Some Refinements of the Hodge Decomposition and Applications

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(Received and accepted December 1999)

Abstract—We extend the nonlinear Hodge decomposition of Iwaniec et al. [1] to other vector fields. We give applications to Leray-Lions type equations with very weak assumptions on data similar to those developed in [1]. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Hodge decomposition, Quasilinear equation, Measures.

1. INTRODUCTION

We give some slight new extensions of the Hodge decomposition due to Iwaniec [2] (see, also [3-5]) by proving essentially that for \( v \in W^{1,n-\epsilon n}(\Omega), \Omega \subset \mathbb{R}^n \) the vector field \( Dv(\|Dv\| + \epsilon)^{-\epsilon n} \) can be written as \( D\psi_z + K_z \) with \( \psi_z \in W^{1,r}(\Omega), r > n \) and \( K_z \) is a free divergence vector and is small, i.e., \( \|K_z\|_{r_z} \leq c\epsilon \|Dv\|_{n-\epsilon n}^{1-\epsilon n} + c\epsilon^2 - \epsilon^n \). In [1], the functions do vanish on the boundary and they decompose the vector field, \( DvDv^{-\epsilon n} \). The main motivations of such decomposition is the study of equivalence problems for nonlinear boundary value problems with measured data. Many authors studied the existence and uniqueness of the solutions of such problems and different formulations have been introduced (see [1,3,6-11]). In particular, in [1], the authors consider the homogeneous Dirichlet problem for the equation

\[
- \text{div}(\hat{a}(x, u, Du)) + b(u) = \mu, \quad \text{in } D'(\Omega),
\]

where \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \), \( \hat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a Carathéodory function with usual upper growth and such that \( \hat{a}(x, \eta, \xi)ξ \geq \alpha|\xi|^n \) for a.e., \( x \in \Omega \), for all \( (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n \) and \( b(u) \in L^1(\Omega) \). They prove that it is equivalent to find solutions in one of the three spaces.
V₀, W₀, L₀¹ⁿ(Ω) (see Section 3 for their definitions). A basic tool in their proofs is the classical Hodge decomposition, unfortunately their proofs do not work in the case where the coercivity assumption on ̂a is replaced by the weaker coercivity assumption

\[ ̂a(x, \eta, \xi) \geq \alpha |\xi|^n - \beta |\eta|^s - a_1(x), \]  

where \( \alpha > 0, \beta \geq 0, s \geq 0, a_1 \in L¹(Ω) \). For example the function ̂a \( \equiv (\hat{a}_1, \cdots, \hat{a}_n) \) with

\[ \hat{a}_i(x, \eta, \xi) = |\xi|^{n-2}\xi_i + b_i(x)\eta_i, i = 1, \ldots, n \]

satisfies (1.2). The new types of Hodge decomposition that we introduce allow us to say that solving the equation −div(̂a(x, u, Du)) + F(x, u, Du) = µ in \( \mathcal{D}'(Ω) \) if ̂a satisfies (1.2) and F verifies suitable assumptions, in \( V₀, W₀, L₀¹ⁿ(Ω) \) is totally equivalent. Actually, we will prove the results in the case of the Neumann problem. Moreover, in the case \( µ \in L¹(Ω) \), besides weak, solutions we also consider renormalized solutions in the sense of Di Perna-Lions (see [8,11,12]) and entropy solutions introduced in [6]. By similar arguments, we have that all these different formulations are equivalent.

2. SOME REFINEMENTS OF THE HODGE DECOMPOSITION

The following theorem holds.

