\phi(z,c) = z\prod_{n=0}^{\infty}\left(1+\frac{c}{f_c^n(z)}\right)^{\frac{d}{(d+1)^{n+1}}}$$

the Böttcher function $\phi(z,c)$ is analytic of two variables in Ω. It follows that Φ is analytic on $\mathbb{C} - \mathcal{C}_d$. Furthermore, for $c \in \mathbb{C} - \mathcal{C}_d$ we have

$$\log|\Phi(c)| = \log|\phi_c(f_c(c_0))| = G_c(f_c(c_0)) = (d+1)G_c(c_0) > 0,$$

and $|\Phi(c)| > 1$. On the other hand, by Proposition 3, the Green function G_c is continuous. Since $G_c|_{\mathcal{C}_d} = 0$, we conclude that $G_c(f_c(c_0)) \to 0$ as c tends to the boundary of $\mathbb{C} - \mathcal{C}_d$. Consequently $|\Phi(c)| \to 1$ as c tends to the boundary of $\mathbb{C} - \mathcal{C}_d$. Near infinity we have $\phi_c(f_c(c_0)) = (\phi_c(c_0))^{d+1} \sim \frac{-c}{d+1}$, which implies that Φ is injective near ∞, has a simple pole at ∞, and has no zero in $\mathbb{C} - \mathcal{C}_d$. The argument principle can apply: Φ assumes every value in $\mathbb{C} - \overline{\mathbb{D}}$ exactly once on $\mathbb{C} - \mathcal{C}_d$. Hence Φ maps $\hat{\mathbb{C}} - \mathcal{C}_d$ conformally onto $\hat{\mathbb{C}} - \overline{\mathbb{D}}$. In particular, $\hat{\mathbb{C}} - \mathcal{C}_d$ is simply connected, which implies the assertion. $\square$

4. Bicomplex Numbers

We recall some of the basic results of the theory of bicomplex numbers. The bicomplex numbers are defined as follows:

$$\mathbb{C}_2 := \{a + bi_1 + ci_2 + dj : \; i_1^2 = i_2^2 = -1, \; j^2 = 1,$$
$$i_2 j = j i_2 = -i_1, \;\; i_1 j = j i_1 = -i_2, \;\; i_2 i_1 = i_1 i_2 = j\}$$

where $a, b, c, d \in \mathbb{R}$. The norm used on $\mathbb{C}_2$ will be the Euclidean norm (also denoted by $|\;|$) of $\mathbb{R}^4$.

We remark that we can write a bicomplex number as

$$a + bi_1 + ci_2 + dj \quad \text{as} \quad (a + bi_1) + (c + di_1)i_2 = z_1 + z_2 i_2,$$

where $z_1; z_2 \in \mathbb{C}_1 := \{x + yi_1 : i_1^2 = -1\}$. Thus, $\mathbb{C}_2$ can be viewed as the complexification of the usual complex numbers $\mathbb{C}_1$ and a bicomplex number can be regarded as an element of $\mathbb{C}_2$. Moreover, the norm of the bicomplex number is the same as the norm of the associated element (z_1, z_2) of $\mathbb{C}_2$. It is easy to see that $\mathbb{C}_2$ is a commutative unitary ring with the following characterization for noninvertible elements.

Proposition 4. *Let $w = a + bi_1 + ci_2 + dj \in \mathbb{C}_2$. Then w is noninvertible if and only if ($a = -d$ and $b = c$) or ($a = d$ and $b = -c$) if and only if $z_1^2 + z_2^2 = 0$.*

It is also important to know that every bicomplex number $z_1 + z_2 i_2$ has the following unique idempotent representation:

$$z_1 + z_2 i_2 = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2, \quad \text{where } e_1 = \frac{1+j}{2} \text{ and } e_2 = \frac{1-j}{2}.$$

This representation is very useful because: addition, multiplication and division can be done term-by-term. Also, an element will be noninvertible if and only if $z_1 - z_2 i_1 = 0$ or $z_1 + z_2 i_1 = 0$. The next definition will be useful to construct a natural "disc" in $\mathbb{C}_2$.

Definition 2. We say that $X \subset \mathbb{C}_2$ is a $\mathbb{C}_2$-Cartesian set determined by X_1 and X_2 if $X = X_1 \times_e X_2 := \{z_1 + z_2 i_2 \in \mathbb{C}_2 : z_1 + z_2 i_2 = w_1 e_1 + w_2 e_2; (w_1; w_2) \in X_1 \times X_2\}$.

