Local numerical range for a class of $2 \otimes d$ hermitian operators

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Abstract
A local numerical range is analyzed for a family of circulant observables and states of composite $2 \otimes d$ systems. It is shown that for any $2 \otimes d$ circulant operator $O$ there exists a basis giving rise to the matrix representation with real non-negative off-diagonal elements. In this basis the problem of finding extremum of $O$ on product vectors $|x⟩ \otimes |y⟩ \in \mathbb{C}^2 \otimes \mathbb{C}^d$ reduces to the corresponding problem in $\mathbb{R}^2 \otimes \mathbb{R}^d$. The final analytical result for $d = 2$ is presented.

1 Introduction
For any linear operator $O$ acting in the Hilbert space $\mathcal{H}$ one defines its numerical range \[ \text{NR}(O) := \{ ⟨ψ|O|ψ⟩ | ψ ∈ \mathcal{H}, ||ψ|| = 1 \} . \] (1) Clearly, NR($O$) defines a subset of the complex plane. Now, if $O$ is hermitian then $\text{NR}(O) = [λ_{\text{min}}, λ_{\text{max}}]$, where $λ_{\text{min}}$ and $λ_{\text{max}}$ denote the minimal and maximal eigenvalue of $O$. Recently, more specific characterization of the hermitian operator called restricted numerical range has been introduced in order to describe the interval of expectation values for some specific sets of vectors in $\mathcal{H}$. In particular, if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ one introduces the notion of local (product) numerical range \[ \text{LNR}(O) = \{ ⟨x \otimes y|O|x \otimes y⟩ : ||x|| = ||y|| = 1 , |x⟩ ∈ \mathcal{H}_1 , |y⟩ ∈ \mathcal{H}_2 \} . \] (2) It is clear that if $O$ is hermitian then \[ \text{LNR}(O) = [γ_{\text{min}}, γ_{\text{max}}] \subseteq \text{NR}(O) = [λ_{\text{min}}, λ_{\text{max}}] . \] It turns out that the notions of various restricted numerical ranges are useful in many branches of quantum information theory (see [3, 5, 6] for details). For example any entanglement witness $W$ can be written in the following form \[ W = χ\mathbb{I} - O , \] for some hermitian operator $O$ and a positive number $χ$. Now, the necessary condition for $W$ to be an entanglement witness is $χ > γ_{\text{max}}$. In practice, it is very hard to determine LNR for a given hermitian operator. In this paper we limit ourselves to the case when $O$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^d$ belongs to a class of circulant operators [11] (see also [12, 13]).
The paper is organized as follows. Sect. 2 is devoted to some basic definitions and properties of circulant bipartite operators. In Sect. 3, we emphasize that it is always possible to bring a matrix representing the circulant operator to the so-called real form using a local unitary transformation. In Sect. 4 we show how to carry out calculations of the local numerical range for circulant operators. The final analytical result for $d = 2$ is presented in Sect. 5 together with some instructive examples.

## 2 Circulant operators in $C^2 \otimes C^d$

Let $\mathcal{H} = C^2 \otimes C^d$ and let $\{|g_i\rangle \otimes |f_k\rangle\}$ ($i = 1, 2, k = 1, \ldots, d$) be an orthonormal product basis in $\mathcal{H}$. One defines the family of 2-dimensional subspaces $\Sigma_k$ in $\mathcal{H}$:

\[
\Sigma_1 = \text{span}\{|g_1\rangle \otimes |f_1\rangle, |g_2\rangle \otimes |f_2\rangle\},
\]

\[
\Sigma_2 = \text{span}\{|g_1\rangle \otimes |f_2\rangle, |g_2\rangle \otimes |f_3\rangle\}
\]

\[\vdots\]

\[
\Sigma_d = \text{span}\{|g_1\rangle \otimes |f_d\rangle, |g_2\rangle \otimes |f_1\rangle\}.
\]

It is clear that $\Sigma_k$ give rise to the direct sum decomposition $[11, 19]

\[C^2 \otimes C^d = \bigoplus_{k=1}^{d} \Sigma_k.\tag{3}\]

