Research Article

On the Neutrix Composition of the Delta and Inverse Hyperbolic Sine Functions

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1. Introduction

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$, and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

Now, let $\rho(x)$ be a function in $\mathcal{D}[-1,1]$ having the following properties:

(i) $\rho(x) \geq 0$,
(ii) $\rho(x) = \rho(-x)$,
(iii) $\int_{-1}^{1} \rho(x) \, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if $F$ is
an arbitrary distribution in $\mathcal{S}'$ and $F_n(x) = F(x) \ast \delta_n(x) = \langle F(x-t), \varphi(t) \rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

Since the theory of distributions is a linear theory, thus we can extend some of the operations which are valid for ordinary functions to the space of distributions and such operations are called regular operations such as: addition, multiplication by scalars; see [1]. Other operations can be defined only for a particular class of distributions or for certain restricted subclasses of distributions; these are called irregular operations such as: multiplication of distributions, convolution products, and composition of distributions; see [2–4]. Thus, there have been several attempts recently to define distributions of the form $F(f(x))$ in $\mathcal{S}'$, where $F$ and $f$ are distributions in $\mathcal{S}$; see for example [5–8]. In the following, we are going to consider an alternative approach. As a starting point, we look at the following definition which is a generalization of Gel’fand and Shilov’s definition of the composition involving the delta function [9], and was given in [6].

**Definition 1.1.** Let $F$ be a distribution in $\mathcal{S}'$ and let $f$ be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to $h$ on the open interval $(a, b)$, with $-\infty < a < b < \infty$, if

$$ N - \lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x) dx = \langle h(x), \varphi(x) \rangle, \quad (1.1) $$

for all $\varphi$ in $\mathcal{S}[a, b]$, where $F_n(x) = F(x) \ast \delta_n(x)$ for $n = 1, 2, \ldots$ and $N$ is the neutrix, see [10], having domain $N'$ the positive and range $N''$ the real numbers, with negligible functions which are finite linear sums of the functions

$$ n^{1/\ln\eta} n^{r-1}, \quad \ln n : \lambda > 0, \quad r = 1, 2, \ldots \quad (1.2) $$

and all functions which converge to zero in the usual sense as $n$ tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to $h$ on the open interval $(a, b)$ if

$$ \lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x) dx = \langle h(x), \varphi(x) \rangle, \quad (1.3) $$

for all $\varphi$ in $\mathcal{S}[a, b]$.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard’s finite part of $f(n)$. The definition of the neutrix composition of distributions was originally given in [10] but was then simply called the composition of distributions.

The following three theorems were proved in [11], [8], and [12], respectively.

**Theorem 1.2.** The neutrix composition $\delta^{(s)}(\text{sgn} x|x|^\lambda)$ exists and

$$ \delta^{(s)}(\text{sgn} x|x|^\lambda) = 0, \quad (1.4) $$
for $s = 0, 1, 2, \ldots$ and $(s + 1)\lambda = 1, 3, \ldots$, and
\[ \delta^{(s)}\left(\text{sgn } x|x|^\lambda\right) = \frac{(-1)^{(s+1)(\lambda+1)}}{\lambda[(s+1)\lambda - 1]} \delta^{(\lambda+1)}(x), \quad (1.5) \]
for $s = 0, 1, 2, \ldots$, and $(s + 1)\lambda = 2, 4, \ldots$

**Theorem 1.3.** The neutrix compositions $\delta^{(2n-1)}(\text{sgn } x|x|^{1/s})$ and $\delta^{(s-1)}(|x|^{1/s})$ exist and
\[ \delta^{(2n-1)}(\text{sgn } x|x|^{1/s}) = \frac{1}{2}(2s)!\delta'(x), \]
\[ \delta^{(s-1)}(|x|^{1/s}) = (-1)^{s-1}\delta(x), \quad (1.6) \]
for $s = 1, 2, \ldots$

**Theorem 1.4.** The neutrix composition $\delta^{(s)}(\sinh^{-1}x_+^{1/r})$ exists and
\[ \delta^{(s)}\left[(\sinh^{-1}x_+)^{1/r}\right] = \sum_{k=0}^{(s+1)/r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^k r c_{s,k,i} \delta^{(k)}(x)}{2^{k+1}k!}, \quad (1.7) \]
for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, where
\[ c_{r,s,k,i} = \frac{(-1)^s s! \left[(k-2i+1)^{r+s-1} + (k-2i-1)^{r+s-1}\right]}{2(rs + r - 1)!} \]
The next two theorems were proved in [13].

