Second-Order Oscillation of Forced Functional Differential Equations with Oscillatory Potentials

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Abstract—New oscillation criteria are established for second-order differential equations containing both delay and advanced arguments of the form,

\[(k(t)x'(t))' + p(t)|x(\tau(t))|^\alpha - 1 x(\tau(t)) + q(t)|x(\sigma(t))|^\beta - 1 x(\sigma(t)) = e(t), \quad t \geq 0,\]

where \(\alpha \geq 1\) and \(\beta \geq 1\); \(k, p, q, e, \tau, \sigma\) are continuous real-valued functions; \(k(t) > 0\) is nondecreasing; \(\tau\) and \(\sigma\) are nondecreasing, \(\tau(t) \leq t, \sigma(t) \geq t\), and \(\lim_{t \to \infty} \tau(t) = \infty\). The potentials \(p, q,\) and \(e\) are allowed to change sign and the information on the whole half-line is not required as opposed to the usual case in most articles. Among others, as an application of the results we are able to deduce that every solution of

\[x''(t) + m_1 \sin t x \left( t - \frac{\pi}{12} \right) + m_2 \cos t x \left( t + \frac{\pi}{6} \right) = \cos 2t, \quad m_1, m_2 \geq 0\]

is oscillatory provided that either \(m_1\) or \(m_2\) is sufficiently large. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

We consider second-order forced equations with both retarded and advanced arguments in the form,

\[(k(t)x'(t))' + p(t)|x(\tau(t))|^\alpha - 1 x(\tau(t)) + q(t)|x(\sigma(t))|^\beta - 1 x(\sigma(t)) = e(t), \quad (E_{\alpha,\beta})\]

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where $t \geq 0$, $\alpha \geq 1$ and $\beta \geq 1$; $k, p, q, e, \tau, \sigma$ are continuous real-valued functions; $k(t) > 0$ is nondecreasing; $\tau$ and $\sigma$ are nondecreasing, $\tau(t) \leq t$, $\sigma(t) \geq t$, and $\lim_{t \to \infty} \tau(t) = \infty$. If $\alpha = \beta = 1$, then $(E_{\alpha, \beta})$ reduces to a linear equation,

$$(k(t)x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) = e(t). \tag{E_{1,1}}$$

As usual, a solution of $(E_{\alpha, \beta})$ is said to be oscillatory if it is defined and nontrivial on some half-line $[t_0, \infty)$ and has an unbounded set of zeros, where $t_0 \geq 0$ may depend on the solution. The equation will be called oscillatory if every solution is oscillatory. As is customary, we tacitly assume that solutions of $(E_{\alpha, \beta})$ exist and nontrivial on some half-line $[t_0, \infty)$. For the general theory, particularly the existence of solutions of such equations, we refer to [1-3]. We should also point out that equations in the form of $(E_{\alpha, \beta})$ are not only of theoretical interest but also of practical importance. For instance, in studying electrodynamic systems, see [2], one has to deal with a second-order differential equation of the form,

$$x''(t) + m_1 x(t - \tau) + m_2 x(t + \tau) = \phi(t),$$

which is clearly a special case of $(E_{1,1})$.

The oscillation of $(E_{\alpha, \beta})$ when $\tau(t) = t$ and $q(t) = 0$ has been studied by many authors, see [4-9] and the references cited therein. If $q(t) \equiv 0$, then the equation becomes a delay differential equation for which there exist also numerous results in the literature [10-13]. Unlike delay differential equations, those containing advanced arguments or both delay and advanced arguments are quite rare in literature [14-19], and in most cases $p(t)$ and $q(t)$ are assumed to be nonnegative.

Nasr [7] and Wong [9] have studied the oscillation of

$$(k(t)x')(') + p(t)|x|^\alpha x = e(t).$$

Recently, Sun [12] has extended these results to delay differential equations of the form,

$$x''(t) + p(t)|x(\tau(t))|^{\alpha-1} x(\tau(t)) = e(t),$$

where $\alpha \geq 1$ and the potentials $p$ and $e$ are allowed to change sign. Later, employing the arguments in [12], Çakmak and Tiryaki [13] have established similar oscillation criteria for equations of the form,

$$x''(t) + q(t)f(x(\tau(t))) = e(t),$$

where $f(x)$ is assumed to satisfy certain growth conditions.

