Option valuation model with adaptive fuzzy numbers

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Abstract

In this paper, we consider moment properties for a class of quadratic adaptive fuzzy numbers defined in Dubois and Prade [D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980]. The corresponding moments of Trapezoidal Fuzzy Numbers (Tr.F.N's) and Triangular Fuzzy Numbers (T.F.N's) turn out to be special cases of the adaptive fuzzy number [S. Bodjanova, Median value and median interval of a fuzzy number, Information Sciences 172 (2005) 73–89]. A numerical example is presented based on the Black–Scholes option pricing formula with quadratic adaptive fuzzy numbers for the characteristics such as volatility parameter, interest rate and stock price. Our approach hinges on a characterization of imprecision by means of fuzzy set theory.

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1. Introduction

Most stochastic models involve uncertainty arising mainly from lack of knowledge or from inherent vagueness. Traditionally those stochastic models are solved using probability theory and fuzzy set theory. There exist many practical situations where both types of uncertainties are present. For example, if the price of an option depends upon the nature of the volatility which changes randomly, then the volatility of the stock price movement which is estimated from the sample data is a random variable as well as a fuzzy number. Recently there has been a growing interest in using fuzzy numbers to deal with impreciseness (see Appadoo et al. [3] and Thavaneswaran et al. [4] for more details). Viewing the fuzzy numbers as random sets Dubois and Prade [5] introduced the mean value as a closed interval bounded by the expectations calculated from its upper and lower distribution functions. In this section we provide a literature survey of some of the research done on fuzzy option pricing. Many authors have tried to deal fuzziness along with randomness in option pricing models. For example, recently, Cherubini [6] determined the price of a corporate debt contract and provided a fuzzified version of the Black and Scholes model by means of a special class of fuzzy measures. On the other hand, Ghaziri et al. [7] introduced artificial intelligence approach to price the
options, using neural networks and fuzzy logic. They compare the result of artificial intelligence approach to that of Black–Scholes model, using stock indexes. Since the Black–Scholes option pricing formula is only approximate, which leads to considerable errors, Trenev [8] obtained a refined formula for pricing options. Due to the fluctuation of the financial markets from time to time, some of the input parameters in the Black–Scholes formula cannot always be expected in the precise sense. As a result, Wu [9] applied fuzzy approach to the Black–Scholes formula. Zmeskal [10] applied Black–Scholes methodology of appraising equity as a European call option. He used the input data in a form of fuzzy numbers in his approach to option pricing. Carlson and Fuller [11] use possibility theory to fuzzy real option valuation. Applications of fuzzy sets theory to volatility models have been studied by Thavaneswaran et al. [12] and Thiagarajah and Thavaneswaran [13].

Fuzzy set theory has attracted considerable interest in other fields as well. As an illustration, in an attempt to introduce an element of uncertainty or vagueness into conventional actuarial models, Ostaszewski [14] proposed fuzzy set theory as an alternative approach to stochastic methods for modeling uncertainty. Also, in [14] several applications of fuzzy set theory to Actuarial Science are discussed, for example, fuzzy net single premium for term insurance.

The famous formula of Black and Scholes [15] for the fair prices of European options follows from several assumptions, which are discussed in Merton [16]. Asset prices \( S_t \) are assumed to follow geometric Brownian motion and are represented by the equation

\[
dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}
\]

where the process \( W_t \) is a standard Brownian motion, \( \mu \) is the drift and \( \sigma \) is the volatility of the underlying stock. Generally, a call (put) option is the right to buy (sell) a particular asset for a specified amount at the strike price \( K \) at a specified time in the future with the expiration time \( T \). If the option is of such a type that it can be exercised only on the expiration date itself, then it is called a European option. Let \( S_T \) be the price of the underlying asset at expiration time \( T \). Then the payoff, \( g \), of a European style call option at time \( T \) is given by

\[
g(S_T) = \max(S_T - K, 0) = (S_T - K)^+. \tag{1.2}
\]