**Theorem 1.5.** The neutrix composition $\delta^{(s)}[\ln^r(1 + |x|)]$ exists and
\[ \delta^{(s)}[\ln^r(1 + |x|)] = \sum_{k=0}^{s r + r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^s s! [1 + (-1)^k]^{r+s+r-1}}{2(r s + r - 1)! k!} \delta^{(k)}(x), \quad (1.9) \]
for $s = 0, 1, 2, \ldots$, and $r = 1, 2, \ldots$

In particular, the composition $\delta[\ln(1 + |x|)]$ exists and
\[ \delta[\ln(1 + |x|)] = \delta(x). \quad (1.10) \]

**Theorem 1.6.** The neutrix composition $\delta^{(s)}[\ln^r(1 + |x^{1/r}|)]$ exists and
\[ \delta^{(s)}[\ln^r(1 + |x^{1/r}|)] = \sum_{k=0}^{m-1} \sum_{i=0}^{kr + r-1} \binom{kr + r-1}{i} \frac{(-1)^{r+s+i-1} [1 + (-1)^k]^{r(s+i-1)}}{2^{k-1}} \delta^{(k)}(x), \quad (1.11) \]
for $s = 0, 1, 2, \ldots$ and $r = 2, 3, \ldots$, where $m$ is the smallest non-negative integer greater than $(s-r+1)r^{-1}$. 
In particular, the composition $\delta^{(s)}[\ln(1 + |x^{1/r}|)]$ exists and

$$\delta^{(s)}[\ln\left(1 + \left|x^{1/r}\right|\right)] = 0,$$  \hspace{1cm} (1.12)

for $s = 0, 1, 2, \ldots, r - 2$ and $r = 2, 3, \ldots$ and

$$\delta^{(r-1)}[\ln\left(1 + \left|x^{1/r}\right|\right)] = (-1)^{r-1}r!\delta(x),$$  \hspace{1cm} (1.13)

for $r = 2, 3, \ldots$.

2. Main Results

We now prove the following theorem.

**Theorem 2.1.** The neutrix composition $\delta^{(s)}[(\sinh^{-1}x_{+})']$ exists and

$$\delta^{(s)}\left[(\sinh^{-1}x_{+})'\right] = \sum_{k=0}^{s+r-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^k r c_{s,k,i}}{2^{k+1}k!} \delta^{(k)}(x),$$  \hspace{1cm} (2.1)

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, where

$$c_{r,s,k,i} = \frac{(-1)^r s! [k - 2i + 1]^{r+s-1} + (k - 2i - 1)^{r+s-1}}{2(rs + r - 1)!}.$$  \hspace{1cm} (2.2)

In particular, the neutrix composition $\delta(\sinh^{-1}x_{+})$ exists and

$$\delta(\sinh^{-1}x_{+}) = \frac{1}{2}\delta(x).$$  \hspace{1cm} (2.3)

**Proof.** To prove (2.1), we first of all evaluate

$$\int_{-1}^{1} \delta^{(s)}_n\left[(\sinh^{-1}x_{+})'\right] x^k \, dx.$$  \hspace{1cm} (2.4)

We have

$$\int_{-1}^{1} \delta^{(s)}_n\left[(\sinh^{-1}x_{+})'\right] x^k \, dx = n^{s+1} \int_{-1}^{1} \rho^{(s)}\left[n(\sinh^{-1}x_{+})'\right] x^k \, dx$$

$$= n^{s+1} \int_{0}^{1} \rho^{(s)}\left[n(\sinh^{-1}x_{+})'\right] x^k \, dx$$

$$+ n^{s+1} \int_{-1}^{0} \rho^{(s)}(0)x^k \, dx$$

$$= I_1 + I_2.$$  \hspace{1cm} (2.5)
It is obvious that

\[ N - \lim_{n \to \infty} I_2 = N - \lim_{n \to \infty} \int_{-1}^{0} \delta_n^{(s)} \left( \sinh^{-1} x \right)^r x^k \, dx = 0, \quad (2.6) \]

for \( k = 0, 1, 2, \ldots \).