In this article, we consider second-order differential equation containing not only delay but also advanced arguments and derive new oscillation criteria which are similar to ones obtained in [12]. We also allow the potentials $p$, $q$, and $e$ to oscillate. To the best of our knowledge, no such criteria is known concerning the oscillation of $(E_{\alpha, \beta})$, even for the case $p(t) \equiv 0$. Therefore, our results extend and improve the main results of [6,7,9,12].

2. PRELIMINARIES

Suppose that for any given $T \geq 0$, there exist intervals $[\tau(a_1), b_1]$, $[\tau(a_2), b_2]$, $[c_1, \sigma(d_1)]$, and $[c_2, \sigma(d_2)]$ contained in $[T, \infty)$ such that $a_1 < b_1$, $a_2 < b_2$, $c_1 < d_1$, $c_2 < d_2$, and

$$p(t) \geq 0, \quad \text{for } t \in [\tau(a_1), b_1] \cup [\tau(a_2), b_2], \tag{2.1}$$

and

$$q(t) \geq 0, \quad \text{for } t \in [c_1, \sigma(d_1)] \cup [c_2, \sigma(d_2)]. \tag{2.2}$$
Let the function $e(t)$ satisfy
\begin{align}
    e(t) &\leq 0, \quad \text{for } t \in [\tau(a_1), b_1], \\
    e(t) &\geq 0, \quad \text{for } t \in [\tau(a_2), b_2],
\end{align}
and
\begin{align}
    e(t) &\leq 0, \quad \text{for } t \in [c_1, \sigma(d_1)], \\
    e(t) &\geq 0, \quad \text{for } t \in [c_2, \sigma(d_2)].
\end{align}

We may also require that
\begin{align}
    c_i = \tau(a_i), \quad d_i = a_i, \quad b_i = \sigma(d_i), \quad \text{for } i = 1, 2,
\end{align}
which simply means that the intervals $[\tau(a_i), b_i]$ and $[c_i, \sigma(d_i)]$ are the same. It is not difficult to see that if $p(t) \equiv 0$ or $q(t) \equiv 0$, then condition (2.5) is not necessary, and if $e(t) \equiv 0$ then one may take $a_2 = a_1, b_2 = b_1, c_2 = c_1$, and $d_2 = d_1$.

For our purpose, we denote
\[ D(a, b) = \{ u \in C^1[a, b] : u(a) = u(b) = 0, \ u(t) \neq 0 \}. \]

Let $H \in D(a, b)$. Following Wong [20], we define a linear functional $A^b_a : C[0, \infty) \to \mathbb{R}$ as
\[ A^b_a (h) = \int_a^b H^2(t) h(t) \, dt. \]

By using the integration by parts formula it is not difficult to see that
\[ A^b_a (h') = -A^b_a \left( \frac{2hH'}{H} \right), \quad h \in C^1[0, \infty). \]

3. THE MAIN RESULTS

We begin with the linear case, namely $\alpha = 1$ and $\beta = 1$.

**Theorem 3.1.** Suppose that (2.1)–(2.5) hold. If there exist $H_1 \in D(a_i, b_i)$ and $H_2 \in D(c_i, d_i)$ such that either
\begin{align}
    \int_{a_i}^{b_i} \left[ H^2_1(t) p(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - H^2_1(t) k(t) \right] \, dt &\geq 0 \tag{3.1}
\end{align}
or
\begin{align}
    \int_{c_i}^{d_i} \left[ H^2_2(t) q(t) \frac{\sigma(t) - \sigma(d_i)}{\sigma(d_i) - t} - H^2_2(t) k(t) \right] \, dt &\geq 0, \tag{3.2}
\end{align}
for $i = 1, 2$, then $(E_{1, i})$ is oscillatory.

