Abstract—The paper introduces a controls coefficient generalized inversion attitude tracking design methodology for realization of desired linear spacecraft attitude deviation dynamics. A prescribed stable linear second order time-invariant ordinary differential equation in a spacecraft attitude deviation norm measure is evaluated along the solution trajectories of the spacecraft equations of motion, yielding a linear relation in the control variables. Generalized inversion of the relation results in a control law that consists of particular and auxiliary parts. The particular part resides in the range space of the controls coefficient row vector, and it works to drive the spacecraft attitude variables in order to nullify the attitude deviation norm measure. The auxiliary part resides in the complementary orthogonal complement subspace, and therefore it does not affect realization of the desired trajectory. Nevertheless, the auxiliary part is crucial in the control design, because it provides the necessary spacecraft internal stability by proper design of the null-control vector. The null-control vector construction is made by solving a state dependent Lyapunov equation, yielding global internal stability. The control design utilizes a damped controls coefficient generalized inverse to limit the growth of the controls coefficient generalized inverse as the steady state response is approached. The design provides uniformly ultimately bounded attitude trajectory tracking errors, and reveals the tradeoff between generalized inversion stability and tracking performance.

I. INTRODUCTION

Generalized inversion is a well known control system design methodology, with numerous applications in the fields of aerospace engineering and robotics. A fundamental advantage of generalized inversion is that it overcomes the limitations of dimensionality and rank that are related to the notion of inversion, which makes the inversion process capable of solving underdetermined problems where requirements can be satisfied in more than one course of action, as well as solving for best approximating solutions to overdetermined problems, when the requirements cannot be satisfied.

Another fundamental property of generalized inversion is the characterization of solution nonuniqueness. This is depicted by the Greville formula [1], which uses the Moore-Penrose generalized matrix inverse (MPGI) [2], [3] to obtain the general solution of a linear algebraic system equations, and parameterizes the corresponding linear algebraic system matrix nullspace via a free nullvector that appears explicitly in the solution expression.

The generalized inversion feature of nullspace parametrization has been utilized in engineering analysis and design for the purpose of modeling, control, and optimization. At the level of analysis, utilization of this feature in the field of analytical dynamics was made by deriving the Udwadia-Kalaba equations of motion for constrained dynamical systems [4]. The corresponding free nullvector was chosen in order to optimize acceleration energy of the system, yielding its natural accelerations, i.e., those obeying Gauss’ principle of least constraints [5], or equivalently yielding constraint forces that satisfy D’Alembert’s principle of virtual work [6].

At the design level, nullspace parametrization has been insightful in viewing the active perspective of servo-constrained motion, where the design requirements are formulated in terms constraints on the system dynamics that are to be imposed using the available control authority, and the control forces are possibly obtained by utilizing the same fundamental variational principles of mechanics that are used in modeling passively constrained dynamical systems. This has been used in attempts to synthesize the most natural control laws that realize servo-constraints [7], [8], and to optimize control laws for multibody systems in context of the operational space framework [9], [10].

By observing that the control problem of a controllable dynamical system is a problem of nonuniqueness, and based on generalized inversion nullspace parametrization provided by the Greville formula, the concept of controls coefficient generalized inversion has been introduced in [11] for spacecraft attitude control, considering nulling the deviation from desired spacecraft kinematics to be a desired servo-constraint that is to be realized.

Generalized inversion of the controls coefficient guarantees stable attitude dynamics feedback linearization. To fulfill internal stability requirement, and inspired by the control law’s affinity in the null-control vector, the later has been chosen in [11] to be linear in the angular velocity vector, resulting in a stable perturbed feedback linearization of the spacecraft internal dynamics. The control design provides simultaneous perturbed feedback linearizing transformation of the spacecraft global dynamics and realization of the prescribed linear attitude deviation dynamics.

A controls coefficient generalized inversion law consists of auxiliary and particular parts, residing in the nullspace of the controls coefficient and the range space of its generalized inverse, respectively. The free null-control vector in the auxiliary part is projected onto the controls coefficient nullspace by a nullprojection matrix. Therefore, the choice of the null-control vector does not affect dynamics of the deviation measure function, and it furnishes a design freedom in constructing the control law that is capable of realizing

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that dynamics.

Based on the null-control vector design, this paper presents a Lyapunov based spacecraft control design methodology, following the controls coefficient generalized inversion paradigm. The null-control vector is chosen to be proportional to the spacecraft body angular velocity vector error from a desired angular velocity vector that is determined from the desired attitude trajectory. The proportionality matrix is obtained by solving a novel Lyapunov equation type that involves the controls coefficient nullprojector (CCNP). The control design results in global internal dynamics stability.

The control design procedure begins by defining a norm measure function of the spacecraft’s attitude variables deviations from their desired values, and prespecifying a stable second-order linear differential equation in the measure function, resembling the desired attitude deviation dynamics. The differential equation is then transformed to a relation that is linear in the control vector by differentiating the norm measure function along the trajectories defined by the solution of the spacecraft’s state space mathematical model. The Greville formula is utilized thereafter to invert this relation for the control law required to realize the desired stable linear dynamics.

