A note on solutions of wave, Laplace’s and heat equations with convolution terms by using a double Laplace transform

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ABSTRACT

In this study we consider general linear second-order partial differential equations and we solve three fundamental equations by replacing the non-homogeneous terms with double convolution functions and data by a single convolution.

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1. Introduction

The wave equation, heat equation and Laplace’s equations are known as three fundamental equations in mathematical physics and occur in many branches of physics, in applied mathematics as well as in engineering. It is also known that there are two types of these equations: The homogeneous equations that have constant coefficients with many classical solutions such as by means of separation of variables [see [1]], the method of characteristics [see [2,3]], the single Laplace transform and the Fourier transform [see [4]]; and the non-homogeneous equations with constant coefficients solved by means of the double Laplace transform [see [8]] and operation calculus [see [5–7]].

In this study we use the double Laplace transform to solve a second-order partial differential equation. In special cases we solve the non-homogeneous wave, heat and Laplace’s equations with non-constant coefficients by replacing the non-homogeneous terms by double convolution functions and data by single convolutions.

For example, we prove that if $F_1, F_2, \ldots, F_i$ are solutions of non-homogeneous equations with constant coefficients and $G_1, G_2, \ldots, G_i$ are solutions of non-homogeneous equations with non-constant coefficients then $\sum_{i=1}^{n} F_i(x, t) \ast \ast G_i(x, y)$ is the solution of

$$u_{tt}(x, t) - u_{xx}(x, t) - h(x, t) = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+$$

where the non-constant coefficient is a polynomial in the form of $k(x, t) = \sum_{j=1}^{m} \sum_{i=1}^{n} x^j t^i$.

Now we consider a linear second-order partial differential equation with constant coefficients

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = \sum_{i=1}^{n} f_i(x, y) \ast \ast g_i(x, y)$$

and an equation with non-constant coefficients of the following form:

$$k(x, y) \ast \ast (au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu) = \sum_{i=1}^{n} f_i(x, y) \ast \ast g_i(x, y)$$

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under boundary conditions

\[ u(x, 0) = h_1(x), \quad u(0, y) = h_2(y) \]

\[ u_x(x, 0) = h'_1(x), \quad u_x(0, y) = h'_2(y) \quad \text{and} \quad u(0, 0) = 0 \]

where the symbol \( \ast \ast \) means double convolution (see [5]) and \( a, b, c, d, e \) and \( f \) are constant coefficients. In this study we consider that Eq. (1.1) has a solution on using the Laplace transform and Eq. (1.2) also has a solution on using the double Laplace transform; further the inverse double Laplace transform exists.

We note that all three fundamental equations with constant coefficients are particular forms of Eq. (1.1). In the following sections we will discuss the solution for three fundamental equations with non-constant coefficients where the non-constant coefficients are produced by convolution of a polynomial.

### 2. Wave equation and double Laplace transform

Consider the following non-homogeneous wave equation with non-constant coefficients in one dimension:

\[ k(x, t) \ast \ast (u_{tt} - u_{xx}) = \sum_{i=1}^{n} f_i(t, x) \ast \ast g_i(t, x) \quad (t, x) \in \mathbb{R}^2_+ \quad (2.1) \]

where \( k(x, t) \) is polynomial as defined above and the boundary conditions are given by

\[ u(x, 0) = r_1(x) \ast r_2(x), \quad u_t(0, x) = \frac{\partial }{\partial x} (r_1(x) \ast r_2(x)) \]

\[ = \frac{\partial }{\partial x} r_1(x) \ast r_2(x) \quad \text{or} \quad r_1(x) \ast \frac{\partial }{\partial x} r_2(x) \quad (2.2) \]

\[ u(0, t) = h_1(t) \ast h_2(t), \quad u_{xx}(0, t) = \frac{\partial }{\partial t} (h_1(t) \ast h_2(t)) \]

\[ = \frac{\partial }{\partial t} h_1(t) \ast h_2(t) \quad \text{or} \quad h_1(t) \ast \frac{\partial }{\partial t} h_2(t) \quad (2.3) \]

where the non-homogeneous terms of Eq. (2.1) are double-convolution terms and non-homogeneous initial conditions are single-convolution ones. By taking the double Laplace transform for Eq. (2.1) and the single Laplace transform for Eqs. (2.2) and (2.3) we obtain

\[ U(p, s) = \frac{sR_1(p)R_2(p)}{(s^2 - p^2)} - \frac{pH_1(s)H_2(s)}{(s^2 - p^2)} + \frac{R_1(p) (pR_2(p) - R_2(0))}{(s^2 - p^2)} - \frac{H_1(s) (sH_2(p) - H_2(0))}{(s^2 - p^2)} + \frac{\sum_{i=1}^{n} F_i(s, p)G_i(s, p)}{(s^2 - p^2)} K(p, s) \quad (2.4) \]