We shall call (3) a circulant decomposition. Now, we call a linear operator $O \in B(\mathcal{H})$ to be circulant operator with respect to a circulant decomposition (3) iff

\[O = O_1 \oplus \ldots \oplus O_d,\tag{4}\]

where $O_k$ is supported on $\Sigma_k$, that is,

\[O_k = \sum_{i,j=1}^{2} a_{ij}^{(k)} |g_i\rangle \langle f_{i+k} | \otimes | g_{j+k} \rangle \langle f_j |, \mod d,\tag{5}\]

and $||a_{ij}^{(k)}||$ is a $2 \times 2$ complex matrix. In particular, for $d = 2$ and $d = 3$ we obtain the following matrix representations of the circulant operators (in the basis $|g_i\rangle \otimes |f_k\rangle$)

\[\begin{pmatrix}
    a_{11}^{(2)} & \cdots & a_{12}^{(2)} \\
    \cdots & a_{11}^{(1)} & a_{12}^{(1)} \\
    a_{21}^{(2)} & \cdots & a_{22}^{(2)}
\end{pmatrix}, \quad \begin{pmatrix}
    a_{11}^{(3)} & \cdots & a_{12}^{(3)} \\
    \cdots & a_{11}^{(2)} & a_{12}^{(2)} \\
    a_{21}^{(3)} & \cdots & a_{22}^{(3)}
\end{pmatrix}, \quad \begin{pmatrix}
    a_{11}^{(1)} & \cdots & a_{12}^{(1)} \\
    \cdots & a_{11}^{(2)} & a_{12}^{(2)} \\
    a_{21}^{(1)} & \cdots & a_{22}^{(1)}
\end{pmatrix}, \quad \begin{pmatrix}
    a_{11}^{(1)} & \cdots & a_{12}^{(1)} \\
    \cdots & a_{11}^{(2)} & a_{12}^{(2)} \\
    a_{21}^{(1)} & \cdots & a_{22}^{(1)}
\end{pmatrix}, \tag{6}\]

where to make the picture more transparent we replaced all zeros by dots. Interestingly for $d = 2$ the circulant matrix displays characteristic X-shape. Such 2-qubit states have been recently investigated in [14, 15, 16, 17, 18]. In the following we limit ourselves to circulant states and
observables, i.e. hermitian circulant matrices, only. Let us introduce a more convenient notation and denote by
\[ w_{ik} = a_{ik}^{(k-1)} \pmod d , \]
for \( i = 1, 2, k = 1, \ldots, d \), and
\[ a_{12}^{(k+2)} = u_k e^{i\alpha_k} , \]
where \( u_k = |a_{12}^{(k+2)}| \geq 0 \), and \( \alpha_k \in (-\pi, \pi] \). As a consequence, the general circulant observable reads
\[ \mathcal{O} = \sum_{i=1}^{2} \sum_{k=1}^{d} w_{ik} |g_i\rangle \langle g_i| \otimes |f_k\rangle \langle f_k| + \left( \sum_{k=1}^{d} u_k e^{i\alpha_k} |g_1\rangle \langle g_2| \otimes |f_k\rangle \langle f_{k+1}| + \text{h.c.} \right) , \]
where as usual h.c. stands for hermitian conjugation.

3 Real representation of circulant operators

Let \( \mathcal{O} \) be an hermitian circulant operator living in \( \mathbb{C}^2 \otimes \mathbb{C}^d \). One has the following

**Proposition 1** There exists an orthonormal product basis \( \{ |g_i\rangle \otimes |f_{k}\rangle \} \) such that

1. \( \mathcal{O} \) is circulant with respect to the circulant decomposition constructed out of \( \{ |g_i\rangle \otimes |f_{k}\rangle \} \).
2. matrix elements of \( \mathcal{O} \) with respect to \( \{ |g_i\rangle \otimes |f_{k}\rangle \} \) satisfy:

\[ w'_{ik} = w_{ik} , \quad a_{12}^{(k+2)\prime} = |a_{12}^{(k+2)}| = u_k . \]

