A NEW METHOD FOR SOLVING THE AMBIPOLAR DIFFUSION EQUATION (ADE) BASED ON A FINITE ELEMENT FORMULATION

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Keywords- Power bipolar devices, diffusion problems, modeling and simulation.

Abstract
This paper presents a new method for simulation of power bipolar devices. The method is based on the solution of the Ambipolar Diffusion Equation (ADE) with a finite element (FE) approach. This one allows analog solutions for ADE through an equivalent electrical network. So the method could constitute the core for power bipolar devices simulation into any electrical circuit simulator.

1. INTRODUCTION

The charge (holes and electrons) dynamics within the bases of power bipolar devices are described by the well known ADE:

\[ D^* \frac{d^2 p(x,t)}{dx^2} = \frac{dp(x,t)}{dt} + \frac{p(x,t)}{\tau} \]

For power devices with large base regions, such as Diodes, BJTs, IGBTs and Thyristors, the classical charge control concepts no longer apply because the transit time of carriers is usually of the same order of the transient phenomena itself. In fact the usual quasi-static approach produces unrealistic results. This implies that the ADE must be solved exactly, which can be done by specialized programs, using finite difference or finite elements techniques. From the point of view of the power electronics engineering this approaches are cumbersome, difficult to develop into general electrical circuit simulators and very time consuming. To overcome these drawbacks several authors have proposed alternatives for the solution of the ADE based either on assumed geometric solutions of the distribution (piece-wise, trigonometric...) or on solutions for the distribution \( p(x,t) \) that enables the application of the method of separation of variables, or, more recently, in spectral methods.

The proposed method to solve (Eq. 1) is the following:

1. We will take an equivalent variational formulation for (Eq. 1) namely the following functional, \( \Pi \):
\[
\Pi = \int \left\{ \frac{1}{2} \left( \frac{dp(x,t)}{dx} \right)^2 - \frac{c \times p(x,t)^2}{2} - p(x,t) \times Q \right\} dV - \int \left[ (f(t) + g(t)) \times p(x,t) \right] dS
\]

with:
\[
c = -\frac{1}{D \times \tau}
\]
\[
Q = -\frac{dp(x,t)}{D \times dt}
\]

It can be verified that the minimization of this functional \( (\Delta \Pi = 0) \) corresponds to solve (Eq. 1) with the boundary conditions (Eq. 2 and Eq. 3).

2. We divide the domain into \( n \) elements (Fig. 1). For sake of simplicity we will use equally spaced elements and will assume fixed parameters \( D \) and \( \tau \). It becomes effective to implement more sophisticated assumptions, such as non equidistant elements, and \( D, \tau \), functions of space and time.

3. We apply the FE formulation to our domain, namely:

\[ P_e = N_i^e p_i + N_j^e p_j \]

and using simplex shape functions:

\[ N_i^e = \frac{X_j - x}{L_e} \]
\[ N_j^e = \frac{x - X_i}{L_e} \]

In principle for the presented hypothesis of \( n \) elements we have \( (n+1) \) nodes and:

\[ P(\beta) = N_0^\beta p_0 + N_1^\beta p_1 + \cdots + N_n^\beta p_{n+1} \]

with \( P(\beta) \) a generic element, \( p_\beta \) the (unknown) value of \( p(x,t) \) in \( x = x_\beta \) and \( N_0^\beta, N_1^\beta, \ldots, N_n^\beta \) simplex shape functions with values:

\[ N_0^\beta = -\frac{x}{L_e} \]
\[ N_1^\beta = \frac{x_1 - x}{L_e} \]
\[ N_n^\beta = \frac{x - x_n}{L_e} \]

Therefore the unknown function \( p(x,t) \) is approached by the sum of \( n \) elementary functions being each element, \( P(\beta) \), of the type (Eq. 8).

In a matrix form:

\[ [P] = [N] [p] \]

with \([N]\) a \([n] \times \left[ n+1 \right]\) matrix with elements only in the diagonal and above the diagonal:

\[
[N] = \begin{bmatrix}
N_1 & N_1^2 & N_2 & N_3 & N_4 & N_5 & \cdots \\
N_1^2 & N_2 & N_3 & N_4 & N_5 & \cdots \\
N_1 & N_2 & N_3 & N_4 & N_5 & \cdots \\
N_1 & N_2 & N_3 & N_4 & N_5 & \cdots \\
N_1 & N_2 & N_3 & N_4 & N_5 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\]

and \([p]\) a column vector:

\[ p = [p_1, p_2, p_3, \ldots, p_{n+1}]^T \]

4. Now we replace \( p(x,t) \) by the finite element approximation, \([P]\), in functional, \( \Pi \):
Substitution for \( P(\beta) \) and calculating will give:

\[
P = \sum_{\beta} \left\{ \frac{Ae}{2 \times Le} (p_{\beta+1} - p_{\beta})^2 - \frac{c \times Ae \times Le}{6} \left( p_{\beta}^2 + p_{\beta+1}^2 + p_{\beta}^2 + p_{\beta+1}^2 \right) \right\} - f(t) \times p_{\beta} \times A_1 - g(t) \times p_{\beta} \times A_n \]