**Proof.** Assume to the contrary that $x(t)$ is a nonoscillatory solution of equation $(E_{1, i})$. Let $x(t) > 0, x(\tau(t)) > 0, x(\sigma(t)) > 0$, for all $t \geq t_1$ for some $t_1 \geq 0$. Define
\[ w(t) = \frac{k(t) x'(t)}{x(t)}, \quad t \geq t_1. \]

In view of $(E_{1, i})$, we see that
\begin{align}
    w'(t) = \frac{1}{k(t)} w^2(t) + p(t) \frac{x(\tau(t))}{x(t)} + q(t) \frac{x(\sigma(t))}{x(t)} - \frac{e(t)}{x(t)}.
\end{align}
Choose \( a_1 \) sufficiently large so that \( \tau(\tau(a_1)) \geq t_1 \). In view of (2.1)-(2.5), we see from (3.3) that

\[
 w'(t) \geq \frac{1}{k(t)} w^2(t) + p(t) \frac{x(\tau(t))}{x(t)}, \quad t \in [\tau(a_1), b_1],
\]

and

\[
 w'(t) \geq \frac{1}{k(t)} w^2(t) + q(t) \frac{x(\sigma(t))}{x(t)}, \quad t \in [c_1, \sigma(d_1)].
\]

Note that \((k(t)x'(t))' \leq 0\) on \([\tau(a_1), b_1]\). Therefore, by a slight modification of the arguments used in the proof of Theorem 1 in [12], see also the lines following (3.7), we arrive at

\[
 \frac{x(\tau(t))}{x(t)} \geq \frac{\tau(t) - \tau(a_1)}{t - \tau(a_1)}, \quad t \in (a_1, b_1].
\]

In a similar manner, we have

\[
 \frac{x(\sigma(t))}{x(t)} \geq \frac{\sigma(d_1) - \sigma(t)}{\sigma(d_1) - t}, \quad t \in [c_1, d_1).
\]

Indeed, if \( t \in [c_1, \sigma(d_1)] \), then by the mean-value theorem there exists a real number \( z \in (t, \sigma(d_1)) \) such that

\[
 x(\sigma(d_1)) - x(t) = \frac{k(z)x'(z)}{k(z)} (\sigma(d_1) - t),
\]

from which, on using

(i) the positivity of \( x(\sigma(d_1)) \),

(ii) the monotonicity of \( k(t) \), and

(iii) the nonincreasing nature of \( k(t)x'(t) \) on \([c_1, \sigma(d_1)]\),

we have

\[-x(t) \leq x'(t)(\sigma(d_1) - t), \quad t \in [c_1, \sigma(d_1)],
\]

and hence,

\[
 \frac{x'(t)}{x(t)} \geq \frac{1}{t - \sigma(d_1)}, \quad t \in [c_1, \sigma(d_1)].
\]

Integration of (3.8) over \([t, \sigma(t)]\) for \( t \in [c_1, d_1) \) results in (3.7).

In view of (3.6) and (3.7), it follows from (3.4) and (3.5) that

\[
 w'(t) \geq \frac{1}{k(t)} w^2(t) + p(t) \frac{\tau(t) - \tau(a_1)}{t - \tau(a_1)}, \quad t \in (a_1, b_1],
\]

and

\[
 w'(t) \geq \frac{1}{k(t)} w^2(t) + q(t) \frac{\sigma(d_1) - \sigma(t)}{\sigma(d_1) - t}, \quad t \in [c_1, d_1).
\]

Suppose that (3.1) holds. Noting that the functional \( A^b_n \) defined by (2.6) is monotone, we apply it to (3.9) with \( H = H_1, a = s_1 > a_1, \) and \( b = b_1 \). Then, by using (2.7), we get

\[
 A^b_n \left( p(t) \frac{\tau(t) - \tau(a_1)}{t - \tau(a_1)} - \frac{kH'^2}{H^2} \right) \leq - \left( H^2 w \right) (s_1) - A^b_n \left( \left[ \frac{w}{\sqrt{k} + \sqrt{kH'}} \right]^2 \right).
\]