**Proof.** Let \( |g_i\rangle = U_1 |g_i\rangle \) and \( |f_{k}\rangle = U_2 |f_k\rangle \), where \( U_1 \) and \( U_2 \) are unitary operators with the following matrix representations in the original basis \( |g_i\rangle \) and \( |f_k\rangle \):
\[ U_1 = D[1, e^{i\mu_1}] , \quad U_2 = D[1, e^{i\mu_2}, \ldots, e^{i\mu_k}] , \]
where \( D[a_1, \ldots, a_k] \) denotes diagonal \( k \times k \) matrix with diagonal entries \( a_1, \ldots, a_k \). One has
\[ \mathcal{O} = \sum_{i=1}^{2} \sum_{k=1}^{d} w_{ik} |g_i\rangle \langle g_i| \otimes |f_{k}\rangle \langle f_{k}| + \left( \sum_{k=1}^{d} u_k e^{i\vartheta_k} |g_1\rangle \langle g_2| \otimes |f_{k}\rangle \langle f_{k+1}| + \text{h.c.} \right) , \]
where the phases \( \vartheta_k \) satisfying the following relations (mod(2\( \pi \))
\[ \vartheta_1 = \alpha_1 - \mu_1 - \mu_2 , \]
\[ \vartheta_k = \alpha_k - \mu_1 + \mu_k - \mu_{k+1} , \quad k = 2, \ldots, d - 1 \]
\[ \vartheta_d = \alpha_d - \mu_1 + \mu_d . \]

Formula (10) proves that \( \mathcal{O} \) is circulant with respect to the circulant decomposition constructed out of \( \{ |g_i\rangle \otimes |f_{k}\rangle \} \). Now, we show that one can remove all the phases \( \vartheta_k \) by the appropriate choice of \( \mu_k \). Note, that (11) may be rewritten as a matrix equation \( \alpha - \vartheta = W \mu \), where the matrix \( W \) is defined by
\[ W_{k1} = 1 , \]
\[ W_{kk} = -W_{k,k+1} , \quad k > 1 , \]
and the remaining elements vanish. Note that taking $d$-vector $\mu = (\mu_1, \ldots, \mu_d)$ which satisfies the matrix equation

$$\alpha = W\mu,$$

one finds $\vartheta = 0$. It can be done due to the fact that $\det W = d(-1)^{d+1} \neq 0$ which ends the proof.

We will call the corresponding matrix representation of $O$ with respect to $\{|g'_i \otimes |f'_k\rangle\}$ real representation.

4 Local Numerical Range for a Circulant Operator

Let $O$ be an hermitian circulant operator with respect to a fixed basis $|g_i\rangle \otimes |f_k\rangle$ in $C^2 \otimes C^d$, and let us define

$$F(x, y) = \frac{\langle x \otimes y | O | x \otimes y \rangle}{\langle x \otimes y | x \otimes y \rangle}. \quad (14)$$

Now to provide $\text{LNM}(O)$ one has to find $\gamma_{\min} = \inf F(x, y)$ and $\gamma_{\max} = \sup F(x, y)$. Let

$$\gamma_{\min} = F(x^-, y^-), \quad \gamma_{\max} = F(x^+, y^+). \quad (15)$$

One has the following

**Proposition 2** The corresponding vectors $|x^\pm\rangle \in C^2$ and $|y^\pm\rangle \in C^d$ have the following components with respect to basis $|g'_i\rangle$ and $|f'_k\rangle$ provided in Proposition 1

$$|x^\pm\rangle = (x_1^\pm, x_2^\pm), \quad |y^\pm\rangle = (y_1^\pm, \ldots, y_d^\pm), \quad (16)$$

where

$$x_1^\pm \geq 0, \quad y_k^\pm \geq 0. \quad (17)$$

**Proof.** Consider e.g. $\gamma_{\min}$ and to simplify notation let us write simply $|x\rangle$ and $|y\rangle$ instead of $|x^-\rangle$ and $|y^-\rangle$, respectively. Moreover, let us introduce the following parametrization of vectors $|x\rangle \in C^2$ and $|y\rangle \in C^d$ in the original basis $|g_i\rangle \otimes |f_k\rangle$:

$$|x\rangle = (x_1, x_2 e^{i\beta_1}), \quad |y\rangle = (y_1, y_2 e^{i\beta_2}, \ldots, y_d e^{i\beta_d}), \quad x_1, x_2 \geq 0, \quad y_1, \ldots, y_d \geq 0. \quad (18)$$