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II. SPACECRAFT MATHEMATICAL MODEL

The spacecraft mathematical model is given by the following system of kinematical and dynamical differential equations

\begin{align}
\dot{\rho} &= G(\rho)\omega, \quad \rho(0) = \rho_0 \tag{1} \\
\dot{\omega} &= J^{-1}\omega^\times J\omega + \tau, \quad \omega(0) = \omega_0 \tag{2}
\end{align}

where \( \rho \in \mathbb{R}^{3\times 1} \) is the spacecraft vector of modified Rodrigues attitude parameters (MRPs) \[14\], \( \omega \in \mathbb{R}^{3\times 1} \) is the vector of spacecraft angular velocity components in its body reference frame, \( J \in \mathbb{R}^{3\times 3} \) is a diagonal matrix containing spacecraft’s body principal moments of inertia, and \( \tau := J^{-1}u \in \mathbb{R}^{3\times 1} \) is the vector of scaled control torques, where \( u \in \mathbb{R}^{3\times 1} \) contains the applied gas jet actuator torque components about the spacecraft’s principal axes. The cross product matrix \( x^\times \) which corresponds to a vector \( x \in \mathbb{R}^{3\times 1} \) is skew symmetric of the form

\[
x^\times = \begin{bmatrix}
0 & x_3 & -x_2 \\
-x_3 & 0 & x_1 \\
x_2 & -x_1 & 0
\end{bmatrix}
\]

and the matrix valued function \( G(\rho) : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 3} \) is given by

\[
G(\rho) = \frac{1}{2} \left( \frac{1 - \rho^T \rho}{2} I_{3\times 3} - \rho \rho^T + \rho \rho^T \right). \tag{3}
\]

The MRPs are used as the attitude state variables, because of their validity in describing any angular displacement about the spacecraft’s body axes up to \( 2\pi \) rad, such that \( G(\rho) \) remains finite and invertible for any value of \( \rho \) that corresponds to such spacecraft angular displacement.

III. ATTITUDE DEVIATION NORM MEASURE DYNAMICS

Let \( \rho_r(t) \in \mathbb{R}^{3\times 1} \) be a prescribed desired spacecraft attitude vector such that \( \rho_r(t) \) is twice continuously differentiable in \( t \). The spacecraft attitude deviation vector from \( \rho_r(t) \) is defined as

\[
z(\rho, t) := \rho - \rho_r(t). \tag{4}
\]

Consequently, the scalar attitude deviation norm measure function \( \phi : \mathbb{R}^{3\times 1} \to \mathbb{R} \) is defined to be the squared norm of \( z(\rho, t) \)

\[
\phi = \| z(\rho, t) \|_2^2 = \| \rho - \rho_r(t) \|_2^2. \tag{5}
\]

The first two time derivatives of \( \phi \) along the spacecraft trajectories given by the solution of (1) and (2) are

\[
\dot{\phi} = \frac{\partial \phi}{\partial \rho} G(\rho)\omega + \frac{\partial \phi}{\partial t} = 2z^T(\rho, t) [G(\rho)\omega - \dot{\rho}_r(t)] \tag{6}
\]

and

\[
\ddot{\phi} = 2 [G(\rho)\omega - \dot{\rho}_r(t)]^T [G(\rho)\omega - \dot{\rho}_r(t)] + 2z^T(\rho, t) \left[ \dot{G}(\rho, \omega)\omega + G(\rho) [J^{-1}\omega^\times J\omega + \tau] - \dot{\rho}_r(t) \right] \tag{7}
\]

where \( \dot{G}(\rho, \omega) \) is the time derivative of \( G(\rho) \) obtained by differentiating the individual elements of \( G(\rho) \) along the kinematical subsystem given by equations (1). We prespecify a desired stable linear second-order dynamics of \( \phi \) in the form

\[
\ddot{\phi} + c_1 \dot{\phi} + c_2 \phi = 0, \quad c_1, c_2 > 0. \tag{9}
\]

With \( \phi, \dot{\phi}, \dot{c_2} \) given by (5), (7), and (8), it is possible to write (9) in the pointwise-linear form

\[
A(\rho, t) \tau = B(\rho, \omega, t), \tag{10}
\]

where the vector valued function \( A(\rho, t) : \mathbb{R}^{4\times 1} \to \mathbb{R}^{1\times 3} \) is given by

\[
A(\rho, t) = 2z^T(\rho, t) G(\rho) \tag{11}
\]

and the scalar valued function \( B(\rho, \omega, t) : \mathbb{R}^{7\times 1} \to \mathbb{R} \) is

\[
B(\rho, \omega, t) = -2 [G(\rho)\omega - \dot{\rho}_r(t)]^T [G(\rho)\omega - \dot{\rho}_r(t)] - 2z^T(\rho, t) \left[ \dot{G}(\rho, \omega)\omega + G(\rho) J^{-1}\omega^\times J\omega - \dot{\rho}_r(t) \right] - 2c_1z^T(\rho, t) [G(\rho)\omega - \dot{\rho}_r(t)] - c_2 \| z(\rho, t) \|_2^2. \tag{12}
\]
The row vector function $A(\rho, t)$ is the controls coefficient of the attitude deviation norm measure dynamics given by (9) along the spacecraft trajectories, and the scalar function $B(\rho, \omega, t)$ is the corresponding controls load.