**Theorem 2.1.** Let \( Ω \) be an open bounded smooth subset of \( \mathbb{R}^n \) and let \( \varepsilon \in ]0, 1/n[. \) For all \( v \in W₀¹ⁿ⁻\varepsilon(Ω) \) there exist \( \Psi_\varepsilon \in W₀¹ⁿ⁻\varepsilon(Ω) \) and \( K_\varepsilon \in L¹(Ω) \), where \( r_\varepsilon = (n - \varepsilon n)/(1 - \varepsilon n) \) such that

\[ \text{div} K_\varepsilon = 0 \quad \text{and} \quad Dv(IDv + \varepsilon )⁻\varepsilon = DΨ_\varepsilon + K_\varepsilon. \]

Moreover,

\[ \|K_\varepsilon\|_{r_\varepsilon} \leq c(n)\varepsilon \|Dv\|_{n⁻\varepsilon} + c(n, Ω)\varepsilon²⁻\varepsilon, \]  

(2.1)

\[ \|Ψ_\varepsilon\|_∞ \leq c(n) \left( |Ω|^r⁻¹ \right)^{1/(r_\varepsilon)} \|Dv\|_{n⁻\varepsilon}, \]  

(2.2)

where \( (r_\varepsilon)' = r_\varepsilon/r_\varepsilon - 1 \) is the Hölder conjugate of \( r_\varepsilon \). \( \|\cdot\|_r \) denotes the norm in \( L^r \).

**Sketch of the Proof.** If we set \( F_\varepsilon = Dv(|Dv| + \varepsilon )⁻\varepsilon \), since \( F_\varepsilon \in L^r(Ω) \) the problem

\[ \Delta \Psi_\varepsilon = \text{div}(F_\varepsilon), \quad \text{in} \ Ω, \]

\[ \Psi_\varepsilon = 0, \quad \text{in} \ ∂Ω, \]

has a unique solution \( Ψ_\varepsilon \in W₀¹ⁿ⁻\varepsilon(Ω) \) (see [13,14]). Moreover, the following estimate holds

\[ \|DΨ_\varepsilon\|_{r_\varepsilon} \leq c(n)\|F_\varepsilon\|_{r_\varepsilon} \leq c(n, Ω)\|Dv\|_{n⁻\varepsilon}. \]

Then, using techniques similar to those used in [1], we obtain

\[ \|Ψ_\varepsilon\|_∞ \leq c(n) \left( |Ω|^r⁻¹ \right)^{1/(r_\varepsilon)} \|DΨ_\varepsilon\|_{r_\varepsilon} \leq c(n) \left( |Ω|^r⁻¹ \right)^{1/(r_\varepsilon)} \|Dv\|_{n⁻\varepsilon}. \]

If we put \( K_\varepsilon = Dv(|Dv| + \varepsilon )⁻\varepsilon - DΨ_\varepsilon, \) to complete the proof we have only to prove (2.1). To this aim, let us consider the classical Hodge decomposition (see [2,5]) \( Dv|Dv|⁻\varepsilon = DΨ_\varepsilon + H_\varepsilon. \)

We have \( \|K_\varepsilon\|_{r_\varepsilon} \leq \|K_\varepsilon - H_\varepsilon\|_{r_\varepsilon} + \|H_\varepsilon\|_{r_\varepsilon}. \) In order to get the desired estimate, we will use the estimate of \( \|H_\varepsilon\|_{r_\varepsilon} \) given in [2,5] and the fact that \( \|K_\varepsilon - H_\varepsilon\|_{r_\varepsilon} \leq c(n)\|δF_\varepsilon\|_{r_\varepsilon}, \) where

\[ δF_\varepsilon = \frac{Dv}{|Dv|⁻\varepsilon} - \frac{Dv}{(|Dv| + \varepsilon )⁻\varepsilon}, \]

Inequality (2.1) can be then obtained using the following inequality:

\[ \frac{t}{t^\varepsilon} - \frac{t}{(t + \varepsilon)^\varepsilon} \leq n\varepsilon²⁻\varepsilon, \]

in order to get an estimate for \( \|δF_\varepsilon\|_{r_\varepsilon}. \)

**Theorem 2.1 can be extended to the case of functions that do not vanish on the boundary of Ω.**

Using the extension theorem (see [15]) and cut-off functions it is possible to prove the following.
**Theorem 2.2.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$, with smooth boundary $\partial \Omega$, for example, $C^1$ and let $\varepsilon \in [0, 1/n]$. For all $v \in W^{1,n-\varepsilon n}(\Omega)$ there exist $\Psi_\varepsilon \in W^{1,\varepsilon n}(\Omega)$ and $K_\varepsilon \in L^{\varepsilon n}(\Omega)$, where $r_\varepsilon = (n - \varepsilon n)/(1 - \varepsilon n)$ such that

$$\text{div } K_\varepsilon = 0, \quad Dv(\lvert Dv \rvert + \varepsilon)^{-\varepsilon n} = D\Psi_\varepsilon + K_\varepsilon.$$