Now we assume that the inverse double Laplace transform for Eq. (2.4) exists; then one can obtain a solution of Eq. (2.1) as follows:

\[ u(x, t) = L_x^{-1} T_1^{-1} \left[ \frac{sR_1(p)R_2(p)}{(s^2 - p^2)} - \frac{pH_1(s)H_2(s)}{(s^2 - p^2)} + \frac{R_1(p) (pR_2(p) - R_2(0))}{(s^2 - p^2)} - \frac{H_1(s) (sH_2(p) - H_2(0))}{(s^2 - p^2)} + \frac{\sum_{i=1}^{n} F_i(s, p)G_i(s, p)}{(s^2 - p^2)} K(p, s) \right] \quad (2.5) \]

where we assume that the double inverse Laplace transform exists for each term in the right hand side of Eq. (2.5).

Now we let \( F(t, x) \) be a solution of

\[ u_{tt}(x, t) - u_{xx}(x, t) = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+ \quad (2.6) \]

and further we consider \( G(x, t) \) as a solution of

\[ k(x, t) \ast \ast [u_t(x, t) - u_{xx}(x, t)] = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+ \quad (2.7) \]

Then \( F(t, x) \) satisfies Eq. (2.6):

\[ F_{tt}(t, x) - F_{xx}(t, x) = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t) \quad (2.8) \]

and similarly \( G(x, t) \) satisfies Eq. (2.7):

\[ G_{tt}(t, x) - G_{xx}(t, x) = \nu(x, t) \quad (2.9) \]
Now we can easily check weather $F(x, t) \ast \ast G(x, t)$ is a solution or not for Eq. (2.6). By substitution we obtain

$$
(F(x, t) \ast \ast G(x, t))_{tt} - (F(x, t) \ast \ast G(x, t))_{xx} = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t)
$$

(2.10)

on using the partial derivative of the convolution; the left hand side of Eq. (2.10) follows:

$$
F_{tt}(x, t) \ast \ast G(x, t) - F_{xx}(x, t) \ast \ast G(x, t) = F(x, t) \ast \ast G_{tt}(x, t) - F(x, t) \ast \ast G_{xx}(x, t)
$$

and then Eq. (2.10) can be written in the form

$$
F(x, t) \ast \ast [G_{tt}(x, t) - G_{xx}(x, t)] = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t)
$$

(2.11)

or

$$
[F_{tt}(x, t) - F_{xx}(x, t)] \ast \ast G(x, t) = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t).
$$

(2.12)

Now by substituting Eq. (2.9) into (2.11) and Eq. (2.8) into (2.12), we have

$$
F(x, t) \ast \ast v(x, t) \neq \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t)
$$

(2.13)

and

$$
\left(\sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t)\right) \ast \ast G(x, t) \neq \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t)
$$

(2.14)

and thus we can easily see from Eqs. (2.13) and (2.14) that the convolution $F(x, t) \ast \ast G(x, t)$ is not a solution for Eq. (2.6) in general; however it is a solution for another type of equation as in the following theorem.

**Theorem 1.** If $F(x, t)$ is a solution of

$$
u_{tt}(x, t) - u_{xx}(x, t) = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+
$$

(2.15)

under the initial conditions

$$
u(x, 0) = r_1(x) \ast r_2(x), \quad \nu_t(0, x) = \frac{\partial}{\partial x} (r_1(x) \ast r_2(x))
$$

or

$$
u_t(x, 0) = r_1(x) \ast r_2(x)
$$

$$
u(0, t) = h_1(t) \ast h_2(t), \quad \nu_t(0, t) = \frac{\partial}{\partial t} (h_1(t) \ast h_2(t))
$$

or

$$
u_t(h_1(t) \ast h_2(t))
$$

and if $G(x, t)$ is a solution of

$$
k(x, t) \ast [u_{tt}(x, t) - u_{xx}(x, t)] = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+
$$

(2.16)

under the same conditions, then the convolution $F(x, t) \ast \ast G(x, t)$ is a solution of

$$
u_{tt}(x, t) - u_{xx}(x, t) - h(x, t) = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+
$$

(2.17)

where $g(t, x)$ is an exponential function and $k(x, t) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i t^j$.