Letting \( s_1 \to a_1 \) in the above inequality leads to

\[
 A^b_n \left( p(t) \frac{\tau(t) - \tau(a_1)}{t - \tau(a_1)} - \frac{kH'^2}{H^2} \right) \leq - A^b_n \left( \left[ \frac{w}{\sqrt{k} + \sqrt{kH'}} \right]^2 \right) < 0.
\]

(3.11)
Note that

\[ A^1_{\alpha_1} \left( \frac{w}{\sqrt{k}} + \sqrt{\frac{H'}{H}} \right)^2 = 0 \]

is possible only if \( Hw/\sqrt{k} + \sqrt{H'} = 0 \), and in this case it is easy to show that \( x(t) \) is a multiple of \( H(t) \), which however contradicts the positivity of \( x(t) \). Now, (3.11) contradicts (3.1).

If (3.2) holds, then we repeat the above process from inequality (3.10) with \( a = c_1 \) and \( b = s_2 < d_1 \) to obtain a contradiction with (3.2). Thus, the proof is complete when \( x(t) \) is eventually positive. The proof when \( x(t) \) is eventually negative can be accomplished similarly by working with intervals \([a_2, b_2] \) and \([c_2, d_2] \) instead of \([a_1, b_1] \) and \([c_1, d_1] \), respectively.

In what follows, we set

\[ P_{\alpha}(t) = \alpha (\alpha - 1)^{\frac{1}{\alpha} - 1} \left| p(t) \right|^{\frac{1}{\alpha}} \left| e(t) \right|^{1 - \frac{1}{\alpha}} \]  

and

\[ Q_{\beta}(t) = \beta (\beta - 1)^{\frac{1}{\beta} - 1} \left| q(t) \right|^{\frac{1}{\beta}} \left| e(t) \right|^{1 - \frac{1}{\beta}}. \]

We first consider the case \( \alpha = 1 \) and \( \beta > 1 \).

**THEOREM 3.2.** Suppose that \( \alpha = 1 \) and \( \beta > 1 \), and (2.1)–(2.5) hold. If there exist \( H_1 \in D(a_1, b_1) \) and \( H_2 \in D(c_1, d_1) \) such that either

\[ \int_{a_i}^{b_i} \left[ H_1^2(t) p(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - H_1^2(t) k(t) \right] dt > 0 \]

or

\[ \int_{c_i}^{d_i} \left[ H_2^2(t) Q_\beta(t) \frac{\sigma(d_i) - \sigma(t)}{\sigma(d_i) - t} - H_2^2(t) k(t) \right] dt > 0, \]

for \( i = 1, 2 \), then \((E_{1,\beta})\) is oscillatory.

**Proof.** Let \( x(t) \) be nonoscillatory solution of \((E_{1,\beta})\) so that \( x(t) > 0, x(\sigma(t)) > 0, x(\tau(t)) > 0 \) for all \( t \geq t_1 \) for some \( t_1 \geq 0 \). We may choose \( a_1 \) sufficiently large so that \( \tau(\tau(a_1)) \geq t_1 \). Clearly, if \( t \in [\tau(a_1), b_1] \), then

\[ q(t) x^\beta(\sigma(t)) - e(t) \geq \beta (\beta - 1)^{\frac{1}{\beta} - 1} q(t)^{\frac{1}{\beta}} \left| e(t) \right|^{1 - \frac{1}{\beta}} x^{\frac{1}{\beta} - 1} x(\sigma(t)). \]  

Inequality (3.16) follows from the fact that if \( A \geq 0, B \geq 0, \) and \( \gamma > 1 \) are real numbers, then the function,

\[ f(x) = A x^\gamma - \gamma (\gamma - 1)^{\frac{1}{\gamma} - 1} A^{\frac{1}{\gamma}} B^{1 - \frac{1}{\gamma}} x, \quad x \in [0, \infty), \]

takes on its absolute minimum value as \(-B\), see also [7,12].