This means that the call option is exercised if \( S_T > K \) and is abandoned otherwise. The above mentioned call and put options are sometimes called plain vanilla or standard options. Let \( r \) be the risk-free interest rate. Then a probability measure \( Q \) is called an equivalent martingale measure to the probability measure \( P \) for the discounted price process \( \tilde{S}_t = e^{-rt}S_t \) if

\[
E_Q[\tilde{S}_t | F_s] = \tilde{S}_s \tag{1.3}
\]

for each \( s \leq t \leq T \) and \( Q \sim P \), where \( F_t \) is the history of the process up to time \( t \). This is the discounted price process \( \tilde{S}_t \) which is a martingale under the probability measure \( Q \). According to the Fundamental Theorem of Asset Pricing, an arbitrage-free price \( C_t \) of an option at time \( t \) is given by the conditional expectation of the discounted payoff under an equivalent martingale measure \( Q \),

\[
C_t = E_Q[e^{-r(T-t)}g(S_T)|F_t]. \tag{1.4}
\]

Together with this equivalent martingale approach one needs a so-called equivalent portfolio (a combination of other traded assets). Then the price of the option has to coincide with the price of the corresponding equivalent portfolio. For a general overview on financial derivatives from a mathematical and an economic point of view, we refer readers to Hull [17]. Following the results of Black and Scholes [15], we take a geometric Brownian motion as a stochastic process for modeling the stock price. This model is based on the assumption that the log-returns

\[
X_t = \log S_t - \log S_{t-1} \tag{1.5}
\]

are normally distributed. Now Eq. (1.1) becomes

\[
S_t = S_0e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t} \tag{1.6}
\]

with \( 0 \leq t \leq T \). Using Itos lemma the model can equivalently be described as

\[
dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1.7}
\]
For this model, there exists a unique martingale measure \( Q \) which is given by Girsanov’s theorem

\[
\frac{dQ}{dP} = \exp \left( \frac{r - \mu}{\sigma} W_T - \frac{(r - \mu)^2}{2\sigma^2} T \right).
\]

(1.8)

Some calculations yield the Black–Scholes formula as,

\[
C_{BS} = S_0 \Phi \left( \frac{\log \left( \frac{S_0}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} \right) - K e^{-r \tau} \Phi \left( \frac{\log \left( \frac{S_0}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}} \right),
\]

(1.9)

where \( \Phi \) denotes the cumulative distribution function of a standard normal variable, and \( \tau = T - t \) denotes the time to expiration. In the literature two different ways of calculating volatility have been discussed. The first method is the empirical estimation from the historical data. The second is to calculate the implied volatility by equating the theoretical call price from the Black–Scholes formula with the market price.

In the literature, nonconstant volatility had been modeled by GARCH processes and for calculation purpose, volatility \( \sigma \), in the Black–Scholes formula had been replaced by \( \hat{\sigma} \).

In this paper we use adaptive fuzzy numbers to model the uncertainty of \( \sigma \) in the Black–Scholes model. The plan of the paper is as follows. In Section 2, we summarize some basic results of fuzzy set theory. In Section 3, we derive the moments properties for a class of quadratic adaptive fuzzy numbers. Finally, in Section 4 the valuation of European call option with quadratic adaptive fuzzy numbers for some characteristics such as volatility parameter, interest rate and stock price is illustrated through a numerical example.

2. Basic results of fuzzy set theory

We introduce certain terminology, notations and definitions that will be used in the sequel. Let a classical set \( X \) be a subset of a universal set \( U \) and \( R \subset X \) be the set of real numbers. Sometimes, in the literature \( \alpha \)-cut is also referred to as \( \gamma \)-level set, in which case we use \( \gamma \) instead of \( \alpha \). Most of these related definitions and properties can be found in the following references [18–21].

Definition 2.1. Fuzzy set \( A \) in \( X \) is a set of ordered pairs \( A = \{ x, \mu(x) : x \in X \} \), where \( \mu \) maps \( x \in X \) on the real interval \([0, 1]\) and is known as the membership function.

Definition 2.2. Let \( A \) be a fuzzy set in \( X \). Then the support of \( A \), denoted by \( S(A) \), is a crisp set given by

\[
S(A) = \{ x \in X : \mu_A(x) > 0 \}.
\]

Definition 2.3. Let \( A \) be a fuzzy set in \( X \). The height \( h(A) \) of \( A \) is defined as

\[
h(A) = \sup_{x \in X} \mu_A(x).
\]

If \( h(A) = 1 \) then the fuzzy set \( A \) is called a normal fuzzy set.