IV. DESIRED INTERNAL DYNAMICS

Invertibility of the matrix $G(\rho)$ makes it possible to solve explicitly for the angular velocity vector $\omega$, which takes the form

$$\omega = G^{-1}(\rho)\hat{\rho}. \quad (13)$$

Therefore, a desired vector of dynamic variables $\omega_r(t)$ can be obtained from (13) by substituting the desired vector of kinematic variables $\rho_r(t)$ and its time derivative $\dot{\rho}_r(t)$ in place of $\rho$ and $\dot{\rho}$, respectively, such that

$$\omega_r(t) = G^{-1}(\rho_r(t))\dot{\rho}_r(t). \quad (14)$$

V. LINEARLY PARAMETERIZED ATTITUDE CONTROL LAWS

The quasi-linear form given by Eq. (10) makes it feasible to assess realizability of the linear attitude deviation norm measure dynamics given by Eq. (9) in a pointwise manner.

Definition 1 (Realizability of linear attitude deviation dynamics). For a given desired spacecraft attitude vector $\rho_r(t)$, the linear attitude deviation norm measure dynamics given by (9) is said to be realizable by the spacecraft equations of motion (1) and (2) at specific values of $\rho$ and $t$ if there exists a control vector $\tau$ that solves (10) for these values of $\rho$ and $t$. If this is true for all $\rho$ and $t$ such that $z(\rho, t) \neq 0_{3 \times 1}$, then the linear attitude deviation norm measure dynamics is said to be globally realizable by the spacecraft equations of motion.

Remark 1. The notion of global attitude deviation realizability does not imply global stability in the sense of Lyapunov. Rotational dynamics is not globally stabilizable due to the topological obstruction of the attitude rotation matrix, which is known to preclude the existence of globally stable equilibria for attitude dynamics [15].

For proof of the following proposition, the reader is referred to [11].

Proposition 1 (Linearly parameterized attitude control laws). For any desired spacecraft attitude vector $\rho_r(t)$, the linear attitude deviation norm measure dynamics given by (9) is globally realizable by the spacecraft equations of motion (1) and (2). Furthermore, the infinite set of all control laws realizing that dynamics by the spacecraft equations of motion is parameterized by an arbitrarily chosen null-control vector $y \in \mathbb{R}^{3 \times 1}$ as

$$\tau = A^+(\rho, t)B(\rho, \omega, t) + P(\rho, t)y \quad (15)$$

where “$A^+$” stands for the MPGI of the controls coefficient null projector given by

$$A^+(\rho, t) = \frac{A^T(\rho, t)}{\|A(\rho, t)\|_2}, \quad A(\rho, t) \neq 0_{1 \times 3} \quad (16)$$

and $P(\rho, t) \in \mathbb{R}^{3 \times 3}$ is the corresponding controls coefficient null projector (CCNP) given by

$$P(\rho, t) = I_{3 \times 3} - A^+(\rho, t)A(\rho, t). \quad (17)$$

Any choice of the null-control vector $y$ in the control law expression given by (15) yields a solution to (10). Therefore, the choice of $y$ does not affect realizability of the linear attitude deviation norm measure dynamics given by (9). Nevertheless, the choice of $y$ substantially affects the spacecraft transient state response [8]. In particular, an inadequate choice of $y$ can destabilize the spacecraft internal dynamics given by (2) or causes unsatisfactory closed loop performance. Substituting the control laws expressions given by (15) in the spacecraft’s equations of motion (1) and (2) yields the following parametrization of the infinite set of spacecraft closed loop systems equations that realize the dynamics given by (9)

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0 \quad (18)$$

$$\dot{\omega} = J^{-1}\omega^\times J\omega + A^+(\rho, t)B(\rho, \omega, t) + P(\rho, t)y, \quad \omega(0) = \omega_0. \quad (19)$$

VI. PERTURBED CONTROLS COEFFICIENT NULL PROJECTOR

Definition 2 (Perturbed controls coefficient null projector). The perturbed CCNP $\tilde{P}(\rho, \delta, t)$ is defined as

$$\tilde{P}(\rho, \delta, t) := I_{3 \times 3} - h(\delta)A^+(\rho, t)A(\rho, t) \quad (20)$$

where $h(\delta) : \mathbb{R}^{1 \times 1} \rightarrow \mathbb{R}^{1 \times 1}$ is any continuous function such that

$$h(\delta) = 1 \quad \text{if and only if} \quad \delta = 0.$$

Properties of the Perturbed Controls Coefficient Null Projector

The following properties of the perturbed CCNP are utilized in the present development of the controls coefficient generalized inversion based attitude tracking control.