Using (3.16) in \((E_{1,\beta})\), we easily obtain

\[ (k(t) x'(t))' + p(t) x(\tau(t)) + Q_\beta(t) x(\sigma(t)) \leq 0, \quad t \in [\tau(a_1), b_1]. \]

The remainder of the proof proceeds along the lines of that of Theorem 3.1.

In a similar manner, we obtain the following theorem.
Theorem 3.3. Suppose that $\alpha > 1$ and $\beta = 1$, and (2.1)-(2.5) hold. If there exist $H_1 \in D(a_i, b_i)$ and $H_2 \in D(c_i, d_i)$ such that either

$$\int_{a_i}^{b_i} \left[ H_1^2(t) P_\alpha(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - H_1^2(t) k(t) \right] dt \geq 0 \tag{3.17}$$

or

$$\int_{c_i}^{d_i} \left[ H_2^2(t) q(t) \frac{\sigma(d_i) - \sigma(t)}{\sigma(d_i) - t} - H_2^2(t) k(t) \right] dt \geq 0, \tag{3.18}$$

for $i = 1, 2$, then $(E_{\alpha,1})$ is oscillatory.

Finally, we consider the case $\alpha > 1$ and $\beta > 1$.

Theorem 3.4. Suppose that $\alpha > 1$ and $\beta > 1$, and (2.1)-(2.5) hold. If there exist $\lambda \in (0,1)$, $H_1 \in D(a_i, b_i)$ and $H_2 \in D(c_i, d_i)$ such that either

$$\int_{a_i}^{b_i} \left[ \lambda^{1-1/\alpha} H_1^2(t) P_\alpha(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - H_1^2(t) k(t) \right] dt \geq 0 \tag{3.19}$$

or

$$\int_{c_i}^{d_i} \left[ (1-\lambda)^{1-1/\beta} H_2^2(t) Q_\beta(t) \frac{\sigma(d_i) - \sigma(t)}{\sigma(d_i) - t} - H_2^2(t) k(t) \right] dt \geq 0, \tag{3.20}$$

for $i = 1, 2$, then $(E_{\alpha,\beta})$ is oscillatory.

Proof. Suppose that there exists a nonoscillatory solution $x(t)$ of $(E_{\alpha,\beta})$. We may assume that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$ for all $t \geq t_1$ for some $t_1 \geq 0$. Choose $a_1$ sufficiently large so that $\tau(\tau(a_1)) \geq t_1$. Let $t \in [\tau(a_1), b_1]$ and write

$$e(t) = \lambda e(t) + (1-\lambda) e(t).$$

Then, as in (3.16), we have

$$p(t) x^\alpha(\tau(t)) - \lambda e(t) \geq \lambda^{1-1/\alpha} P_\alpha(t) x(\tau(t)) \tag{3.21}$$

and

$$q(t) x^\beta(\sigma(t)) - (1-\lambda) e(t) \geq (1-\lambda)^{1-1/\beta} Q_\beta(t) x(\sigma(t)). \tag{3.22}$$

Using (3.21) and (3.22) in $(E_{\alpha,\beta})$, we get

$$(k(t) x'(t))' + \lambda^{1-1/\alpha} P_\alpha(t) x(\tau(t)) + (1-\lambda)^{1-1/\beta} Q_\beta(t) x(\sigma(t)) \leq 0.$$ 

The rest of the proof is similar to that of Theorem 3.1, hence, it is omitted.

It is easy to see that

$$\lim_{\alpha \to 1^+} P_\alpha(t) = p(t), \quad \lim_{\beta \to 1^+} Q_\beta(t) = q(t).$$

Therefore, we may consider Theorems 3.1–3.3 as special cases of Theorem 3.4 provided that $\alpha = 1$ and $\beta = 1$ are interpreted as $\alpha \to 1^+$ and $\beta \to 1^+$, respectively.