Definition 2.4. LR-Fuzzy Number: A fuzzy number \( \tilde{M} \) is of the LR-type if there exist shape functions \( L \) and \( R \) and four parameters \((m, \overline{m}) \in \bigcup\{-\infty, +\infty\}, \alpha, \beta \) and the membership function of \( \tilde{M} \)

\[
\mu_{\tilde{M}}(x) = \begin{cases} 
L \left( \frac{m - x}{\alpha} \right) & x \leq m, \alpha > 0 \\
1 & m \leq x \leq \overline{m}, \alpha > 0 \\
R \left( \frac{x - \overline{m}}{\beta} \right) & x \geq \overline{m}, \beta > 0.
\end{cases}
\]

The LR-fuzzy number is then denoted by \( \tilde{M} = (m, \overline{m}, \alpha, \beta)_{LR} \), where \( \alpha \) is the left spread and \( \beta \) is the right spread. This definition is very general and allows quantification of quite different types of information. If \( \tilde{M} \) is supposed to be a real crisp number for \( m \in R \), then \( \tilde{M} = (m, m, 0, 0)_{LR}, \forall L \) and \( \forall R \). If \( \tilde{M} \) is a crisp interval, then
\( \tilde{M} = (a, b, 0, 0)_{LR}, \forall L \text{ and } \forall R \text{ and } a \neq b. \) If \( \tilde{M} \) is a trapezoidal fuzzy number, then \( L(x) = R(x) = \text{Max}(0, 1 - x) \) is implied, Zimmermann [19]. If \( L \) and \( R \) are strictly decreasing functions then we can easily compute the \( \gamma \)-level sets of \( M. \) \([m, \bar{m}]\) is the peak of \( M \) and \( m \) and \( \bar{m} \) are the lower and upper modal values. Also, \( L, R : [0, 1] \rightarrow [0, 1] \) with \( L(0) = R(0) = 1 \) and \( L(1) = R(1) = 0 \) are nonincreasing and continuous mappings [1].

Now, we consider the nonlinear fuzzy number given in Bodjanova [2] which we are going to use in the fuzzy option pricing model in Section 4.

**Definition 2.5.** Let \( \mathcal{R} \) be the set of all real numbers. A fuzzy number \( A(x), x \in \mathcal{R} \) is of the form

\[
A(x) = \begin{cases} 
g(x) & \text{when } x \in [a, b) \\
1 & \text{when } x \in [b, c] \\
h(x) & \text{when } x \in [c, d) \\
0 & \text{otherwise} \end{cases}
\]  

(2.1)

where \( g \) is a real valued, increasing and right continuous function, \( h \) is a real valued, decreasing and left continuous function, and \( a, b, c, d \) are real numbers such that \( a < b < c < d \). Notice that (2.1) is an LR fuzzy number with strictly monotone shape functions as described in Bodjanova [2]. (For more details on LR fuzzy numbers the reader is referred to Dubois and Prade [1] and Zimmermann [19]). A fuzzy number \( A \) with shape functions \( g \) and \( h \) defined by

\[
g(x) = \left( \frac{x - a}{b - a} \right)^n \]  

(2.2)

and

\[
h(x) = \left( \frac{d - x}{d - c} \right)^n \]  

(2.3)

respectively, where \( n > 0 \), will be denoted by \( A = [a, b, c, d]_n \). If \( n = 1 \), we simply write \( A = [a, b, c, d] \), which is known as a trapezoidal fuzzy number. If \( n \neq 1 \), a fuzzy number \( A^* = [a, b, c, d]_n \) is a modification of a trapezoidal fuzzy number \( A = [a, b, c, d] \). If \( n > 1 \), then \( A^* \) is a concentration of \( A \) as in Fig. 1. Concentration of \( A \) by \( n = 2 \) is often interpreted as the very linguistic hedge. If \( 0 < n < 1 \), then \( A^* \) is a dilation of \( A \). Dilation of \( A \) by \( n = 0.5 \) is often interpreted as the linguistic hedge more or less. More details on linguistic hedges can be found in [1,5,6, 22]. Each fuzzy number \( A \) described by (2.2) and (2.3) has the following \( \alpha \)-level sets (\( \alpha \)-cuts), \( A(\alpha) = [a(\alpha), b(\alpha)], a_\alpha, b_\alpha \in \mathcal{R}, \alpha \in [0, 1] \) and

\[ A_\alpha = [g^{-1}(\alpha), h^{-1}(\alpha)], \quad A_1 = [b, c], \quad A_0 = [a, d]. \]

If \( A = [a, b, c, d]_n \) then, for all \( \alpha \in [0, 1] \),

\[
A(\alpha) = [a + \alpha^n (b - a), d - \alpha^n (d - c)].
\]  

(2.4)

In the next section, in the line of Fuller and Majlender [23] we discuss weighted possibilistic moment of the adaptive fuzzy number.
3. Weighted possibilistic mean, variance and covariance

Viewing the fuzzy numbers as random sets, Dubois and Prade [5] defined their interval-valued expectation. However, Carlson and Fuller [24] defined a possibilistic interval-valued mean value of fuzzy numbers by viewing them as possibility distributions. Furthermore, weighted possibilistic mean, variance and weighted interval-valued possibilistic mean value of fuzzy numbers are all introduced in Fuller and Majlender [23].