5. The next step is the minimization of (Eq. 14). To do so we will differentiate it in order to the various values \( p_{\beta} \) through a few simplifications the result is the following one:

\[
\text{Eq. 15} \quad [M] \times \frac{dp}{dt} + [K] \times [p] + [F] = 0
\]

with the following matrices:

\[
[K] = \frac{Ae \times Le}{2 \times Le} \begin{bmatrix}
2 & -2 & 0 & 0 & 0 & \cdots \\
-2 & 4 & -2 & 0 & 0 & \cdots \\
0 & 0 & -2 & 4 & -2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\[
[M] = \frac{Ae \times Le \times c}{6} \begin{bmatrix}
1 & 0.5 & 0 & 0 & \cdots \\
0.5 & 2 & 0.5 & 0 & \cdots \\
0 & 0.5 & 2 & 0.5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\[
[F] = \begin{bmatrix}
f(t) \times A_1 & 0 & \cdots & \cdots & g(t) \times A_{n+1}
\end{bmatrix}^T
\]

\[
p = \begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
\vdots \\
p_{n+1}
\end{bmatrix}^T
\]

\[
[p] = \begin{bmatrix}
\frac{dp_1}{dt} \\
\frac{dp_2}{dt} \\
\frac{dp_3}{dt} \\
\vdots \\
\frac{dp_{n+1}}{dt}
\end{bmatrix}^T
\]

We have now a system of \((n+1)\) ODE to be solved, which can be done with, for instance, some Runge-Kutta routines. Fig. 2 shows the solution for the turn-off of a power p-i-n diode with a steady state current of 300A and a gradient of -1000A/µs.

If we consider the steady-state \( \frac{dp}{dt} = 0 \) we can compare the FE solution with the real (known) solution.
This can be watched in Fig. 3, for the steady state current of 300 A. The system of (n+1) ODEs is solved, in both situations, using some MATLAB scripts.

3. ELECTRICAL CIRCUIT ANALOGY

The conversion of the ADE into a system of (n+1) ODEs is established, being the number of equations a compromise between accuracy and time of computation.

It is interesting to conclude that the system of equations (Eq. 15) has an immediate electrical analogy. This can be shown with an example of only 2 elements (for simplicity), \( P(1) \) and \( P(2) \), and 3 nodes \( x_1 \), \( x_2 \) and \( x_3 \), with corresponding values of the unknown function \( p_1 \), \( p_2 \) and \( p_3 \).

We will have the following matrices:

\[
[M] = \frac{Ae \times Le}{3 \times D} \times \begin{bmatrix} 1 & 0.5 & 0.5 & 0 \ 0 & 2 & 0.5 & 1 \end{bmatrix}
\]

\[
[K] = \frac{Ae \times Le}{2 \times Le} \times \begin{bmatrix} 2 & -2 & 0 & 0 \ 0 & 4 & 0 & 0 \ -2 & 0 & 2 & 0 \ 0 & 0 & 2 & 0 \end{bmatrix} - \frac{Ae \times Le \times c}{6} \times \begin{bmatrix} 2 & 1 & 0 \ 1 & 4 & 1 \ 0 & 1 & 2 \end{bmatrix}
\]

Fig. 4 - Electrical circuit analogy of the system (Eq. 15).

\[
[F] = \begin{bmatrix} f(t) \times A_1 \ 0 \ g(t) \times A_3 \end{bmatrix}
\]

Now consider the electrical circuit of Fig. 4. Solving it with a nodal analysis we get:

\[
[G] \times [V] + [C] \times \left[ \frac{dV}{dt} \right] + [I] = 0
\]

with the values for \([G], [I]\) and \([C]\) given below.

\[
\begin{bmatrix} G_1 \ G_2 \ G_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_{12}} & \frac{1}{R_{12}} & \frac{1}{R_{12}} \ \
-\frac{1}{R_{23}} & \frac{1}{R_{23}} & \frac{1}{R_{23}} \ 
\frac{1}{R_{30}} & \frac{1}{R_{30}} & \frac{1}{R_{30}} \end{bmatrix}
\]

This immediately identifies the factors of the different matrices for equality of solutions.

Table 1 shows the steady state results of PSPICE in a modelization with 11 nodes (10 elements, 21 Ri//Ci).
4. CONCLUSIONS-

The paper demonstrates the possibility of solving the ADE using a FE modelization based on a variational formulation. This approach, being implementable in a circuit simulator, enables the power electronics engineer to establish his own time/accuracy compromises. It is also easily achieved (once we have the matrices \([M]\), \([K]\) and \([F]\)) and the most important aspect of all being the ability to simulate the ADE with parameters \((D, \tau)\) varying in time and position. We can also, for an increase in the accuracy of the solution, use nonequally spaced elements (using more elements near the borders where the concentration of carriers moves faster) and complex shape functions.

This work will constitute the core for simulation of all power bipolar devices. In future work the author intends to extend the application of this method to the modeling of power Diodes and IGBTs.

5. REFERENCES