Using (7) one obtains

$$\langle x \otimes y | O | x \otimes y \rangle = \sum_{i=1}^{2} \sum_{k=1}^{d} w_{ik} x_i^2 y_k^2 + 2x_1 x_2 \sum_{k=1}^{d} y_k y_{k+1} u_k \cos \varphi_k, \quad (19)$$

where

$$\varphi_1 = \alpha_1 + \beta_1 + \beta_2, \quad \varphi_k = \alpha_k + \beta_1 - \beta_k + \beta_{k+1}, \quad k = 2, \ldots, d - 1 \quad (20)$$

The extremalization procedure leads to the set of equations for real positive variables $x_1, x_2, y_1, \ldots, y_d$ and for the phases $\beta_1, \ldots, \beta_d$ (see Appendix for details). In particular, phases $\beta_k$ can be easily obtained in the generic case, i.e., for $x_i \neq 0$, and $y_k \neq 0$, as shown in (54). Using simple algebra (see the Appendix) one finds

$$\beta_k = -\mu_k, \quad k = 1, \ldots, d. \quad (21)$$
where $\mu_k$ are solutions of (13). Hence in the new basis $|g'_i\rangle \otimes |f'_k\rangle$ the phases $\beta_k$ are completely removed and the components of $|x\rangle$ and $|y\rangle$ are non-negative.

Hence, essentially LNR($O$) calculations can be done in $\mathbb{R}^2 \otimes \mathbb{R}^d$ instead of $\mathbb{C}^2 \otimes \mathbb{C}^d$. Unfortunately, solving the set of $d + 2$ polynomial equations (14), (15) is in general very hard. Keeping in mind that in the basis $\{|g'_i\rangle \otimes |f'_k\rangle\}$ all $\varphi_k = 0$, we can rewrite (14) as

$$\begin{cases}
(A_1(y) - \lambda_1)x_1 + B(y)x_2 = 0, \\
B(y)x_1 + (A_2(y) - \lambda_1)x_2 = 0,
\end{cases}$$

with

$$A(y) = \sum_{k=1}^d w_{1k} y_k^2, \quad B(y) = \sum_{k=1}^d u_k y_{k+1}.$$ 

Now, we obtain the nonzero solution for $x_1, x_2$ from a linear set of equations (22) if

$$\lambda_1^\pm = \frac{1}{2} \left( A_1(y) + A_2(y) \pm \sqrt{(A_1(y) - A_2(y))^2 + 4B(y)^2} \right).$$

Let us write this solution as

$$x_1 = \frac{1}{\sqrt{1 + C_1^2}}, \quad x_2 = C_\pm x_1 = \frac{C_\pm y_1}{\sqrt{1 + C_\pm^2}},$$

where

$$C_\pm = \frac{\lambda_1^\pm - A_1(y)}{B(y)}$$

and the normalization of $|x\rangle$ has been taken into account. Putting (23) into (15) we arrive at the following set of $d$ nonlinear equations for $y_1, \ldots, y_d$:

$$\left[ \frac{1}{C_\pm} w_{1k} + C_\pm w_{2k} - \frac{1 + C_\pm^2}{C_\pm} y_k + u_{k-1} y_{k-1} + u_k y_{k+1} = 0, \quad k = 1, \ldots, d. \right.$$ (24)

Clearly, in general the solution of (24) is not feasible. Note however that when $A_1(y) = A_2(y)$, i.e. $w_{1k} = w_{2k}$ for $k = 1, \ldots, d$, one gets $C_\pm = \pm 1$ and the set of equations (24) becomes linear.

**Example 1** Let us consider circulant hermitian operator $O$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$ represented in the standard computational basis by the following real matrix

$$M_O = \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
1 & 0 & 0 & 2
\end{pmatrix}. $$

The spectrum of $M_O$ is $\{0, 1, 2, 3\}$. As a consequence, NR($O$) = $[0, 3]$, whereas, as we shall see, LNR($O$) = $[0.5, 2.5]$. Moreover, the upper bound $\gamma_{\text{max}}$ is achieved at complex vectors $|x\rangle = \frac{1}{\sqrt{2}} (1, i)$ and $|y\rangle = \frac{1}{\sqrt{2}} (1, -i)$ and when calculating expectation values on normalized vectors from $\mathbb{R}^2 \otimes \mathbb{R}^2$ we do not go beyond 2.