Definition 3.1. Let $A \in F$ be a fuzzy number with $A(\alpha) = [a_1(\alpha), a_2(\alpha)]$, $\alpha \in [0, 1]$. A function $f : [0, 1] \rightarrow \mathbb{R}$ is said to be a weight function if $f$ is nonnegative, monotone increasing and satisfies the following normalization condition.

$$\int_0^1 f(\alpha) d\alpha = 1. \quad (3.1)$$

Definition 3.2. Let $f$ be a weight function and $A \in F$ be a fuzzy number with $A(\alpha) = [a_1(\alpha), a_2(\alpha)]$, $\alpha \in [0, 1]$. Then the $f$-weighted interval-valued possibilistic mean of $A$ is defined as follows:

$$E_f(A) = [E_f^-(A), E_f^+(A)] \quad (3.2)$$

where

$$E_f^-(A) = \int_0^1 a_1(\alpha) f(\alpha) d\alpha \quad (3.3)$$
$$E_f^+(A) = \int_0^1 a_2(\alpha) f(\alpha) d\alpha. \quad (3.4)$$

Note that expression (3.3) is called the $f$-weighted lower possibilistic mean value of fuzzy number $A$ and expression (3.4) is called the $f$-weighted upper possibilistic mean value of fuzzy number $A$.

Definition 3.3. The $f$-weighted possibilistic mean value of fuzzy number $A$ is defined as

$$\overline{E}_f(A) = \frac{E_f^-(A) + E_f^+(A)}{2} = \frac{\int_0^1 (a_1(\alpha) + a_2(\alpha)) f(\alpha) d\alpha}{2}. \quad (3.5)$$

For example, if $f(\alpha) = 2\alpha$, $\alpha \in [0, 1]$ then

$$\overline{E}_f(A) = \int_0^1 \frac{a_1(\alpha) + a_2(\alpha)}{2} 2\alpha d\alpha = \int_0^1 [a_1(\alpha) + a_2(\alpha)] \alpha d\alpha = E(A). \quad (3.6)$$

This $f$-weighted possibilistic mean value defined by (3.5) can be viewed as a generalization of possibilistic mean value introduced by Carlson and Fuller [24]. By introducing different weighting functions we can give different importance to the $\alpha$-cuts of fuzzy numbers.

Definition 3.4. Let $A$ and $B$ be two fuzzy numbers and $f$ be a weight function. We define the $f$-weighted possibilistic variance of $A$ and the $f$-weighted possibilistic covariance of $A$ and $B$ as follows:

$$\text{Var}_f(A) = \int_0^1 \left( \frac{a_2(\alpha) - a_1(\alpha)}{2} \right)^2 f(\alpha) d\alpha \quad (3.7)$$
$$\text{Cov}_f(A, B) = \int_0^1 \left( \frac{a_2(\alpha) - a_1(\alpha)}{2} \right) \left( \frac{b_2(\alpha) - b_1(\alpha)}{2} \right) f(\alpha) d\alpha. \quad (3.8)$$

From (3.7) and (3.8), we observe that the $f$-weighted possibilistic variance and covariance can be considered as a generalization of possibilistic variance and covariance.