1) The perturbed CCNP $\tilde{P}(\rho, \delta, t)$ is of full rank for all $\delta \neq 0$.
2) The CCNP $P(\rho, t)$ commutes with the perturbed CCNP $\tilde{P}(\rho, \delta, t)$ for all $\delta \in \mathbb{R}$. Furthermore, their matrix multiplication yields the CCNP itself, i.e.,

$$P(\rho, t)\tilde{P}(\rho, \delta, t) = \tilde{P}(\rho, \delta, t)P(\rho, t) = P(\rho, t). \quad (21)$$

3) The CCNP $P(\rho, t)$ commutes with its inverted perturbation $P^{-1}(\rho, \delta, t)$ for all $\delta \neq 0$. Furthermore, their matrix multiplication yields the CCNP itself, i.e.,

$$P^{-1}(\rho, \delta, t)P(\rho, t) = P(\rho, t)P^{-1}(\rho, \delta, t) = P(\rho, t). \quad (22)$$

Proofs of first and third properties are found in [11]. Second property is verified by direct evaluation of $P(\rho, t)$ and $P(\rho, \delta, t)$ expressions given by (17) and (20).
VII. DAMPED CONTROLS COEFFICIENT GENERALIZED INVERSE

The expression of \( A^+(\rho, t) \) given by (16) implies that
\[
\lim_{A(\rho, t) \to 0_{3 \times 3} } \| A^+(\rho, t) \| = \infty. \tag{23}
\]

Implications of the controls coefficient singularity on closed loop stability is depicted by the following singularity analysis.

Controls Coefficient Singularity Analysis

The definition of \( A(\rho, t) \) given by (11) implies that
\[
\lim_{z(\rho, t) \to 0_{3 \times 1} } A(\rho, t) = 0_{1 \times 3} \tag{24}
\]
for all finite values of \( \rho \in \mathbb{R}^3 \). Therefore, a control law \( \tau \) globally realizes the linear attitude deviation norm measure dynamics of (9) by the spacecraft equations of motion (1) and (2) only if
\[
\lim_{t \to \infty} A(\rho, t) = 0_{1 \times 3} \tag{25}
\]
which implies from (23) that
\[
\lim_{z(\rho, t) \to 0_{3 \times 1} } \| A^+(\rho, t) \| = \infty. \tag{26}
\]

However, a fundamental property of the matrix \( G(\rho) \) is [16]
\[
\sigma_{\min}(G(\rho)) = \sigma_{\max}(G(\rho)) = \sigma(G(\rho)) \geq \frac{1}{4}. \tag{27}
\]
Therefore,
\[
\| G^T(\rho) z(\rho, t) \| = \sigma(G(\rho)) \| z(\rho, t) \|, \tag{28}
\]
and the definition of \( A(\rho, t) \) given by (11) implies that
\[
\| A^+(\rho, t) \| = \frac{1}{2\sigma(G(\rho))} \| z(\rho, t) \| \tag{29}
\]
and that
\[
\| A^+(\rho, t) z^T(\rho, t) \| \leq \| A^+(\rho, t) \| \| z(\rho, t) \| \leq \frac{1}{2\sigma(G(\rho))} \| z(\rho, t) \| \tag{30}
\]
and that
\[
\| A^+(\rho, t) z^T(\rho, t) \| \leq \| A^+(\rho, t) \| \| z(\rho, t) \| \leq \frac{1}{2\sigma(G(\rho))} \leq 2 \tag{31}
\]
and that
\[
\| A^+(\rho, t) z^T(\rho, t) z(\rho, t) \| \leq \| A^+(\rho, t) \| \| z(\rho, t) \| \| z(\rho, t) \| \leq 2 \| z(\rho, t) \| \tag{32}
\]
Inequalities (31) and (32) imply that
\[
\lim_{z(\rho, t) \to 0_{3 \times 1} } \| A^+(\rho, t) z^T(\rho, t) \| \leq 2 \tag{33}
\]
and
\[
\lim_{z(\rho, t) \to 0_{3 \times 1} } \| A^+(\rho, t) z^T(\rho, t) z(\rho, t) \| = 0. \tag{34}
\]

Damped Controls Coefficient Generalized Inverse

For the purpose of controlling the growth of the CCGI \( A^+(\rho, t) \) in the control law given by (15), the damped CCGI \( A_d^+(\rho, \beta, t) \) is introduced.

Definition 3 (Damped controls coefficient generalized inverse). The damped CCGI is defined as
\[
A_d^+(\rho, \beta, t) := \begin{cases} \frac{A^+(\rho, t)}{\| A(\rho, t) \|} : \| A(\rho, t) \| \geq \beta \\ 0 : \| A(\rho, t) \| < \beta \end{cases} \tag{35}
\]
where the scalar \( \beta \) is a positive generalized inverse damping factor.