In order to proof that the upper bound of LNR($O$) is indeed 2.5, let us bring the observable $O$ into the real form by a local unitary transformation (which does not change the ranges but does change the extremal vectors),

$$U_1 = D[1, -i] \quad U_2 = D[1, i].$$
Similar proof can be carried out for the lower bound $y$. Obviously, this way we arrive at two separate two-dimensional linear problems. In order to obtain the lower bound $\gamma_{\min}$.

5 Local Numerical Range for $d = 2$

Consider now 2-qubit case corresponding to $d = 2$. The set of nonlinear equations (24) reduces to

\[
\begin{align*}
\frac{1}{\lambda_{\pm}} w_{11} + C_{\pm} w_{21} - \lambda_2 \left( \frac{1 + C_{\pm}^2}{C_{\pm}} \right) y_1 + (u_1 + u_2) y_2 &= 0, \\
(u_1 + u_2) y_1 + \frac{1}{\lambda_{\pm}} w_{12} + C_{\pm} w_{22} - \lambda_2 \left( \frac{1 + C_{\pm}^2}{C_{\pm}} \right) y_2 &= 0.
\end{align*}
\]

(27)

Consider normalized vector $|q\rangle = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$. It is separable iff $q_1 q_4 = q_2 q_3$. Hence, we define

\[
\tilde{G}(q) = \langle q | M_O^\prime | q \rangle - \lambda_1 \left( \sum_{j=1}^{4} q_j^2 - 1 \right) - 2 \lambda_2 (q_1 q_4 - q_2 q_3),
\]

where $M_O^\prime$ represents matrix of $O$ in the basis $|g_i^\prime\rangle \otimes |f_j^\prime\rangle$, that is,

\[
M_O^\prime = \begin{pmatrix}
w_{11} & 0 & 0 & u_1 \\
0 & w_{12} & u_2 & 0 \\
u_1 & w_{21} & 0 & 0 \\
0 & 0 & 0 & w_{22}
\end{pmatrix}, \quad u_1, u_2 \geq 0, \quad w_{ij} \in \mathbb{R}.
\]

(31)

Now, $d \tilde{G} = 0$ leads to a linear matrix equation

\[
M |q\rangle = |0\rangle,
\]

(32)

where

\[
M = \begin{pmatrix}
-\lambda_1 + w_{11} & 0 & 0 & u_1 - \lambda_2 \\
0 & w_{12} - \lambda_1 & u_2 + \lambda_2 & 0 \\
u_1 - \lambda_2 & 0 & -\lambda_1 + w_{12} & 0 \\
0 & 0 & 0 & w_{22} - \lambda_1
\end{pmatrix}.
\]

Obviously, this way we arrive at two separate two-dimensional linear problems. In order to obtain nonzero solutions the following condition should be fulfilled:

\[
\det M = d_{1}(\lambda_{1}, \lambda_{2}) \cdot d_{2}(\lambda_{1}, \lambda_{2}) = 0,
\]
where
\[ d_1(\lambda_1, \lambda_2) = u_2^2 - w_1 w_{21} + (w_{12} + w_{21}) \lambda_1 - \lambda_1^2 + 2u_2 \lambda_2 + \lambda_2^2, \quad (33) \]
\[ d_2(\lambda_1, \lambda_2) = u_1^2 - w_1 w_{22} + (w_{11} + w_{22}) \lambda_1 - \lambda_1^2 - 2u_1 \lambda_2 + \lambda_2^2. \quad (34) \]