In the following example, we compute weighted possibilistic variance, covariance of a class of adaptive fuzzy number and discuss some special cases.
Example 1. Let $A$ and $B$ be two quadratic adaptive fuzzy numbers. Let $f(\alpha) = (\omega + 1)\alpha^\omega$ be a weighted function, then the $f$-weighted possibilistic covariance between $A$ and $B$ is computed as follows:

$$\text{Cov}_f(A, B) = \int_0^1 \left(\frac{a_2(\alpha) - a_1(\alpha)}{2}\right) \left(\frac{b_2(\alpha) - b_1(\alpha)}{2}\right) f(\alpha)d\alpha. \tag{3.9}$$

Using expression (3.9) we find an expression for the weighted possibilistic covariance between two adaptive fuzzy numbers as follows

$$= \frac{1}{4} \int_0^1 [a_4(b_4 - b_1) + a_1(b_1 - b_4)](\omega + 1)\alpha^\omega d\alpha$$

$$+ \frac{1}{4} \int_0^1 [(a_4 - a_1)(b_1 - b_4 + b_3 - b_2) + (a_1 - a_4 + a_3 - a_2)(b_4 - b_1)]\alpha \frac{1}{(\omega + 1)\alpha^\omega} d\alpha$$

$$+ \frac{1}{4} \int_0^1 [(a_1 - a_4 + a_3 - a_2)(b_1 - b_4 + b_3 - b_2)](\omega + 1)\alpha \frac{2}{(\omega + 1)\alpha^\omega} d\alpha$$

$$= \left[\frac{a_4(b_4 - b_1) + a_1(b_1 - b_4)}{4}\right]$$

$$+ \left[\frac{[(a_4 - a_1)(b_1 - b_4 + b_3 - b_2) + (a_1 - a_4 + a_3 - a_2)(b_4 - b_1)]n(\omega + 1)}{4(n(\omega + 1) + 1)}\right]$$

$$+ \left[\frac{[(a_1 - a_4 + a_3 - a_2)(b_1 - b_4 + b_3 - b_2)]n(\omega + 1)}{4(n(\omega + 1) + 2)}\right]. \tag{3.10}$$

When $b_1 = a_1, b_2 = a_2, b_3 = a_3, b_4 = a_4$ in (3.10), the $f$-weighted possibilistic variance of $A$ is given by

$$\text{Var}_f(A) = \left[\frac{(a_1 - a_4)^2}{4}\right] + \left[\frac{[2(a_4 - a_1)(a_1 - a_4 + a_3 - a_2)]n(\omega + 1)}{4(n(\omega + 1) + 1)}\right]$$

$$+ \left[\frac{[(a_1 - a_4 + a_3 - a_2)^2]n(\omega + 1)}{4(n(\omega + 1) + 2)}\right]. \tag{3.11}$$

When $\omega = 1$ in (3.10) the possibilistic covariance of fuzzy numbers $A$ and $B$ reduces to

$$\text{Cov}(A, B) = \left[\frac{a_4(b_4 - b_1) + a_1(b_1 - b_4)}{4}\right]$$

$$+ \left[\frac{n[(a_4 - a_1)(b_1 - b_4 + b_3 - b_2) + (a_1 - a_4 + a_3 - a_2)(b_4 - b_1)]}{2(n + 1)}\right]$$

$$+ \left[\frac{n[(a_1 - a_4 + a_3 - a_2)(b_1 - b_4 + b_3 - b_2)]}{2(2n + 2)}\right]. \tag{3.12}$$

Similarly, when $\omega = 1$ in (3.11) the possibilistic variance of $A$ reduces to

$$\text{Var}(A) = \left[\frac{(a_1 - a_4)^2}{4}\right] + \left[\frac{n[(a_4 - a_1)(a_1 - a_4 + a_3 - a_2)]}{2(n + 1)}\right] + \left[\frac{n[(a_1 - a_4 + a_3 - a_2)^2]}{2(2n + 2)}\right]. \tag{3.13}$$

3.1. Possibilistic mean, variance and covariance of fuzzy numbers

The crisp possibilistic mean and the crisp possibilistic variance are defined as follows:

$$E(A) = \int_0^1 (a_1(\alpha) + a_2(\alpha))\alpha d\alpha \tag{3.14}$$
we obtain the possibilistic variance of a trapezoidal fuzzy number

\[
\text{Var}(A) = \int_0^1 \text{Pos}[A \leq a_1] \left( \frac{a_1(\alpha) + a_2(\alpha)}{2} - a_1(\alpha) \right)^2 d\alpha \\
+ \int_0^1 \text{Pos}[A \geq a_2] \left( \frac{a_1(\alpha) + a_2(\alpha)}{2} - a_2(\alpha) \right)^2 d\alpha
\]

\[
= \int_0^1 \frac{1}{2} (a_2(\alpha) - a_1(\alpha))^2 \alpha d\alpha.
\]  

(3.15)