The above definition implies that
\[
\| A_d^+(\rho, \beta, t) \| \leq \frac{1}{\beta} \tag{36}
\]
and that
\[
\lim_{z(\rho, t) \to 0_{3 \times 1} } \| A_d^+(\rho, \beta, t) \| = \| \frac{2}{\beta^2} G^T(\rho) z(\rho, t) \| = 0 \tag{37}
\]
and that \( A_d^+(\rho, \beta, t) \) pointwise converges to \( A^+(\rho, t) \) as \( \beta \) vanishes (see Figure 1).

![Fig. 1. Damped CCGI](image)

Damped Controls Coefficient Nullprojector

The damped controls coefficient nullprojector is a modified controls coefficient nullprojector with vanishing dependency on the steady state attitude variables.

Definition 4 (Damped controls coefficient nullprojector). The damped CCNP \( P_d(\rho, \beta, t) \) is defined as
\[
P_d(\rho, \beta, t) := I_{3 \times 3} - A_d^+(\rho, \beta, t) A(\rho, t) \tag{38}
\]
where \( A_d^+(\rho, \beta, t) \) is given by (35).
The above definition implies that
\[
\mathcal{P}_d(\rho, \beta, t) = \begin{cases} 
I_{3\times3} - \frac{G^T(\rho)G(\rho)}{\|G(\rho)\|^2} & : \|A(\rho, t)\| = \|2G^T(\rho)z(\rho, t)\| \geq \beta \\
I_{3\times3} - \frac{4G^T(\rho)z(\rho, t)G(\rho)}{\beta^2} & : \|A(\rho, t)\| = \|2G^T(\rho)z(\rho, t)\| < \beta
\end{cases}
\]
and consequently,
\[
\lim_{z(\rho, t) \to 0} \mathcal{P}_d(\rho, \beta, t) = I_{3\times3}.
\]
Hence, the damped CCNP maps the null-control vector to itself during steady state phase of response, during which the auxiliary part of the control law converges to the null-control vector.

VIII. CONTROL COEFFICIENT GENERALIZED INVERSION CONTROL

For convenience, let the control law expression given by (15) be written as
\[
\tau = \mathcal{H}_1(\rho, \omega, \beta, t)\omega + \mathcal{H}_2(\rho, t) + \mathcal{P}(\rho, t)y
\]
where
\[
\mathcal{H}_1(\rho, \omega, \beta, t) = -2A^+(t)z^T(\rho, t)G(\rho)\left[\dot{\rho}(t) + c_1\hat{\rho}_r(t)\right] - \rho(\rho, t) \|\hat{\rho}_r(t)\|^2.
\]
and
\[
\mathcal{H}_2(\rho, t) = c_2A^+(\rho, t)z^T(\rho, t)\left[\hat{\rho}_r(t) + c_1\hat{\rho}_r(t)\right] - 2A^+(\rho, t) \|\hat{\rho}_r(t)\|^2.
\]
To avoid closed loop instability due to the CCNP unstable dynamics described by (26), we define \( \mathcal{H}_{1d}(\rho, \omega, \beta, t) \) and \( \mathcal{H}_{2d}(\rho, \beta, t) \) by replacing \( A^+(\rho, t) \) in the last terms of the \( \mathcal{H}_1(\rho, \omega, \beta, t) \) and \( \mathcal{H}_2(\rho, t) \) expressions given by Eqs. (42) and (43) with the damped CCGI \( A^*_d(\rho, \beta, t) \), such that
\[
\mathcal{H}_{1d}(\rho, \omega, \beta, t) = -2A^+_d(\rho, t)z^T(\rho, t)G(\rho)\left[\dot{\rho}(t) + c_1\hat{\rho}_r(t)\right] - \rho(\rho, t) \|\hat{\rho}_r(t)\|^2.
\]
and
\[
\mathcal{H}_{2d}(\rho, \beta, t) = c_2A^+_d(\rho, t)z^T(\rho, t)\left[\hat{\rho}_r(t) + c_1\hat{\rho}_r(t)\right] - 2A^+_d(\rho, \beta, t) \|\hat{\rho}_r(t)\|^2.
\]
The other terms in \( \mathcal{H}_1(\rho, \omega, t) \) and \( \mathcal{H}_2(\rho, t) \) involving the CCGI \( A^+_d(\rho, \beta, t) \) are left unaltered, because they remain bounded according to inequality (33) and Eq. (34) as the closed loop system reaches steady state.

The control vector \( \tau_d \) is defined as
\[
\tau_d = \mathcal{H}_{1d}(\rho, \omega, \beta, t) + \mathcal{H}_{2d}(\rho, \beta, t) + \mathcal{P}_{\phi}(\rho, \beta, t)\phi.
\]

**Theorem 1** (Implication on attitude stability). Let the control law \( \tau_d \) be given by (46), where the null-control vector \( y \) is arbitrary. Then the desired attitude deviation dynamics given by (9) is realized by the spacecraft equations of motion (1) and (2) for all values of \( \phi \) in the domain \( D_\phi \) given by
\[
D_\phi : \phi \geq \frac{\beta^2}{4\sigma^2(G(\rho))},
\]
and the resulting attitude trajectory tracking errors are UUB.