It is shown that if X_1 and X_2 are domains of $\mathbb{C}_1$, then $X_1 \times_e X_2$ is also a domain of $\mathbb{C}_2$. Then, a manner to construct a natural "disc" in $\mathbb{C}_2$ is to take the $\mathbb{C}_2$-Cartesian product of two discs in $\mathbb{C}_1$. Hence, we define the natural "disc" of $\mathbb{C}_2$ as follows:

$$\begin{aligned} D(0; r) &:= B^1(0; r) \times_e B^1(0; r) \\ &= \{z_1 + z_2 i_2 : z_1 + z_2 i_2 = w_1 e_1 + w_2 e_2; |w_1| < r, |w_2| < r\} \end{aligned}$$

where $B^n(0; r)$ is an open ball of $\mathbb{C}_1^n \equiv \mathbb{C}^n$ with radius r.

5. The Generalized Bicomplex Mandelbrot Set of Polynomials of Type E_d

In this section, we want to give a version of the Mandelbrot set of polynomials of type E_d for bicomplex numbers.

Now, to give a version of the Mandelbrot set of polynomials of type E_d for bicomplex numbers we have only to reproduce the algorithm of Definition 2 for bicomplex numbers. This is the next definition.

Definition 3. Let $f_c(w) = w(w + c)^d$ where $w; c \in \mathbb{C}_2$ and $f_c^n(w) := (f_c^{(n-1)} \circ f_c)(w)$. Then the generalized Mandelbrot set for bicomplex numbers is defined as follows: $\mathcal{C}_d^2 = \{c \in \mathbb{C}_2 : f_c^n(c_0) \quad \text{is bounded for every} \quad n \in \mathbb{N}\}$.

The next lemma is a characterization of $\mathcal{C}_d^2$ using only $\mathcal{C}_d$. This lemma will be useful to prove that $\mathcal{C}_d^2$ is also a connected set.

Proposition 5. $\mathcal{C}_d^2 = \mathcal{C}_d \times_e \mathcal{C}_d$.

Proof. First, we prove that $\mathcal{C}_d^2 \subset \mathcal{C}_d \times_e \mathcal{C}_d$. Let $c_2 \in \mathbb{C}_2$ such that $f_c^n(c_0)$ is bounded for every $n \in \mathbb{N}$. We have $f_c(w) = w(w + c)^d = [(z_1 - z_2 i_1)((z_1 - z_2 i_1) + (c_1 - c_2 i_1))^d]e_1 + [(z_1 + z_2 i_1)((z_1 + z_2 i_1) + (c_1 + c_2 i_1))^d]e_2$, where $w = (z_1 - z_2 i_1)e_1 + (z_1 + z_2 i_1)e_2$ and $c = (c_1 - c_2 i_1)e_1 + (c_1 + c_2 i_1)e_2$. Then $f_c^n(w) = f^n_{(c_1 - c_2 i_1)}(z_1 - z_2 i_1)e_1 + f^n_{(c_1 + c_2 i_1)}(z_1 + z_2 i_1)e_2$.

By hypothesis, $f_c^n(c_0) = f^n_{(c_1 - c_2 i_1)}(c_0)e_1 + f^n_{(c_1 + c_2 i_1)}(c_0)e_2$ is bounded for every $n \in \mathbb{N}$.

Hence, $f^n_{(c_1-c_2i_1)}(c_0)$ and $f^n_{(c_1+c_2i_1)}(c_0)$ are also bounded for $\forall n \in \mathbb{N}$. Then $c_1 - c_2i_1, c_1 + c_2i_1 \in \mathcal{C}_d$ and $c = (c_1 - c_2i_1)e_1 + (c_1 + c_2i_1)e_2 \in \mathcal{C}_d \times_e \mathcal{C}_d$.

Conversely, if we take $c \in \mathcal{C}_d \times_e \mathcal{C}_d$, we have $c = (c_1 - c_2i_1)e_1 + (c_1 + c_2i_1)e_2$ with $c_1 - c_2i_1, c_1 + c_2i_1 \in \mathcal{C}_d$. Hence $f^n_{(c_1-c_2i_1)}(c_0)$ and $f^n_{(c_1+c_2i_1)}(c_0)$ are also bounded for every $n \in \mathbb{N}$. Then $f^n_c(c_0)$ is bounded for every $n \in \mathbb{N}$, that is $c \in \mathcal{C}^2_d$. □

Theorem 2. *The generalized Mandelbrot set $\mathcal{C}^2_d$ of polynomials of type E_d is connected.*

Proof. Define a mapping e as follows:

$$e : \mathbb{C}_1 \times \mathbb{C}_1 \to \mathbb{C}_1 \times_e \mathbb{C}_1, \quad (w_1; w_2) \mapsto w_1e_1 + w_2e_2.$$

The mapping e is clearly a homeomorphism. Then, if X_1 and X_2 are connected subsets of $\mathbb{C}_1$, we have that $e(X_1 \times X_2) = X_1 \times_e X_2$ is also connected. Now, by Proposition 5, $\mathcal{C}^2_d = \mathcal{C}_d \times_e \mathcal{C}_d$.. Moreover, by Theorem 1, $\mathcal{C}_d$ is connected. It follows, if we let $X_1 = X_2 = \mathcal{C}_d$, that $\mathcal{C}^2_d$ is connected. □

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