Let \( A, B \in \mathcal{F} \) be fuzzy numbers with \( A(\alpha) = [a_1(\alpha), a_2(\alpha)] \) and \( B(\alpha) = [b_1(\alpha), b_2(\alpha)], \alpha \in [0, 1] \). Goetschel et al. [25] introduced a method for ranking fuzzy numbers as

\[
A \leq B \iff \int_0^1 (a_1(\alpha) + a_2(\alpha))\alpha d\alpha \leq \int_0^1 (b_1(\alpha) + b_2(\alpha))\alpha d\alpha.
\]  

(3.16)

As pointed out by Goetschel et al. [25] the definition given in (3.16) for ordering fuzzy numbers was motivated by the desire to give less importance to the lower levels of fuzzy numbers. In the sequel, we will refer to the \( \alpha \)-cut for \( A \) and \( B \) as follows:

\[
A(\alpha) = [a_1(\alpha), a_2(\alpha)] = [a_1 + \alpha \frac{1}{2} (a_2 - a_1), a_4 - \alpha \frac{1}{2} (a_4 - a_3)]
\]  

(3.17)

\[
B(\alpha) = [b_1(\alpha), b_2(\alpha)] = [b_1 + \alpha \frac{1}{2} (b_2 - b_1), b_4 - \alpha \frac{1}{2} (b_4 - b_3)].
\]  

(3.18)

If \( A \) is a trapezoidal fuzzy number, then the possibilistic expected value of \( A \) is given by

\[
E(A) = \int_0^1 \alpha(a_1(\alpha) + a_2(\alpha))d\alpha = \int_0^1 \alpha(a_1 + \alpha \frac{1}{2} (a_2 - a_1) + a_4 - \alpha \frac{1}{2} (a_4 - a_3))d\alpha
\]

\[
= \left[ \frac{(a_1 + a_4)}{2} \right] + \left[ \frac{n(a_2 - a_1 - a_4 + a_3)}{1 + 2n} \right].
\]  

(3.19)

Thus, we have a formula for the possibilistic expectation of an adaptive fuzzy number for different values of \( n \). As an illustration, if we set \( n = 1 \) in (3.19), the possibilistic mean and the possibilistic variance of a trapezoidal fuzzy number that we obtain \( A \) are as follows:

\[
E(A) = \left[ \frac{(a_1 + a_4)}{2} \right] + \left[ \frac{(a_2 - a_1 - a_4 + a_3)}{3} \right]
\]  

(3.20)

and

\[
\text{Var}(A) = \frac{1}{2} \int_0^1 \alpha(a_2(\alpha) - a_1(\alpha))^2 d\alpha
\]

\[
= \frac{(a_4 - a_1)^2}{4} + \frac{n(a_4 - a_1)(a_3 - a_4 - a_2 + a_1)}{1 + 2n} + \frac{n(a_3 - a_4 - a_2 + a_1)^2}{2(2 + 2n)}.
\]  

(3.21)

Similarly, if \( n = 1 \) in Eq. (3.21) we obtain the possibilistic variance of a trapezoidal fuzzy number \( A \) as

\[
\text{Var}(A) = \left[ \frac{(a_3 - a_1)^2}{4} \right] + \left[ \frac{(a_3 - a_1)(a_1 - a_3)}{3} \right] + \left[ \frac{(a_1 - a_3)^2}{8} \right].
\]  

(3.22)

Now, the possibilistic covariance of two adaptive fuzzy numbers can be given as

\[
\text{Cov}(A, B) = \frac{1}{2} \int_0^1 \alpha [(a_2(\alpha) - a_1(\alpha))(b_2(\alpha) - b_1(\alpha))] d\alpha
\]

\[
= \left( \frac{a_4b_4 - a_1b_4 + a_1b_1 + a_4b_1}{4} \right) + \left( \frac{n((a_3b_4 - a_4b_4 + a_4b_3) + (a_1b_2 - 2a_1a_1 + a_2b_1))}{2(1 + 2n)} \right)
\]

\[
- \left( \frac{n((a_3b_1 + a_4b_2 - 2a_4b_1) + (a_3b_4 - 2a_1b_4 + a_1b_3))}{2(1 + 2n)} \right).
\]
then Cov $S$, interest rate, the fuzzy stock price and the fuzzy volatility by possibilistic mean value in the fuzzy Black–Scholes formula for a dividend paying stock with exercise price $K$.