**Proof:** Let \( \phi_d \) be a norm measure function of the attitude deviation obtained by applying the control law given by (46) to the spacecraft equations of motion (1) and (2), and let \( \phi_d, \tilde{\phi}_d \) be its first two time derivatives. Hence,
\[
\phi_d := \phi_d(\rho, t) = \phi(\rho, t)
\]
\[
\tilde{\phi}_d := \tilde{\phi}_d(\rho, \omega, \beta, t) = \phi(\rho, \omega, \beta, t)
\]
\[
\tilde{\phi}_d := \tilde{\phi}_d(\omega, \phi_d, \tau_d) = \phi(\rho, \omega, \beta, t)
\]
\[
+ \mathcal{A}(\rho, \beta, t)\tau_d - \mathcal{A}(\rho, \beta, t)\tau
\]
where \( \tau \) and \( \tau_d \) are given by (41) and (46), respectively. Adding \( c_1\phi_d + c_2\tilde{\phi}_d \) to both sides of (50) yields
\[
\tilde{\phi}_d + c_1\phi_d + c_2\tilde{\phi}_d = \phi + c_1\dot{\phi} + c_2\dot{\phi}
\]
\[
+ \mathcal{A}(\rho, \beta, t)\tau_d - \mathcal{A}(\rho, \beta, t)\tau
\]
\[
= \mathcal{A}(\rho, \beta, t)\tau_d - \mathcal{A}(\rho, \beta, t)\tau.
\]
Let the state vector \( \Phi_d \in \mathbb{R}^{2\times1} \) be defined as
\[
\Phi_d = [\phi_d \quad \tilde{\phi}_d]^T.
\]
The attitude deviation norm measure closed loop dynamics can be written in the state space form
\[
\dot{\Phi}_d = \Lambda_{11}\Phi_d + \Lambda_{12}(\rho, \omega, \beta, t) + \Delta_1(\rho, \beta, t)
\]
where the strictly stable system matrix \( \Lambda_{11} \in \mathbb{R}^{2\times2} \) is
\[
\Lambda_{11} = \begin{bmatrix} 0 & 1 \\ -c_2 & -c_1 \end{bmatrix}
\]
and the matrix valued function \( \Lambda_{12}(\rho, \omega, \beta, t) : \mathbb{R}^{7\times1} \rightarrow \mathbb{R}^{2\times3} \) is
\[
\Lambda_{12}(\rho, \omega, \beta, t) = \begin{bmatrix} \mathcal{A}(\rho, t) \mathcal{H}_{1d}(\rho, \omega, \beta, t) \\ -\mathcal{H}_1(\rho, \omega, t) \end{bmatrix}
\]
and the matrix valued function \( \Delta_1(\rho, \beta, t) : \mathbb{R}^{5\times1} \rightarrow \mathbb{R}^{2\times1} \) is
\[
\Delta_1(\rho, \beta, t) = \begin{bmatrix} 0 \\ \mathcal{A}(\rho, t) \mathcal{H}_{2d}(\rho, \beta, t) - \mathcal{P}(\rho, t) \end{bmatrix}
\]
The expression of $A^+(\rho, t)$ given by (16) is identical to the expression of $A^+_{d}(\rho, \beta, t)$ given by (35) for values of $\rho$ and $t$ that satisfy $\|A(\rho, t)\| \geq \beta$. These values of $\rho$ and $t$ can be described in terms of $\phi$ by noticing that the singular values of the matrix $G(\rho)$ are repeated, so that

$$\|A(\rho, t)\| = 2\sigma^2(\rho)(\|G(\rho)\|z(\rho, t))$$

which implies from the definition of $\phi$ that the corresponding set $D_{\phi}$ of $\phi$ values is given by (47). Therefore, for values of $\phi$ in $D_{\phi}$, $\Lambda_{12}(\rho, \omega, \beta, t) = 0_{2 \times 3}$ and $\Delta_{1}(\rho, \beta, t) = 0_{2 \times 1}$, the expressions (46) for $\tau_d$ and (41) for $\tau$ are identical, and the desired linear attitude deviation dynamics are not realized. Nevertheless, for values of $\phi$ in the bounded open complement set given by

$$D_{\phi_c} : \phi < \frac{\beta^2}{4\sigma^2(\rho)}$$

the definition of $A^+(\rho, t)$ is different from the definition of $A^+_{d}(\rho, \beta, t)$, and the desired attitude dynamics is not realized. Instead, the dynamics given by (52) is the one that is realized over this bounded domain, resulting in uniformly ultimately bounded attitude trajectory tracking errors rather than in asymptotic attitude tracking.

Stability of internal dynamics is the most important factor to be considered in designing the null-control vector $y$. The structure of the control law $\tau_d$ has a special feature, namely the affinity of its auxiliary part in $y$, which provides pointwise-linear parametrization to the nonlinear control law.

In the following section, we provide a Lyapunov control function-based design of the null-control vector, guaranteeing global internal closed loop control stability.