**Proof.** Since $F(x, t)$ is solution of Eq. (2.15) then

$$
F_{tt}(x, t) - F_{xx}(x, t) = \sum_{i=1}^{n} f_i(x, t) \ast \ast g(x, t)
$$

(2.18)

holds true and $G(x, t)$ is a solution of Eq. (2.16); then

$$
G_{tt}(x, t) - G_{xx}(x, t) = v(x, t)
$$

(2.19)
is also true and then by substitution we have

\[ (F(x, t) * \ast G(x, t))_{tt} - (F(x, t) * \ast G(x, t))_{xx} - h(x, t) = \sum_{i=1}^{n} f_i(x, t) * \ast g(x, t) \] (2.20)

and on using the partial derivative of the convolutions we obtain

\[ F_{tt}(x, t) * \ast G(x, t) - F_{xx}(x, t) * \ast G(x, t) = F(x, t) * \ast G_{tt}(x, t) - F(x, t) * \ast G_{xx}(x, t) \] (2.21)

and then Eq. (2.20), followed by

\[ [F_{tt}(x, t) - F_{xx}(x, t)] * \ast G(x, t) - h(x, t) = \sum_{i=1}^{n} f_i(x, t) * \ast g(x, t). \] (2.22)

By substituting Eq. (2.18) in (2.22) we have

\[ \left( \sum_{i=1}^{n} f_i(x, t) * \ast g(x, t) \right) * \ast G(x, t) - h(x, t) = \sum_{i=1}^{n} f_i(x, t) * \ast g(x, t). \] (2.23)

This shows that the convolution \( F(x, t) * \ast G(x, t) \) is a solution of Eq. (2.17). □

In the next part we apply a similar technique to non-homogeneous one-dimensional heat equations and Laplace’s equation.

### 3. Heat equation and double Laplace transform

Consider the non-homogeneous heat equation in one dimension in a normalized form:

\[ k(x, t) * \ast [u_t(x, t) - u_{xx}(x, t)] = \sum_{i=1}^{n} f_i(x, t) * \ast g(x, t) \] (3.1)

where \( k(x, t) \) is a polynomial as defined above under initial conditions

\[
\begin{align*}
  u(x, 0) &= q_1(x) \ast q_2(x), \\
  u(0, t) &= w_1(t) \ast w_2(t), \quad u_x(0, t) = \frac{\partial}{\partial t} (w_1(t) \ast w_2(t)).
\end{align*}
\] (3.2)

By using the double Laplace transform for Eq. (3.1) and the single Laplace transform for Eq. (3.2) we can obtain

\[ U(p, s) = \frac{Q_1(p)Q_2(p)}{(s - p^2)} - \frac{pW_1(s)W_2(s)}{(s - p^2)} - \frac{W_1(s)(sW_2(s) - W_2(0))}{(s - p^2)} + \sum_{i=1}^{n} \frac{F_i(p, s)G(p, s)}{(s - p^2)K(p, s)} \] (3.3)

and similarly if we take the inverse double Laplace transform for Eq. (3.3) with respect to \( s, p \), we obtain the solution of (3.1) in the form

\[ u(t, x) = L_p^{-1}L_s^{-1} \left[ \frac{Q_1(p)Q_2(p)}{(s - p^2)} - \frac{pW_1(s)W_2(s)}{(s - p^2)} \right] - L_p^{-1}L_s^{-1} \left[ \frac{W_1(s)(sW_2(s) - W_2(0))}{(s - p^2)} \right] + L_p^{-1}L_s^{-1} \left[ \sum_{i=1}^{n} \frac{F_i(p, s)G(p, s)}{(s - p^2)K(p, s)} \right] \] (3.4)

provided that the double inverse Laplace transform and inverse double Laplace transform exist for Eqs. (3.3) and (3.4) respectively; then we have the following theorem.