If we set $n = 1$ in (3.23) then Cov$(A, B)$ turns out to be the covariance between two Tr.F.N’s.

$$\text{Cov}(A, B) = \left( \frac{4}{a_4 b_4 - a_4 b_1 - a_1 b_4 + a_1 b_1} \right) + \left( \frac{6}{a_3 b_4 - 2 a_4 b_4 + a_4 b_3 + (a_2 b_2 - a_2 b_1 + a_1 b_1)} \right)$$

$$+ \left( \frac{6}{a_3 b_4 - a_3 b_4 + a_3 b_3 + (a_2 b_2 - a_2 b_1 + a_1 b_1)} \right)$$

$$+ \left( \frac{8}{a_4 b_4 - a_4 b_2 + a_3 b_2 - a_3 b_1 + (a_2 b_3 - a_2 b_4 + a_1 b_4 - a_1 b_3)} \right).$$

(3.24)

If we set $n = 1, a_4 = a_3, a_3 = a_2, b_4 = b_3$ and $b_3 = b_2$ in expression (3.23), we obtain the possibilistic covariance between two Tr.F.N’s.

4. Fuzzy option valuation model

The expiry date and exercise price are always known and are nonfuzzy. In this paper we model the uncertainty of the characteristics such as interest rate, volatility, and stock price using adaptive fuzzy numbers. We replace the fuzzy interest rate, the fuzzy stock price and the fuzzy volatility by possibilistic mean value in the fuzzy Black–Scholes formula. The initial stock price cannot be characterized by a single number. Thus, we assume that $S$ is an adaptive fuzzy number of the form $\tilde{S}_0 = (S_1, S_2, S_3, S_4)_n$. In a similar manner $\tilde{r} = (r_1, r_2, r_3, r_4)_n$ for the interest rate and of the form $\tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)_n$ for the volatility can also be modeled. We consider the following Black–Scholes formula for a dividend paying stock with exercise price $K$.

$$\tilde{FCOV} = \tilde{S}_0 e^{-\tilde{\delta} \tau} N(d_1) - K e^{-\tilde{r} \tau} N(d_2),$$

(4.1)

where

$$d_1 = \ln \left( \frac{E(\tilde{S}_0)}{K} \right) + \left( E(\tilde{r}) - \tilde{\delta} + \frac{E(\tilde{\sigma})^2}{2} \right) \tau,$$

$$d_2 = d_1 - E(\tilde{\sigma}) \sqrt{\tau},$$

(4.2, 4.3)

$E(\tilde{S}_0)$ is the possibilistic mean value of the stock price, $E(\tilde{\sigma})$ is the possibilistic mean value of expected volatility. We assume that the stock pays dividends continuously at a known rate $\delta$. We have a specific context for the use of the option valuation method with fuzzy numbers, which is the main motivation for our approach. Thus, the use of assumptions on purely stochastic phenomena is not well-founded.

The $\alpha$-cuts of the fuzzy call option value FCOV are as follows:

$$\tilde{FCOV}(\alpha) = [\text{FCOV}_1(\alpha), \text{FCOV}_2(\alpha)],$$

(4.4)

where

$$\text{FCOV}_1(\alpha) = (\kappa_1 e^{-\delta \tau} N(d_1) S_1 - K N(d_2) e^{-r_1 \tau}) + (K N(d_2) (e^{-r_2 \tau} - e^{-r_1 \tau}) + (S_2 - S_1) e^{-\delta \tau} N(d_1)) \alpha^{1/2},$$

(4.5)

$$\text{FCOV}_2(\alpha) = (\kappa_1 e^{-\delta \tau} N(d_1) S_4 - K N(d_2) e^{-r_4 \tau}) - ((S_4 - S_3) e^{-\delta \tau} N(d_1) + K N(d_2) (e^{-r_4 \tau} - e^{-r_4 \tau})) \alpha^{1/2}.$$
The membership function of the fuzzy call option is given below

\[
\mu \tilde{C} = \begin{cases} 
C - \left( e^{-\delta T} N(d_1)S_1 - KN(d_2)e^{-\rho_1 T} \right) & \forall C \in \left( S_1 e^{-\delta T} N(d_1) - KN(d_2)e^{-\rho_1 T}, S_2 e^{-\delta T} N(d_1) - KN(d_2)e^{-\rho_2 T} \right) \\
1 & \forall C \in \left( S_2 e^{-\delta T} N(d_1) - KN(d_2)e^{-\rho_2 T}, S_3 e^{-\delta T} N(d_1) - KN(d_2)e^{-\rho_3 T} \right) \\
0 & \text{otherwise.}
\end{cases}
\]

In the following example we consider the fuzzy call price on a stock option using the adaptive fuzzy number discussed earlier.