A. Globally Internally Stabilizing Null-Control Vector

Let the null-control vector $y$ be chosen as

$$y = K(\omega - \omega_r(t))$$

where $\omega_r(t)$ is given by (14), and $K \in \mathbb{R}^{3 \times 3}$ is a matrix gain that is to be determined. Hence, a class of control laws that realize the attitude deviation norm measure dynamics given by (9) is obtained by substituting this choice of $y$ in (46).

$$\tau_d = [H_{d}(\rho, \omega, \beta, t) + P_d(\rho, \beta, t)K]\omega + H_{2d}(\rho, \beta, t) - P_d(\rho, \beta, t)K\omega_r(t).$$

Consequently, a class of closed loop dynamical subsystems realizing the dynamics given by (9) is obtained by substituting the control law given by (61) in (2), and it takes the form

$$\dot{\omega} = [J^{-1}(\omega)\omega + H_{1d}(\rho, \omega, \beta, t) + P_d(\rho, \beta, t)K]\omega + H_{2d}(\rho, \beta, t) - P_d(\rho, \beta, t)K\omega_r(t).$$

The last two terms in the above closed loop dynamical subsystem are forcing terms that drive the closed loop internal dynamics of the spacecraft to realize the desired attitude deviation norm measure dynamics.

Theorem 2 (Globally Internally stable CCGI trajectory tracking control). Let the control law $\tau_d$ be given by (46), where the null-control vector $y$ is given by (60). Then the gain matrix $K$ can be designed such that closed loop dynamical subsystem given by (2) is GUUB.

Proof: The closed loop spacecraft internal system is obtained by omitting the two forcing terms in the dynamical subsystem given by (62) such that

$$\dot{\omega} = [J^{-1}J^{-1}\omega \omega + H_{1d}(\rho, \omega, \beta, t) + P_d(\rho, \beta, t)K]\omega.$$

A stabilizing gain $K$ for the spacecraft internal dynamics can be synthesized by utilizing the following squared Euclidean norm of the spacecraft angular velocity vector as a control Lyapunov function

$$V = \|\omega\|^2.$$

Differentiating $V$ along the trajectories of the system given by (63) and noticing skew-symmetry of $\omega$ yields

$$\dot{V} = 2\omega^T[J^{-1}J^{-1}\omega + H_{1d}(\rho, \omega, \beta, t) + P_d(\rho, \beta, t)K]\omega + [H_{1d}(\rho, \omega, \beta, t) + H_{2d}(\rho, \omega, \beta, t) + P_d(\rho, \beta, t)K + KP_d(\rho, \beta, t)]\omega,$$

where the matrix gain $K$ is chosen to be symmetric. Global exponential stability of the equilibrium point $\omega = 0_{3 \times 1}$ is guaranteed if $V$ remains negative-definite as the spacecraft dynamics evolves in time, which implies the existence of a positive-definite constant matrix $Q \in \mathbb{R}_{++}^{3 \times 3}$ such that the Lyapunov equation

$$H_{1d}(\rho, \omega, \beta, t) + H_{2d}^T(\rho, \omega, \beta, t) + P_d(\rho, \beta, t) + Q = 0$$

is satisfied for all $t \geq 0$. Due to rank deficiency of the damped CCNP $P_d(\rho, \beta, t)$, the above written Lyapunov equation does not have a unique stabilizing solution for the matrix gain $K$. Nevertheless, the second perturbed CCNP property given by (21) implies that over the domain given by (47), Eq. (65) can be written as

$$H_{1d}(\rho, \omega, \beta, t) + H_{2d}^T(\rho, \omega, \beta, t) + \tilde{P}(\rho, \delta, t)P_d(\rho, \beta, t)K + KP_d(\rho, \beta, t)\tilde{P}(\rho, \delta, t) + Q = 0$$

where $\delta$ is any nonzero real scalar. Therefore, the unique solution for the gain matrix damped nullprojection $P_d(\rho, \beta, t)K$ that exponentially stabilizes the dynamical subsystem given by (63) is

$$P_d(\rho, \beta, t)K = \left(I_{3 \times 3} \otimes \tilde{P}(\rho, \delta, t) + \tilde{P}(\rho, \delta, t) \otimes I_{3 \times 3}\right)^{-1} \vec{\left[H_{1d}(\rho, \omega, \beta, t) + H_{2d}^T(\rho, \omega, \beta, t) + Q\right]}$$

$$= \vec{\left[\tilde{P}(\rho, \delta, t) \otimes \tilde{P}(\rho, \delta, t)\right]}^{-1} \vec{\left[H_{1d}(\rho, \omega, \beta, t) + H_{2d}^T(\rho, \omega, \beta, t) + Q\right]}.$$
where $\otimes$ and $\oplus$ are respectively the kronecker product and sum of matrices, and vec is the matrix vectorizing operator ([17], p. 251). Since $\delta$ is arbitrary, the control law $\tau_{d1}$ given by

$$
\tau_{d1} = [H_{4d}(\rho, \omega, \beta, t) + \mathcal{P}_d(\rho, \beta, t)K] \omega
$$

renders the equilibrium point $\omega = 0_{3\times 1}$ of the spacecraft dynamical subsystem given by (2) globally exponentially stable for any real value of $\delta$. Global exponential stability of the internal dynamical subsystem given by (63) in addition to global boundedness of the two forcing terms that drive the closed loop dynamical subsystem given by (62) imply global uniform ultimate boundedness of the closed loop dynamical subsystem given by (62).