Now consider the delay equation,

$$(k(t) x'(t))' + p(t) |x(\tau(t))|^{\alpha-1} x(\tau(t)) = e(t), \quad \alpha \geq 1, \tag{3.23}$$

and the advanced equation,

$$(k(t) x'(t))' + q(t) |x(\sigma(t))|^{\beta-1} x(\sigma(t)) = e(t), \quad \beta \geq 1. \tag{3.24}$$

From Theorem 3.4, we immediately obtain the following oscillation criteria.
COROLLARY 3.1. Suppose that (2.1) and (2.3) hold. If there exist $H_i \in D(a_i, b_i)$ such that
\[
\int_{a_i}^{b_i} \left[ H_i^2(t) P_\alpha(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - H_i^2(t) k(t) \right] dt \geq 0,
\]
for $i = 1, 2$, then (3.23) is oscillatory.

COROLLARY 3.2. Suppose that (2.2) and (2.4) hold. If there exist $H_i \in D(c_i, d_i)$ such that
\[
\int_{c_i}^{d_i} \left[ H_i^2(t) Q_\beta(t) \frac{\sigma(d_i) - \sigma(t)}{\sigma(d_i) - t} - H_i^2(t) k(t) \right] dt \geq 0,
\]
for $i = 1, 2$, then (3.24) is oscillatory.

REMARK 3.1. Corollary 3.1 coincides with Theorem 1 of [9] when $r(t) = t$ and $\alpha = 1$, leads to an improvement of Theorem 1 in [7], and reduces to Theorem 2 of [12] when $k(t) \equiv 1$. Corollary 3.2 is new.

REMARK 3.2. We may replace the monotonicity condition on $k(t)$ over the half-line with the one only over the intervals $[\tau(a_1), \sigma(d_1)]$ and $[\tau(a_2), \sigma(d_2)]$. Besides the advanced term in the equation, this is also an improvement and a generalization of the results in [6,7,9,12].

THEOREM 3.5. Theorems 3.1–3.4 remain valid for equations of the form,
\[
(k(t)x'(t))' + p(t)f(x(\tau(t))) + q(t)g(x(\sigma(t))) = e(t),
\]
where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy
\[
x f(x) \geq |x|^{\alpha+1} \quad \text{and} \quad x g(x) \geq |x|^{\beta+1}, \quad \text{for all } x \in \mathbb{R}.
\]
PROOF. We sketch only the proof of Theorem 3.1 for (3.25). The others can be accomplished in a similar manner. Suppose that (3.25) has a nonoscillatory solution. If $x(t)$ is eventually positive, then by using (3.26) in (3.25) on intervals where $p$ and $q$ are nonnegative we see that
\[
(k(t)x'(t))' + p(t)x(\tau(t)) + q(t)x(\sigma(t)) \leq e(t).
\]
Define
\[
w(t) = -\frac{k(t)x'(t)}{x(t)}.
\]
In view of (3.27), we see that
\[
w'(t) \geq \frac{1}{k(t)}w^2(t) + p(t)\frac{x(\tau(t))}{x(t)} + q(t)\frac{x(\sigma(t))}{x(t)} \frac{e(t)}{x(t)}.
\]
If $x(t)$ is eventually negative, then inequality (3.27) is reversed but inequality (3.28) remains the same. The remainder of the proof proceeds exactly as in that of Theorem 3.1 after equality (3.3).

REMARK 3.3. If we take $k(t) \equiv 1$ and $q(t) \equiv 0$ in Theorem 3.5, then we recover Theorem 1 and Theorem 4 in [13], and hence, extend their study to more general equations of the form (3.25).

Finally, we pose some problems for future work. First, observe that in case $\alpha > 1$ and/or $\beta > 1$, we must have $e(t) \not\equiv 0$ on the intervals $[\tau(a_1), b_1]$ and $[c_i, \sigma(d_i)]$, since otherwise $P_\alpha(t) = Q_\beta(t) \equiv 0$ and hence, conditions (3.19) and (3.20) fail to hold. Therefore, it would be interesting to establish oscillation criteria for $(E_{\alpha, \beta})$ with $e(t) \equiv 0$ when $\alpha > 1$ or $\beta > 1$. Further, it is an open problem to find analogous oscillation criteria when $0 < \alpha < 1$ or $0 < \beta < 1$. 
4. EXAMPLES