Now, assuming
\[
\begin{cases}
    d_1(\lambda_1, \lambda_2) = 0 & \text{or} \quad d_2(\lambda_1, \lambda_2) = 0 \\
\end{cases}
\]
and using the separability condition, we get four possible product vectors \(|g_i \otimes |f_j\rangle\),
\[
\Big\{ \Big( \begin{matrix} 0 \\ 1 \end{matrix} \Big) \otimes \Big( \begin{matrix} 1 \\ 0 \end{matrix} \Big), \quad \Big( \begin{matrix} 0 \\ 1 \end{matrix} \Big) \otimes \Big( \begin{matrix} 0 \\ 1 \end{matrix} \Big), \quad \Big( \begin{matrix} 0 \\ 1 \end{matrix} \Big) \otimes \Big( \begin{matrix} 1 \\ 0 \end{matrix} \Big) \Big\},
\]
whereas solving
\[
\begin{cases}
    d_1(\lambda_1, \lambda_2) = 0 \\
    d_2(\lambda_1, \lambda_2) = 0 \\
\end{cases}
\]
we obtain two solutions \((\lambda_1^+, \lambda_2^+)\) and \((\lambda_1^-, \lambda_2^-)\) which inserted into (32) imply the following conditions:
\[
\begin{cases}
    q_1 = a_{\pm} q_4 \\
    q_2 = b_{\pm} q_3 \\
    q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \\
    q_1 q_4 = q_2 q_3. \\
\end{cases}
\]
Solving (37) and factorizing \(|q\rangle = |x\rangle \otimes |y\rangle\) we arrive at
\[
|q\rangle = \left( \frac{\sqrt{\xi_{\pm}}}{\sqrt{1 + \xi_{\pm}}} \right) \otimes \left( \frac{\kappa_{\pm}}{\sqrt{1 + \kappa_{\pm}}} \right), \quad (38)
\]
where
\[ a_{\pm} = \frac{l_{\pm}}{m_{\pm}}, \quad b_{\pm} = \frac{g_{\pm}}{h_{\pm}}, \quad \kappa_{\pm} = \frac{a_{\pm}}{b_{\pm}}, \quad \xi_{\pm} = a_{\pm} \cdot b_{\pm}, \]
and
\[
\begin{align*}
    l_{\pm} &= 2u_1^2 + 8u_1^2 w_{12} + 2u_2^2 \pm (u_1 + u_2) (-w_{11} + w_{12} + w_{21} - w_{22}) \sqrt{\Delta} \\
    &\quad + u_2^2 (-w_{11}^2 - 2w_{12} w_{21} + w_{11} (w_{12} + w_{21}) + (w_{12} + w_{21}) w_{22} - w_{22}^2) \\
    &\quad + 2u_1 u_2 (4u_2^2 - w_{11}^2 - 2w_{12} w_{21} + w_{11} (w_{12} + w_{21}) + (w_{12} + w_{21}) w_{22} - w_{22}^2) \\
    &\quad + u_2^2 (12u_2^2 - w_{11}^2 - 2w_{12} w_{21} + w_{11} (w_{12} + w_{21}) + (w_{12} + w_{21}) w_{22} - w_{22}^2), \\
    m_{\pm} &= (u_1 + u_2) \left( \pm 2 (u_1 + u_2) \sqrt{\Delta} + (w_{11} - w_{12}) (w_{11} - w_{21}) (w_{11} - w_{12} - w_{21} + w_{22}) \\
    &\quad + u_1^2 (-3w_{11} + w_{12} + w_{21} + w_{22}) + 2u_1 u_2 (-3w_{11} + w_{12} + w_{21} + w_{22}) + u_2^2 (-3w_{11} + w_{12} + w_{21} + w_{22}) \right), \\
    g_{\pm} &= 2u_1^2 + 8u_1^2 w_{12} + 2u_2^2 \pm (u_1 + u_2) (w_{11} - w_{12} - w_{21} + w_{22}) \sqrt{\Delta} \\
    &\quad + u_2^2 (-w_{11} - w_{12} - w_{21} + w_{22}) (w_{11} - w_{12} - w_{21} + w_{22}) + (w_{12} + w_{21}) w_{22} \\
    &\quad + 2u_1 u_2 (4u_2^2 - w_{11}^2 - w_{12} + w_{21} + w_{22} + (w_{12} + w_{21}) w_{22}) \\
    &\quad + u_2^2 (12u_2^2 - w_{11}^2 - w_{12} - w_{21} + w_{11} (w_{12} + w_{21}) + (w_{12} + w_{21}) w_{22} - w_{22}^2), \\
    h_{\pm} &= (u_1 + u_2) \left( \pm 2 (u_1 + u_2) \sqrt{\Delta} + (w_{11} - w_{12}) (w_{12} - w_{21}) (w_{11} - w_{12} - w_{21} + w_{22}) \\
    &\quad + u_1^2 (w_{11} - 3w_{12} + w_{21} + w_{22}) + 2u_1 u_2 (w_{11} - 3w_{12} + w_{21} + w_{22}) + u_2^2 (w_{11} - 3w_{12} + w_{21} + w_{22}) \right),
\end{align*}
\]
with
\[ \Delta = \left( (u_1 + u_2)^2 + (u_{11} - u_{21}) (u_{12} - u_{22}) \right) \left( (u_1 + u_2)^2 + (u_{11} - u_{12}) (u_{21} - u_{22}) \right). \]