Making the substitution \( t = n(\sinh^{-1} x)^r \), we have for large enough \( n \)

\[
I_1 = \frac{n^{s-r+1}}{r} \int_0^1 t^{1/(r-1)} \sinh^k \left( \frac{t}{n} \right)^{1/r} \cosh \left( \frac{t}{n} \right)^{1/r} \rho^{(s)}(t) \, dt \\
\times \int_0^1 t^{1/(r-1)} \left\{ \exp \left[ (k - 2i + 1) \left( \frac{t}{n} \right)^{1/r} \right] + \exp \left[ (k - 2i - 1) \left( \frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) \, dt, \quad (2.7)
\]

where

\[
n^{(s-1)/(r+1)} \int_0^1 t^{1/(r-1)} \left\{ \exp \left[ (k - 2i + 1) \left( \frac{t}{n} \right)^{1/r} \right] + \exp \left[ (k - 2i - 1) \left( \frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) \, dt \\
= \sum_{p=0}^{\infty} \int_0^1 \frac{\left[ (k - 2i + 1)^p + (k - 2i - 1)^p \right] t^{(p/r)(1/r-1)-1}}{p! n^{(p/r)(1/r-1)-1}} \rho^{(s)}(t) \, dt. \quad (2.8)
\]

It follows that

\[
N - \lim_{n \to \infty} n^{s-1/r+1} \int_0^1 t^{1/(r-1)} \left\{ \exp \left[ (k - 2i + 1) \left( \frac{t}{n} \right)^{1/r} \right] + \exp \left[ (k - 2i - 1) \left( \frac{t}{n} \right)^{1/r} \right] \right\} \rho^{(s)}(t) \, dt \\
= \frac{(-1)^s s! \left[ (k - 2i + 1)^{rs+r-1} + (k - 2i - 1)^{rs+r-1} \right]}{2(rs + r - 1)!} \\
= c_{r,s,k,i}, \quad (2.9)
\]

and by applying the neutrix limit we obtain

\[
N - \lim_{n \to \infty} I_1 = N - \lim_{n \to \infty} \int_0^1 \delta_n^{(s)} \left( \sinh^{-1} x \right)^r x^k \, dx = \frac{1}{2^{s+1} r} \sum_{i=0}^{k} \binom{k}{i} (-1)^i c_{r,s,k,i} \quad (2.10)
\]

for \( k = 0, 1, 2, \ldots \).
When \( k = sr + r \), we have

\[
|I_k| = \int_0^1 \delta_n^{(s)} \left[ \left( \sinh^{-1} x_n \right)' \right] x^{sr+r} dx
\]

\[
= n^{s+1} \int_0^1 \rho_n^{(s)} \left[ n \left( \sinh^{-1} x_n \right)' \right] x^{sr+r} dx
\]

\[
\leq \frac{n^{(s-1)/(r+1)}}{2^{sr+r}} \exp(sr + r + 1) \int_0^1 \left| \left[ 1 - \exp \left[ -2 \left( \frac{t}{n} \right)^{1/r} \right] \right]^{sr+r} \rho^{(s)}(t) \right| dt
\]

\[
= \frac{n^{(s-1)/(r+1)}}{2^{sr+r}} \exp(sr + r + 1) \int_0^1 \left[ 2 \left( \frac{t}{n} \right)^{1/r} + O \left( n^{-2/r} \right) \right]^{sr+r} \left| \rho^{(s)}(t) \right| dt
\]

\[
\leq n^{-1/r} \exp(sr + r + 1) \int_0^1 \left[ 1 + O \left( n^{-2/r} \right) \right] \left| \rho^{(s)}(t) \right| dt
\]

\[
= O \left( n^{-1/r} \right).
\]

Thus, if \( \varphi \) is an arbitrary continuous function, then

\[
\lim_{n \to \infty} \int_0^1 \delta_n^{(s)} \left[ \left( \sinh^{-1} x_n \right)' \right] x^{rs+r} \varphi(x) dx = 0.
\]

We also have

\[
\int_{-1}^0 \delta_n^{(s)} \left[ \left( \sinh^{-1} x_n \right)' \right] \varphi(x) dx = n^{s+1} \int_{-1}^0 \rho^{(s)}(0) \varphi(x) dx,
\]

and it follows that

\[
N - \lim_{n \to \infty} \int_{-1}^0 \delta_n^{(s)} \left[ \left( \sinh^{-1} x_n \right)' \right] \varphi(x) dx = 0.
\]