In this section, we provide some examples to illustrate the relevance of the results. More precisely, we will see that equations of the form,

$$x''(t) + m_1 \sin t x(t - l_1) + m_2 \cos t x(t + l_2) = e(t), \quad m_1, m_2 \geq 0,$$

where \( l_1 \) and \( l_2 \) are positive real numbers, are oscillatory if either \( m_1 \) or \( m_2 \) is sufficiently large. In the following examples \( n \) belongs to the set of natural numbers. Evaluation of the integrals are carried out by using \textsc{Mathematica} software.

**Example 4.1.** Consider the delay equation,

$$x''(t) + m_1 \sin t x(t - \frac{\pi}{3}) = 0, \quad m_1 > 0.$$  

Let \( a_1 = (2n + 1/2)\pi, \quad b_1 = (2n + 1)\pi, \) and \( H_1(t) = (t - 2n\pi) \sin 2t. \) Since the equation is homogeneous we may take \( a_2 = a_1 \) and \( b_2 = b_1. \) Clearly, \( p(t) = m_1 \sin t \geq 0 \) on \( [(2n + 1/6)\pi, (2n + 1)\pi]. \) Since

$$\int_{a_1}^{b_1} [H_1^2(t) p(t)] \left( \frac{t - \tau(a_1)}{t - \tau(a_1)} - H_1^2(t) \right) dt$$

we may conclude from Theorem 3.1 that if

$$m_1 > \frac{375 \pi (9 + 26 \pi^2)}{16 (-1712 - 260 \pi + 375 \pi^2)} \approx 16.69,$$

then (4.2) is oscillatory.

**Example 4.2.** Consider the advanced equation,

$$x''(t) + m_2 \sin t x(t + \frac{\pi}{4}) = 0, \quad m_2 > 0.$$  

Let \( c_1 = 2n\pi, \quad d_1 = (2n + 1/2)\pi, \quad c_2 = c_1, \quad d_2 = d_1, \) and \( H_2(t) = (2n + 3/4)\pi - t) \sin 2t. \) We see that \( q(t) = m_2 \sin t \geq 0 \) on \( [2n\pi, (2n + 3/4)\pi] \) and

$$\int_{c_1}^{d_1} [H_2^2(t) q(t)] \left( \frac{\sigma(d_1) - \sigma(t)}{\sigma(d_1) - t} - H_2^2(t) \right) dt$$

By Theorem 3.1, (4.3) is oscillatory if

$$m_2 > \frac{1125 \pi (6 + 13 \pi^2)}{16 (-3424 - 390 \pi + 675 \pi^2)} \approx 14.74.$$  

**Example 4.3.** Consider the mixed type equation,

$$x''(t) + m_1 \sin t x(t - \frac{\pi}{3}) + m_2 \sin t x(t + \frac{2\pi}{3}) = 0, \quad m_1, m_2 \geq 0.$$  

Let \( a_1 = (2n + 1/3)\pi, \quad b_1 = (2n + 1)\pi, \quad c_1 = 2n\pi, \quad d_1 = (2n + 1/3)\pi, \quad a_2 = a_1, \quad b_2 = b_1, \quad c_2 = c_1, \quad d_2 = d_1, \) \( H_1(t) = (t - 2n\pi) \sin 3t, \) and \( H_2(t) = ((2n + 1)\pi - t) \sin 3t. \)
By Theorem 3.1, (4.4) is oscillatory if either

$$m_1 \geq \frac{-42875 \pi (3 + 26 \pi^2)}{54 (21492 + 1155 \sqrt{3} \pi - 4900 \pi^2)} \approx 31.46$$

or

$$m_2 \geq \frac{-42875 \pi (3 + 38 \pi^2)}{216 (3582 + 1155 \sqrt{3} \pi - 1225 \pi^2)} \approx 106.04.$$