Note that, in order to have real components of \(|q\rangle\),
\[ \xi_\pm \geq 0, \quad \kappa_\pm \geq 0, \quad (39) \]
should be fulfilled. As a consequence, either both \(a_\pm\), \(b_\pm\) are nonnegative or both are non-positive. Finally, for \(|q\rangle\) given by \((38)\) we obtain
\[ \langle q| M'_O |q \rangle \equiv F_\pm = \frac{2(u_1 + u_2) \sqrt{\xi_\pm \kappa_\pm + \xi_\pm \kappa_\pm w_{11} + \xi_\pm w_{11} + \kappa_\pm w_{21} + w_{22}}}{(1 + \kappa_\pm)(1 + \xi_\pm)} \]
\[ = \frac{2(u_1 + u_2)|a_\pm| + a_\pm^2 w_{11} + \xi_\pm w_{11} + \kappa_\pm w_{21} + w_{22}}{1 + \xi_\pm + \kappa_\pm + a_\pm^2}. \]

Taking into account vectors \((35)\) one obtains
\[ \langle g_i \otimes f_j | M'_O | g_i \otimes f_j \rangle = w_{ij}. \]
Hence, LNR of the circulant observable \(O\) is given by \([\gamma_{\min}, \gamma_{\max}]\), where
\[ \gamma_{\min} = \min \left\{ w_{ij}, F_\pm \right\}, \quad (42) \]
\[ \gamma_{\max} = \max \left\{ w_{ij}, F_\pm \right\}, \quad (43) \]

To summarize, in order to calculate LNR for a given \(C^2 \otimes C^2\) circulant operator, we propose the following procedure

1. if in a given basis a matrix representation of an operator \(M_O\) has complex or negative off-diagonal entries then change the basis due to Proposition 1 and bring the matrix to the real form,
2. determine real vectors \(|x\rangle\) and \(|y\rangle\) (see \((33)\)) together with \(F_\pm\) and compare these values with diagonal elements of \(M'_O\). Then LNR(O) = \([\gamma_{\min}, \gamma_{\max}]\), where \(\gamma_{\min}\) and \(\gamma_{\max}\) are defined in \((42)\) and \((43)\), respectively.

Example 2 As an illustration let us consider a two-parameter family of matrices \(Q_{t,s}\), \(t, s \geq 0\), analyzed in \([3]\).
\[ Q_{t,s} = \begin{pmatrix} 2 & 0 & 0 & t \\ 0 & 1 & s & 0 \\ 0 & s & -1 & 0 \\ t & 0 & 0 & -2 \end{pmatrix}. \]
Denoting by \(p = t + s \geq 0\) one obtains
\[ \Delta = (1 + p^2)(9 + p^2) \]
\[ a_\pm = \frac{4p \pm \sqrt{\Delta}}{p^2 - 3} \]
\[ b_\pm = \frac{2p \pm \sqrt{\Delta}}{p^2 + 3} \]
\[ \kappa_\pm = \frac{p^4 + 2p^2 + 9 \pm 2\sqrt{\Delta}}{(p^2 - 3)(p^2 + 3)} \]
\[ \xi_\pm = \frac{p^4 + 18p^2 + 9 \pm 6\sqrt{\Delta}}{(p^2 - 3)(p^2 + 3)}. \]
Note that $b_+ \geq 0$ and $b_- \leq 0$, hence $a_+ \geq 0$ and $a_- \leq 0$. Finally, $|x|$ and $|y|$ are real under the condition $p \geq \sqrt{3}$ (see (39)) and using (40) we arrive at