IX. STABILITY VERSUS PERFORMANCE TRADEOFFS DUE TO DAMPING FACTOR SIZING

Modifying the definition of the CCGI by means of the damping factor $\beta$ according to (35) provides robustness against generalized inversion instability because it limits the growth of the generalized inverse as steady state response is approached. This results in an approximate realization of the desired spacecraft attitude deviation norm measure dynamics, with globally uniformly ultimately bounded closed loop attitude trajectories tracking errors. According to (47), this ultimate bound is directly proportional to the damping factor by which the generalized inverse is modified. Therefore, decreasing $\beta$ implies better tracking performance. Nevertheless, decreasing $\beta$ causes the divergence of the damped CCGI vector elements in (46) to increase, and causes the control vector signal to reach large values, and to become infinite in the limit as the damping factor vanishes. This is unacceptable for practical reasons that are related to the limited available actuating power. Therefore, the damping factor $\beta$ results in a tradeoff between trajectory tracking accuracy and generalized inversion stability.

X. CONTROL SYSTEM DESIGN PROCEDURE

The procedures for designing CCGI attitude tracking control systems are summarized in the following steps

1) A desired spacecraft attitude trajectory $\rho_i(t)$ is prescribed, where $\rho_i$ is at least twice differentiable in $t$. The desired angular velocity vector $\omega_i(t)$ is accordingly defined by (14).

2) The coefficients $c_1$ and $c_2$ in the attitude deviation norm measure dynamics equation (9) are chosen such that the dynamics of $\phi$ is stable. This implies that both $c_1$ and $c_2$ are strictly positive.

3) The expressions given by (11) and (12) for $A(\rho, t)$ and $B(\rho, \omega, t)$ are obtained, where $G(\rho)$ and $z(\rho, t)$ are given by (3) and (4) respectively.

4) The CCGI $A^+(\rho, t)$ given by (16) is modified in the manner of (35), and $A^+_i(\rho, \beta, t)$ is used to define the expressions of $\mathcal{P}_d(\rho, \beta, t)$, $H_4d(\rho, \omega, \beta, t)$, and $H_2d(\rho, \beta, t)$ according to (38), (44), and (45), respectively.

5) The control law $\tau_d$ given by (46) is applied. Null-control vector is taken to be linear in the difference between the spacecraft angular velocity vector $\omega$ and the desired angular velocity vector $\omega_i(t)$ according to (60). Projection of the proportionality gain matrix $K$ on the damp control coefficients nullspace given by $\mathcal{P}_d(\rho, \beta, t)K$ is given by the expression of (67). The involved perturbed CCNP $\mathcal{P}(\rho, \delta, t)$ is given by (20), and the constant matrix $Q$ is arbitrary but positive definite. The control law takes the form of (61).

6) Integrate (1) and (2) to obtain the trajectories of $\rho(t)$ and $\omega(t)$, where $u = J\tau$. The resulting trajectory tracking errors are uniformly ultimately bounded according to (59).

The controls coefficient generalized inversion design structure is illustrated by the block diagram in Figure (5).

XI. NUMERICAL SIMULATIONS

The spacecraft model used for numerical simulations has inertia parameters $I_1 = 200$ Kg-m$^2$, $I_2 = 150$ Kg-m$^2$, $I_3 = 175$ Kg-m$^2$. The desired MRPs trajectories are chosen to be $\rho_{ci} = \cos 0.1 t$, $i = 1, 2, 3$. Their initial values are given by the vector $\rho(0) = [-0.42, 0.35, -0.26]^T$, and the initial spacecraft body angular velocity vector is $\omega(0) = [0.30, -0.61, -0.30]^T$. Parameter value of $\beta = 0.3$ is chosen. Figures 3, 4, and 5 show plots of the MRPs, the angular velocity components, and the scaled control variables versus time $t$. The steady state attitude attitude deviation dynamics is dependent only on the value of $\beta$, and the values of $c_1$ and $c_2$ affect only transient attitude response. Altering the value of $\beta$ reveals the tradeoff between trajectory tracking accuracy and damped generalized inverse stability [11].

XII. CONCLUSION

A controls coefficient generalized inversion design methodologies for spacecraft attitude trajectory tracking is presented. The control design is based on Lyapunov control function construction of the null-control vector in the generalized inversion-based Greville formula for the general solution of linear algebraic system equations. The closed loop attitude dynamics solely depends on a predetermined attitude deviation servo-constraint dynamics, and therefore
it is independent of the null-control vector design. The construction of the null-control vector is made in order to globally stabilize the spacecraft internal dynamics.

REFERENCES