**Example 2.** Consider a European call option on a stock with the following assumptions. The current stock price, the stock price volatility and the risk-free interest rate are all taken as adaptive fuzzy numbers.

\[\tilde{S}_t = [158, 160, 162, 164]_n, \quad \tilde{\sigma} = [0.03, 0.04, 0.05, 0.06]_n, \quad \tilde{\tau} = [0.29, 0.31, 0.33, 0.35]_n, \quad \tau = 2, \delta = 0.03, K = 140.\] Thus, \(E(\tilde{S}_t) = 161, E(\tilde{\sigma}) = 0.045\) and \(E(\tilde{\tau}) = 0.32.\)

\[d_1 = \frac{\ln \left( \frac{E(\tilde{S}_t)}{K} \right) + (E(\tilde{\sigma}) - \delta + \frac{E(\tilde{\tau})^2}{2}) \tau}{E(\tilde{\tau})\sqrt{\tau}} = 0.6014 \]

\[N(d_1) = N(0.6014) = 0.72621 \]

\[d_2 = d_1 - E(\tilde{\tau})\sqrt{\tau} = 0.6014 - (0.32)\sqrt{2} = 0.14885 \]

\[N(d_2) = N(0.14885) = 0.5591 \]

\[FCOV_1(\alpha) = \left[ e^{-\delta T} N(d_1)S_1 - KN(d_2)e^{-\rho_1 T} \right] + \left[ KN(d_2)(e^{-\rho_1 T} - e^{-\rho_2 T}) + (S_2 - S_1)e^{-\delta T} N(d_1) \right] \alpha^\frac{1}{2} = \left[ e^{-0.03(2)}(0.72621)(158) - 140(0.5591)e^{-0.03(2)} \right] + \left[ 140(0.5591)(e^{-0.03(2)} - e^{-0.04(2)}) + (160 - 158)e^{-0.03(2)}(0.72621) \right] \alpha^\frac{1}{2} = 34.343 + 2.8275\alpha^\frac{1}{2} \] (4.7)

\[FCOV_2(\alpha) = \left[ e^{-\delta T} N(d_1)S_4 - KN(d_2)e^{-\rho_4 T} \right] - \left[ (S_4 - S_3)e^{-\delta T} N(d_1) + KN(d_2)(e^{-\rho_3 T} - e^{-\rho_4 T}) \right] \alpha^\frac{1}{2} = \left[ e^{-0.03(2)}(0.72621)164 - 140(0.5591)e^{-0.06(2)} \right] - \left[ (164 - 162)e^{-0.03(2)}(0.72621) + 140(0.5591)(e^{-0.05(2)} - e^{-0.06(2)}) \right] \alpha^\frac{1}{2} = 42.74 - 2.7703\alpha^\frac{1}{2}. \] (4.8)

Using Eqs. (4.7) and (4.8), we obtain the fuzzy call option values for various levels of \(\alpha\) and \(n\). They are presented in Table 1 and are displayed in Fig. 2.

Therefore, a fuzzy adaptive BS model is sufficiently flexible and can be easily adjusted or tuned for optimal solution.

5. Conclusions

The fuzzy assisted Black–Scholes option pricing model is practical and useful. Without introducing fuzzy financial option models it may not be possible to incorporate this genuine uncertainty. In the literature only the linear fuzzy numbers have been used to model the parameter uncertainty in option pricing applications (see for example...
Table 1
Fuzzy option values for different values of $\alpha$ and $n$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$FCOV_1(\alpha)$</th>
<th>$FCOV_2(\alpha)$</th>
<th>$FCOV_1(\alpha)$</th>
<th>$FCOV_2(\alpha)$</th>
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<td>$n = 5$</td>
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<td>41.91</td>
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</tr>
<tr>
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</tr>
</tbody>
</table>

Fig. 2. Fuzzy option values.

Carlson [11], Cherubini [6], Ghaziri [7], Trenev [8], Wu [9] and Zmeskal [10]). In this paper we have used adaptive nonlinear fuzzy numbers to modeled parameter uncertainty in the option pricing model (see Fig. 2 for illustration). This will allow the financial analyst to choose any European option price with a reasonable certainty for investor’s decision making.

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