**EXAMPLE 4.4.** Consider the mixed type equation,

$$x''(t) + m_1 \sin t x(t - \frac{\pi}{6}) + m_2 \cos t x(t + \frac{\pi}{3}) = 0, \quad m_1, m_2 \geq 0. \quad (4.5)$$

Let \(a_1 = (2n + 1/6)\pi, b_1 = (2n + 1/2)\pi, c_1 = 2n\pi, d_1 = (2n + 1/6)\pi, a_2 = a_1, b_2 = b_1, c_2 = c_1,\)

\(d_2 = d_1, H_1(t) = (t - 2n\pi) \sin 6t,\) and \(H_2(t) = ((2n + 1/2)\pi - t) \sin 6t.\)

By Theorem 3.1, (4.5) is oscillatory if either

$$m_1 \geq \frac{-292407 \pi (3 + 26 \pi^2)}{1944 (27080 \sqrt{3} - 20163 \pi)} \approx 74.63$$

or

$$m_2 \geq \frac{-292407 \pi (3 + 38 \pi^2)}{2592 (40620 - 26884 \pi + 6721 \sqrt{3})} \approx 184.39.$$

**EXAMPLE 4.5.** Consider the mixed type equation

$$x''(t) + m_1 \sin t x(t - \frac{\pi}{12}) + m_2 \cos t x(t + \frac{\pi}{6}) = \cos 2t, \quad m_1, m_2 \geq 0. \quad (4.6)$$

Let \(a_1 = (2n + 4/12)\pi, b_1 = (2n + 1/2)\pi, a_2 = (2n + 1/12)\pi, b_2 = (2n + 3/12)\pi, c_1 = (2n + 3/12)\pi,\)

\(d_1 = (2n + 4/12)\pi, c_2 = 2n\pi, d_2 = (2n + 1/12)\pi, H_1(t) = \sin 12t,\) and \(H_2(t) = \sin 12t.\) Noting that

$$\int_{a_1}^{b_1} \left[ H_1^2(t) p(t) \frac{\tau(t) - \tau(a_1)}{t - \tau(a_1)} - H_1^2(t) \right] dt \geq \int_{c_1}^{d_1} \left[ H_1^2(t) p(t) \frac{\tau(t) - \tau(a_1)}{b_1 - \tau(a_1)} - H_1^2(t) \right] dt$$

and

$$\int_{c_1}^{d_1} \left[ H_2^2(t) q(t) \frac{\sigma(d_1) - \sigma(t)}{\sigma(d_1) - t} - H_2^2(t) \right] dt \geq \int_{c_1}^{d_1} \left[ H_2^2(t) q(t) \frac{\sigma(d_1) - \sigma(t)}{\sigma(d_1) - c_1} - H_2^2(t) \right] dt,$$

we see that all conditions of Theorem 3.1 is fulfilled if either \(m_1 \geq 773\) or \(m_2 \geq 1390.\) In this case, every solution of (4.6) is oscillatory.

**EXAMPLE 4.6.** Consider the mixed type super linear equation,

$$x''(t) + m_1 \sin^3 t x^3(t - \frac{\pi}{12}) + m_2 \cos^5 t x^5(t + \frac{\pi}{6}) = \cos^{15} 2t, \quad (4.7)$$

where \(m_1, m_2 \geq 0.\) With the choices of the previous example one can easily show that if either \(m_1\) or \(m_2\) is sufficiently large, then (4.7) is oscillatory. If the forcing term \(\cos^{15} 2t\) is replaced by \(r_0 \cos^{15} 2t\) and if the amplitude \(r_0\) is sufficiently large, then one can easily see that \(m_1\) and \(m_2\) can be arbitrarily small. For instance, if \(r_0 = 6^{15}\) then it suffices to take \(m_1 \geq 2.57 \times 10^{-7}\) or \(m_2 \geq 0.0057,\) where we have taken \(\lambda = 9/10.\)

Of course, smaller values of \(m_1\) and/or \(m_2\) could be found by making use of different choices for the intervals \([a_i, b_i]\) and \([c_i, d_i]\), and the functions \(H_1\) and \(H_2\) in each of the above examples.
REFERENCES