If now \( \varphi \) is an arbitrary function in \( \mathcal{D}[-1, 1] \), then by Taylor’s Theorem, we have

\[
\varphi(x) = \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{rs+r}}{(rs+r)!} \varphi^{(rs+r)}(\xi),
\]
where \( 0 < \xi < 1 \), and so

\[
N - \lim_{n \to \infty} \left< \delta_n^{(s)} \left[ \left( \sinh^{-1} x_+ \right)^{1/r} \right], \varphi(x) \right>
\]

\[
= N - \lim_{n \to \infty} \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} \int_0^1 \delta_n^{(s)} \left[ \left( \sinh^{-1} x_+ \right)^{r} \right] x^k \, dx
\]

\[
+ N - \lim_{n \to \infty} \sum_{k=0}^{sr+r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{0} \delta_n^{(s)} \left[ \left( \sinh^{-1} x_+ \right)^{r} \right] x^k \, dx
\]

\[
+ \lim_{n \to \infty} \frac{1}{sr + r!} \int_0^1 \delta_n^{(s)} \left[ \left( \sinh^{-1} x_+ \right)^{r} \right] x^{sr+r} \varphi^{(sr+r)}(\xi x) \, dx
\]

\[
+ \lim_{n \to \infty} \frac{1}{sr + r!} \int_{-1}^{0} \delta_n^{(s)} \left[ \left( \sinh^{-1} x_+ \right)^{r} \right] x^{sr+r} \varphi^{(sr+r)}(\xi x) \, dx
\]

\[
= \sum_{k=0}^{sr+r-1} \sum_{i=0}^{k} \left( i \right) \left( \frac{r_{cr,k,i} \varphi^{(k)}(0)}{2^{k+1} k!} \right) + 0
\]

\[
= \sum_{k=0}^{sr+r-1} \sum_{i=0}^{k} \left( i \right) \left( \frac{(-1)^k r_{cr,k,i}}{2^{k+1} k!} \right) \delta^{(k)}(x), \varphi(x)
\]

on using (2.3) to (2.14). This proves (2.1) on the interval \((-1, 1)\).

It is clear that \(\delta^{(s)}[(\sinh^{-1} x_+)^{1/r}] = 0\) for \(x > 0\) and so (2.1) holds for \(x > -1\).

Now, suppose that \(\varphi\) is an arbitrary function in \(\mathcal{A}[a, b]\), where \(a < b < 0\). Then,

\[
\int_a^b \delta_n^{(s)} \left[ \left( \sinh^{-1} x_+ \right)^{r} \right] \varphi(x) \, dx = n^{s+1} \int_a^b \rho^{(s)}(0) \varphi(x) \, dx
\]

and so

\[
N - \lim_{n \to \infty} \int_a^b \delta_n^{(s)} \left[ \left( \sinh^{-1} x_+ \right)^{r} \right] \varphi(x) \, dx = 0.
\]

It follows that \(\delta^{(s)}[(\sinh^{-1} x_+)^{1/r}] = 0\) on the interval \((a, b)\). Since \(a\) and \(b\) are arbitrary, we see that (2.1) holds on the real line. This completes the proof of the theorem. \(\square\)

**Corollary 2.2.** The neutrix composition \(\delta^{(s)}[(\sinh^{-1} |x|)^{r}]\) exists and

\[
\delta^{(s)}[(\sinh^{-1} |x|)^{r}] = \sum_{k=0}^{sr+r-1} \sum_{i=0}^{k} \left( \frac{(-1)^k + 1}{2^{k+1} k!} \right) c_{cr,k,i} \delta^{(k)}(x), \quad (2.19)
\]

for \(s = 0, 1, 2, \ldots\) and \(r = 1, 2, \ldots\)
In particular, the composition $\delta(\sinh^{-1}|x|)$ exists and
\[
\delta(\sinh^{-1}|x|) = \frac{1}{2} \delta(x). \tag{2.20}
\]

**Proof.** To prove (2.19), we note that
\[
\int_{-1}^{1} \delta_n^{(s)} \left[ (\sinh^{-1}|x|)^{\prime} \right] x^k \, dx = n^{s+1} \int_{-1}^{1} \rho^{(s)} \left[ (n \sinh^{-1}|x|)^{\prime} \right] x^k \, dx
\]
\[
= n^{s+1} \left[ 1 + (-1)^k \right] \int_{0}^{1} \rho^{(s)} [n (\sinh^{-1}x)^{\prime}] x^k \, dx, \tag{2.21}
\]
and (2.19) now follows as above.