$$F_{\pm} = \pm \frac{\sqrt{\Delta}}{2p}.$$ 

Because the maximal and minimal values of $w_{ij}$ are equal to 2 and $-2$, respectively, due to (42) and (43) we get

$$\gamma_{\text{max}} = \begin{cases} 
2 & \text{for } 0 \leq p < \sqrt{3} \\
\frac{1}{2p} \sqrt{\Delta} & \text{for } p \geq \sqrt{3}
\end{cases}$$

and $\gamma_{\text{min}} = -\gamma_{\text{max}}$ in complete agreement with the result of [3].

**Appendix**

We are going to carry out an extremalization procedure of (19) with two constraints $|x| = 1$, $|y| = 1$ using a Lagrange function $G = F - \lambda_1(|x|^2 - 1) - \lambda_2(|y|^2 - 1)$. As a result we get the following equations:

$$\frac{\partial G}{\partial x_i} = x_i \left[ \sum_{k=1}^{d} w_{ik} y_k^2 - \lambda_1 \right] + x_{i+1} \sum_{k=1}^{d} y_k y_{k+1} u_k \cos \varphi_k = 0, \quad i = 1, 2 \quad (44)$$

$$\frac{\partial G}{\partial y_k} = y_k \left[ \sum_{i=1}^{2} w_{ik} x_i^2 - \lambda_2 \right] + x_1 x_2 \left( y_{k-1} u_{k-1} \cos \varphi_{k-1} + y_{k+1} u_k \cos \varphi_k \right) = 0, \quad k = 1, \ldots, d \quad (45)$$

$$\frac{\partial G}{\partial \beta_1} = x_1 x_2 \sum_{k=1}^{d} y_k y_{k+1} u_k \sin \varphi_k = 0, \quad (46)$$

$$\frac{\partial G}{\partial \beta_k} = x_1 x_2 \left( y_k y_{k+1} u_k \sin \varphi_k - y_{k-1} y_k u_{k-1} \sin \varphi_{k-1} \right) = 0, \quad k = 2, \ldots, d. \quad (47)$$

From the last two equations one obtains in a generic case, i.e., when $x_i \neq 0$, and $y_k \neq 0$, the following set of equations

$$\begin{cases} 
\sum_{k=1}^{d} z_k = 0 \\
z_{k-1} - z_k = 0, \quad k = 2, \ldots, d. \quad (48)
\end{cases}$$

with $z_k = y_k y_{k+1} u_k \sin \varphi_k$ or in a matrix notation $W^T z = 0$, where $W^T$ is a transposition of the matrix given by (12). Now, according to $\det W^T = d(-1)^{d+1} \neq 0$, the set of homogeneous equations (43) has only zero solution, hence in a generic case, $\sin \varphi_k = 0$ for $k = 1, \ldots, d$. The angles $\beta_k$ can now be easily obtained. It results from $\sin \varphi_k = 0$ that

$$\alpha_1 + \beta_1 + \beta_2 = 0, \quad (49)$$

$$\alpha_k + \beta_1 - \beta_k + \beta_{k+1} = 0, \quad k = 2, \ldots, d - 1 \quad (50)$$

$$\alpha_{d} + \beta_1 - \beta_d = 0. \quad (51)$$

or in a matrix form

$$\alpha = -W \beta \quad (53)$$
with exactly the same $W$ as in (13). Hence solutions for $\beta_1, \ldots, \beta_d$ differ only by a sign from solutions for $\mu_1, \ldots, \mu_d$ (see (13)) and one can easily find that

$$\beta_1 = - \frac{1}{d} \sum_{k=1}^{d} \alpha_k = - \mu_1,$$
$$\beta_2 = - \alpha_1 - \beta_1 = - \mu_2,$$
$$\beta_{k+1} = - \alpha_k - \beta_1 + \beta_k = - \mu_{k+1}, \quad k = 2, \ldots, d - 1. \quad (54)$$

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**References**