Moreover,

$$\|K_\varepsilon\|_{r_\varepsilon} \leq c(n, \lvert \Omega \rvert)\varepsilon\|v\|^{1-\varepsilon n}_{W^{1,n-\varepsilon n}(\Omega)} + c(n, \Omega)\varepsilon^2 - \varepsilon n,$$

$$\|\Psi_\varepsilon\|_{\infty} \leq c(n, \Omega)\varepsilon^{-1/(r_\varepsilon)}\|v\|^{1-\varepsilon n}_{W^{1,n-\varepsilon n}(\Omega)}.$$

### 3. Hypotheses, Sets, and Definitions

Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let us consider the three following spaces: $V = \bigcap_{\varepsilon \in [0, 1/n]} W^{1,n-\varepsilon n}(\Omega)$; $W = \{v \in V : \sup_{\varepsilon \in [0, 1/n]} \int_{\Omega} |Dv|^{n-\varepsilon n} < \infty\}$;

$$L^{1,n}(\Omega) = \{v : \Omega \to \mathbb{R} \text{ measurable} : \Phi(v) \in W^{1,n}(\Omega), \forall \Phi \in Lip_0(\mathbb{R}), |v| < +\infty\},$$

where $|v| = \sup_{\delta > 0} \int_{\Omega} |Dv|^n(1 + |v|^{1+\delta})^{-1}$, where

$$Lip_0(\mathbb{R}) = \{\Phi \in W^{1,\infty}(\mathbb{R}) : \Phi' \in L^n(\mathbb{R}), \Phi(0) = 0\}.$$

The spaces $V_0$, $W_0$, $L^{1,n}_0(\Omega)$ are obtained if we replace in the above definitions $W^{1,\varepsilon n}(\Omega)$ by $W^{1,\varepsilon n}_0(\Omega)$. Let us consider the following three problems:

- \begin{align*}
  -\text{div} (a(x, u, Du)) + F(x, u, Du) &= \mu, & \text{in } \Omega, \\
  \hat{a} \cdot \hat{n} + C(x, u) &= \nu, & \text{on } \partial \Omega, \\
  u &\in V, \quad F(x, u, Du) \in L^1(\Omega), \quad C(x, u) \in L^1(\partial \Omega),
  \end{align*}

  \quad (P_1)

- \begin{align*}
  -\text{div} (a(x, u, Du)) + F(x, u, Du) &= \mu, & \text{in } \Omega, \\
  \hat{a} \cdot \hat{n} + C(x, u) &= \nu, & \text{on } \partial \Omega, \\
  u &\in W, \quad F(x, u, Du) \in L^1(\Omega), \quad C(x, u) \in L^1(\partial \Omega),
  \end{align*}

  \quad (P_2)

- \begin{align*}
  -\text{div} (a(x, u, Du)) + F(x, u, Du) &= \mu, & \text{in } \Omega, \\
  \hat{a} \cdot \hat{n} + C(x, u) &= \nu, & \text{on } \partial \Omega, \\
  u &\in L^{1,n}(\Omega) \cap L^{1-1/n}(\Omega), \quad F(x, u, Du) \in L^1(\Omega), \quad C(x, u) \in L^1(\partial \Omega),
  \end{align*}

  \quad (P_3)

where $\hat{n}$ denotes the outer normal to $\partial \Omega$ and the data satisfy

- **(H1)** $a(x, \eta, \xi)$ is a Carathéodory function such that there are: $\alpha > 0$, $\beta \geq 0$, $s \geq 0$, $a_1 \in L^1(\Omega)$

  $$a(x, \eta, \xi) \geq a_1|\xi|^n - \beta|\eta|^s - a_1(x), \quad \text{for a.e., } x \in \Omega, \quad \forall (\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n,$$

  $$|a(x, \eta, \xi)| \leq c (|\xi|^{n-1} + |\eta|^{n-1} + a_2(x)), \quad a_2 \in L^{n/(n-1)}(\Omega);$$