Equation (2.20) follows on noting that in the particular case $s = 0$, the usual limit holds in (2.10). This completes the proof of the corollary. \qed

**Theorem 2.3.** The neutrix composition $\delta^{(2s-1)} [ \sinh^{-1} (\text{sgn} \, x \cdot x^2) ]$ exists and
\[
\delta^{(2s-1)} [ \sinh^{-1} (\text{sgn} \, x \cdot x^2) ] = \sum_{k=0}^{2s-1+i} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^k b_{s,k,i}}{2^{k+1}(k+1)!} \delta^{(k)}(x), \tag{2.22}
\]
for $s = 1, 2, \ldots$, where
\[
b_{s,k,i} = (k-2i+1)^{2s-1} + (k-2i-1)^{2s-1}. \tag{2.23}
\]

**Proof.** To prove (2.22), we now have to evaluate
\[
\int_{-1}^{1} \delta_n^{(2s-1)} [ \sinh^{-1} (\text{sgn} \, x \cdot x^2) ] x^k \, dx. \tag{2.24}
\]
We have
\[
\int_{-1}^{1} \delta_n^{(2s-1)} [ \sinh^{-1} (\text{sgn} \, x \cdot x^2) ] x^k \, dx = n^{2s} \int_{-1}^{1} \rho^{(2s-1)} [n \sinh^{-1} (\text{sgn} \, x \cdot x^2)] x^k \, dx
\]
\[
= \begin{cases} 
2n^{2s} \int_{0}^{1} \rho^{(2s-1)} [n (\sinh^{-1}x^2)] x^k \, dx, & k \text{ odd}, \\
0, & k \text{ even},
\end{cases} \tag{2.25}
\]
Making the substitution \( t = n(\sinh^{-1}x^2) \), we have for large enough \( n \)

\[
\int_{-1}^{1} \delta_n^{(2s-1)} \left[ \sinh^{-1}(\text{sgn} x \cdot x^2) \right] x^k dx
\]

\[
= 2n^{2s} \int_{0}^{1} \rho^{(2s-1)} \left[ n \left( \sinh^{-1}x^2 \right) \right] x^{2k+1} dx
\]

\[
= \frac{n^{2s-1}}{2^{k+1}} \sum_{i=0}^{k} (-1)^i \int_{0}^{1} \left\{ \exp \left[ \frac{(k-2i+1)t}{n} \right] + \exp \left[ \frac{(k-2i-1)t}{n} \right] \right\} \rho^{(2s-1)}(t) dt,
\]

where

\[
n^{2s-1} \int_{0}^{1} \left\{ \exp \left[ \frac{(k-2i+1)t}{n} \right] + \exp \left[ \frac{(k-2i-1)t}{n} \right] \right\} \rho^{(s)}(t) dt
\]

\[
= \sum_{p=0}^{\infty} \int_{0}^{1} \frac{[(k-2i+1)^p + (k-2i-1)^p] t^p}{p!n^{p+2s+1}} \rho^{(2s-1)}(t) dt.
\]

It follows that

\[
N - \lim_{n \to \infty} n^{2s-1} \int_{0}^{1} \left\{ \exp \left[ \frac{(k-2i+1)t}{n} \right] + \exp \left[ \frac{(k-2i-1)t}{n} \right] \right\} \rho^{(s)}(t) dt
\]

\[
= N - \lim_{n \to \infty} \sum_{p=0}^{\infty} \int_{0}^{1} \frac{[(k-2i+1)^p + (k-2i-1)^p] t^p}{p!n^{p+2s+1}} \rho^{(2s-1)}(t) dt
\]

\[
= \frac{-(k-2i+1)^{2s-1} + (k-2i-1)^{2s-1}}{2}
\]

\[
= \frac{b_{s,k,i}}{2},
\]