**Theorem 2.** If \( F(x, t) \) is a solution for

\[ u_t(x, t) - u_{xx}(x, t) = \sum_{i=1}^{n} f_i(x, t) * \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+ \]

under the initial condition

\[
\begin{align*}
  u(x, 0) &= q_1(x) \ast q_2(x), \\
  u(0, t) &= w_1(t) \ast w_2(t), \quad u_x(0, t) = \frac{\partial}{\partial t} (w_1(t) \ast w_2(t))
\end{align*}
\]

and if \( G(x, t) \) is a solution for

\[ k(x, t) * \ast [u_t(x, t) - u_{xx}(x, t)] = \sum_{i=1}^{n} f_i(x, t) * \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+ \]
then the convolution $F(x, t) \ast G(x, t)$ is a solution of

$$u_t(x, t) - u_{xx}(x, t) - \theta(x, t) = \sum_{i=1}^{n} f_i(x, t) \ast g(x, t) \quad (x, t) \in \mathbb{R}^2_+.$$ 

The proof of this theorem is similar to the above proof of Theorem 1.

4. Laplace equation and double Laplace transform

Similarly, considering the non-homogeneous Laplace’s equation with non-constant coefficient in two dimensions as follows:

$$k(x, y) \ast (u_{xx} + u_{yy}) = \sum_{i=1}^{n} f_i(x, y) \ast g(x, y) \quad (x, y) \in \mathbb{R}^2_+ \tag{4.1}$$

where $k(x, y)$ is polynomial as defined above under the condition

$$u(x, 0) = r_1(x) \ast r_2(x), \quad u_y(0, x) = \frac{\partial}{\partial x} (r_1(x) \ast r_2(x))$$

$$= \frac{\partial}{\partial x} r_1(x) \ast r_2(x) \quad \text{or} \quad r_1(x) \ast \frac{\partial}{\partial x} r_2(x) \tag{4.2}$$

$$u(0, y) = h_1(y) \ast h_2(y), \quad u_y(0, y) = \frac{\partial}{\partial y} (h_1(y) \ast h_2(y))$$

$$= \frac{\partial}{\partial y} h_1(y) \ast h_2(y) \quad \text{or} \quad h_1(y) \ast \frac{\partial}{\partial y} h_2(y) \tag{4.3}$$

we can then easily deduce the following theorem without proof.

**Theorem 3.** If $F(x, y)$ is a solution of

$$u_{xx}(x, y) + u_{yy}(x, y) = \sum_{i=1}^{n} f_i(x, y) \ast g(x, y) \quad (x, y) \in \mathbb{R}^2_+$$

under the boundary conditions

$$u(x, 0) = r_1(x) \ast r_2(x), \quad u_y(0, x) = \frac{\partial}{\partial x} (r_1(x) \ast r_2(x))$$

$$= \frac{\partial}{\partial x} r_1(x) \ast r_2(x) \quad \text{or} \quad r_1(x) \ast \frac{\partial}{\partial x} r_2(x)$$

$$u(0, y) = h_1(y) \ast h_2(y), \quad u_y(0, y) = \frac{\partial}{\partial y} (h_1(y) \ast h_2(y))$$

$$= \frac{\partial}{\partial y} h_1(y) \ast h_2(y) \quad \text{or} \quad h_1(y) \ast \frac{\partial}{\partial y} h_2(y)$$

and if $G(x, y)$ is a solution of

$$k(x, y) \ast [u_{xx}(x, y) + u_{yy}(x, y)] = \sum_{i=1}^{n} f_i(x, y) \ast g(x, y) \quad (x, y) \in \mathbb{R}^2_+$$

then the convolution $F(x, y) \ast G(x, y)$ is a solution of

$$u_{xx}(x, y) + u_{yy}(x, y) - \Psi(x, y) = \sum_{i=1}^{n} f_i(x, y) \ast g(x, y) \quad (x, y) \in \mathbb{R}^2_+$$

where $\Psi(x, y)$ is considered as a remainder function and $g(x, y)$ is an exponential function. The proof is similar to the proof of Theorem 1.

Note that if we take a particular example of a two-dimensional Laplace’s equation using data similar to those used in previous theorems we get the similar result, that is the solution of the non-constant coefficient equation in the form

$$k(x, y) \ast (W_{xx} + W_{yy}) = u_{xx} + u_{yy} + \Psi(x, y).$$

Thus we note that the PDEs with the non-constant coefficients (polynomials), in particular, the heat, wave and Laplace’s equations, give similar results when we use the same initial, boundary conditions having non-homogeneous terms as convolutions.
References