- **(H2)** $\mu \in M_b(\Omega)$, $\nu \in M_b(\partial \Omega)$, being $M_b$ the set of bounded Radon measures; the three formulations are taken in the following sense: $\forall \varphi \in \bigcup_{r > n} W^{1,r}(\Omega)$

  $$\int_{\Omega} a(x, u, Du) D\varphi + \int_{\Omega} F(x, u, Du) \varphi + \int_{\partial \Omega} C(x, u) \varphi = \int_{\Omega} \varphi d\mu + \int_{\partial \Omega} \varphi dv.$$  \quad (3.1)
If $\mu \in L^1(\Omega)$ and $\nu \in L^1(\partial \Omega)$, besides solutions of problems $(P_i)$, $i = 1, 2, 3$ in the sense of (3.1), we will consider renormalized solutions in the sense of [12] and entropy solutions in the sense of [6]. That is, we will take the following formulations:

$$
\int_{\Omega} \tilde{a}(x, u, Du) DT(u)\varphi + \int_{\Omega} F(x, u, Du)T(u)\varphi + \int_{\partial \Omega} C(x, u)T(u)\varphi
= \int_{\Omega} \mu T(u)\varphi + \int_{\partial \Omega} \nu T(u)\varphi
$$

$\forall \varphi \in W^{1,n}(\Omega) \cap L^\infty(\Omega)$, $\forall T \in W^{1,\infty}(\mathbb{R})$, supp $(T)$ is compact

$u \in L^{1,n}(\Omega) \cap L^{1-1/n}(\Omega)$, $F(\cdot, u, Du) \in L^1(\Omega)$, $C(\cdot, u) \in L^1(\partial \Omega)$,

and $\int_{\gamma \in [u] \leq \gamma + 1} |Du|^n = o(1)$, when $\gamma \to +\infty$.

$$
\int_{\Omega} \tilde{a}(x, u, Du) DT_k(u - \varphi) + \int_{\Omega} F(x, u, Du)T_k(u - \varphi) +
\int_{\partial \Omega} C(x, u)T_k(u - \varphi) = \int_{\Omega} \mu T_k(u - \varphi) + \int_{\partial \Omega} \nu T_k(u - \varphi),
$$

$\forall \varphi \in W^{1,n}(\Omega) \cap L^\infty(\Omega)$, $F(\cdot, u, Du) \in L^1(\Omega)$, $C(\cdot, u) \in L^1(\partial \Omega)$

$u \in \bigcap_{\epsilon \in [0,1/n]} W^{1,n-\epsilon n}(\Omega)$, $T_k(u) \in W^{1,n}(\Omega)$, $\forall k > 0$.

where $T_k$ is the truncated function of $u$ at the level $k$, that is $T_k(\sigma) = \sigma$ if $|\sigma| \leq k$, $T_k(\sigma) = k \text{sign}(\sigma)$ if $|\sigma| > k$.

### 4. MAIN RESULTS FOR QUASILINEAR EQUATIONS

All the formulations given in the previous section are equivalent (see also [1,8,16]). More precisely we have the following.

**Theorem 4.1.** Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial \Omega$, for example $C^1$. Then, under Assumptions (H1) and (H2) the three problems $(P_i)$, $i = 1, 2, 3$ are equivalent.

**Theorem 4.2.** Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial \Omega$, for example $C^1$. Then, under Assumption (H1), $\mu \in L^1(\Omega)$, and $\nu \in L^1(\partial \Omega)$, problems $(P_R),(P_E)$ are equivalent to problems $(P_i)$, $i = 1, 2, 3$.

We remark that Theorems 4.1 and 4.2 still hold if we replace the Neumann boundary condition with the homogeneous Dirichlet condition and $V$, $W$, $L^{1,n}(\Omega) \cap L^{1-1/n}(\Omega)$ with $V_0$, $W_0$, $L^{1,n}_0(\Omega)$, respectively. The proofs of the theorems presented above and some more results like uniqueness will be given in a detailed paper [17].

### REFERENCES