and so by using the neutrix limit, we have

\[
N - \lim_{n \to \infty} \int_{-1}^{1} \delta_n^{(2s-1)} \left[ \sinh^{-1}(\text{sgn} x \cdot x^2) \right] x^{2k+1} dx = \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i b_{s,k,i}}{2^{k+1}},
\]

for \( k = 0, 1, 2, \ldots \).
When \( k = 2s \), we have
\[
\int_{-1}^{1} \sinh^{-1}(\text{sgn } x \cdot x^2) x^{4s+1} \, dx = n^{2s-1} \int_{-1}^{1} \rho^{(2s-1)} \left[ n \left( \sinh^{-1} x^2 \right) \right] x^{4s+1} \, dx
\]
\[
\leq n^{2s-1} \exp(s + 1) \int_{-1}^{1} \left[ 1 - \exp \left( -\frac{2t}{n} \right) \right] \rho^{(2s-1)}(t) \, dt
\]
\[
= n^{2s-1} \exp(s + 1) \int_{-1}^{1} \left[ \frac{2t}{n} + O \left( n^{-2} \right) \right] \rho^{(2s-1)}(t) \, dt
\]
\[
\leq 2^{2s+1} n^{-1} \exp(s + 1) \int_{-1}^{1} \left[ 1 + O \left( n^{-2/\rho} \right) \right] \rho^{(2s-1)}(t) \, dt
\]
\[
= O \left( n^{-1} \right).
\] (2.30)

Thus, if \( \varphi \) is an arbitrary continuous function, then
\[
\lim_{n \to \infty} \int_{-1}^{1} \delta_n^{(2s-1)} \left[ \sinh^{-1}(\text{sgn } x \cdot x^2) \right] x^{4s+1} \varphi(x) \, dx = 0.
\] (2.31)

If now \( \varphi \) is an arbitrary function in \( \mathcal{D}[-1,1] \), then by Taylor’s Theorem, we have
\[
\varphi(x) = \sum_{k=0}^{4s} \frac{\varphi^{(k)}(0)}{k!} x^k + x^{4s+1} \frac{\varphi^{(4s+1)}(\xi x)}{(4s+1)!},
\] (2.32)

where \( 0 < \xi < 1 \), and so
\[
N - \lim_{n \to \infty} \left\langle \delta_n^{(2s-1)} \left[ \sinh^{-1}(\text{sgn } x \cdot x^2) \right], \varphi(x) \right\rangle
\]
\[
= N - \lim_{n \to \infty} \sum_{k=0}^{2s-1} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \int_{-1}^{1} \delta_n^{(2s-1)} \left[ \sinh^{-1}(\text{sgn } x \cdot x^2) \right] x^{2k+1} \, dx
\]
\[
+ \lim_{n \to \infty} \frac{1}{(4s+1)!} \int_{-1}^{1} \delta_n^{(4s+1)} \left[ \sinh^{-1}(\text{sgn } x \cdot x^2) \right] x^{4s+1} \varphi^{(4s+1)}(\xi x) \, dx
\] (2.33)
\[
= \sum_{k=0}^{2s-1} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{l+1} b_{s,k,l} \varphi^{(k)}(0)}{2^{k+1}(2k+1)!} + 0
\]
\[
= \sum_{k=0}^{2s-1} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{l+k+1} b_{s,k,l}}{2^{k+1}(2k+1)!} \left\langle \delta_n^{(k)}(x), \varphi(x) \right\rangle,
\]
on using (2.25) to (2.31), proving (2.22) on the interval \((-1,1)\). However, it is clear that \( \delta_n^{(2s-1)} \left[ \sinh^{-1}(\text{sgn } x \cdot x^2) \right] = 0 \) for \( |x| > 0 \) and so (2.22) holds on the real line, completing the proof of the theorem. \( \square \)
Corollary 2.4. The composition $\delta'[\sinh^{-1}\text{sgn }x \cdot x^2]$ exists and

$$\delta'\left[\sinh^{-1}\left(\text{sgn }x \cdot x^2\right)\right] = \frac{\delta'(x)}{4.3!} - 2\delta(x).$$  \hspace{1cm} (2.34)

Proof. To prove (2.34) note that in the particular case $s = 1$, the usual limits hold and then (2.34) is a particular case of (2.22). This completes the proof of the corollary. \qed

For further related results on the neutrix operation of distributions, see [12–22] and [2, 3, 23].

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