Some Exact Solutions of an Elastico-Viscous Fluid

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Abstract—The exact solutions for the unsteady flow of an elastico-viscous fluid caused by general periodic oscillations are obtained. Further, the plate is assumed to be rigid as well as porous executing periodic rotary oscillations. The velocity field and shear stress are established.

Keywords—Elastico-viscous fluid, Periodic oscillations, Porous plate, Porous disk, Temporal Fourier transform

1. INTRODUCTION

Elastico-viscous fluids, in particular the model characterized by Walters [1] (liquid $B'$), exhibit a close resemblance to real fluids like oils and certain high polymer solutions. One of the important aspects of any fluid is the behavior of its unsteady boundary layers. A large number of papers concerning the time-dependent flows of elastico-viscous liquid $B'$ have appeared in the literature, for example, [2–7], etc.

The purpose of the present study is three-fold. First, in Section 2, exact analytic solutions are established for the unsteady flow of an elastico-viscous fluid caused by periodic oscillations of a porous plate. A general periodic function $g(t)$ with period $T_0$ is considered. An expression for shear stress is also given. The flow fields and shear stresses due to certain special values of oscillations are derived as special cases of the periodic oscillations. Second, in Section 3, the wall considered is nonporous and the exact flow generated by general periodic oscillations of a rigid plate is obtained. Third, in Section 4, the problem of a porous disk executing periodic rotary oscillations is discussed. The velocity field and the moment of the shear stresses are established. In order to study the behavior of the solution, a second-order approximation of the appropriate root of the characteristic equation is found and the structure of waves is discussed. A comparison of these results is made among the elastico-viscous, second grade, and Newtonian fluid cases.

2. FORMULATION OF THE PROBLEM

Let $x$ and $y$ denote the space coordinates. The $x$-axis is taken in the direction of oscillation and $y$-axis normal to the porous plate, respectively. Let $V_1$ and $V_2$ be the $x$- and $y$-components...
of fluid velocity and \( t \) the time. As the plate is assumed to be infinite, all the physical variables are functions of \( y \) and \( t \) only. The fluid over a porous plate is assumed to be an elastico-viscous incompressible fluid. The equation of motion for the oscillatory flow field is \[^8\]

\[
\rho \left( \frac{\partial v_k}{\partial t} + v_k \frac{\partial v_k}{\partial x_k} \right) = -\frac{\partial p}{\partial x_k} + \eta_0 \frac{\partial^2 v_k}{\partial x_k \partial x_k} - k_0 \left[ \frac{\partial}{\partial t} \left( \frac{\partial^2 v_1}{\partial x_k \partial x_k} \right) + \nu \frac{\partial^2 v_1}{\partial x_m \partial x_k \partial x_k} \right]
\]

(1)

In the above equation, \( \rho \) is density, \( p \) is pressure, \( \eta_0 \) is viscosity, and \( k_0 \) is an elastic coefficient.

Now, assuming uniform suction, we take

\[
v_1 = u, \quad v_2 = -W, \quad v_3 = 0,
\]

(2)

where \( W > 0 \) is the uniform suction velocity normal to the plate. Thus, in absence of pressure gradient, we have, from equations (1) and (2),

\[
\frac{\partial u}{\partial t} - W \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{k_0}{\rho} \left[ \frac{\partial^2 u}{\partial y^2 \partial t} - W \frac{\partial^3 u}{\partial y^3} \right],
\]

(3)

where \( \nu = \frac{\eta_0}{\rho} \). For the problem under consideration, the plate is making periodic oscillations of the form \( g(t) \) with period \( T_0(=2\pi/\alpha) \). Then the Fourier series representation of \( g(t) \) is given by

\[
g(t) = \sum_{n=-\infty}^{\infty} m_n e^{i\omega_0 t},
\]

(4)

where \( \alpha \) is nonzero fundamental frequency and the Fourier series coefficients \( \{m_n\} \) are given by

\[
m_n = \frac{1}{T_0} \int_{T_0} g(t) e^{-i\omega_0 t} dt
\]

(5)

Thus, the boundary conditions are

\[
u(0, t) = U_0 g(t),
\]

\[
u(y, t) \to 0, \quad \text{as } y \to \infty
\]

(6)

Equations (3) and (6) can be rewritten in the following nondimensional form

\[
\frac{\partial f}{\partial \tau} - \frac{\partial f}{\partial \eta} = \frac{\partial^2 f}{\partial \eta^2} - k \left( \frac{\partial^3 f}{\partial \eta^2 \partial \tau} - \frac{\partial^3 f}{\partial \eta^3} \right),
\]

(7)

\[
f(0, \tau) = \sum_{n=-\infty}^{\infty} m_n e^{i\omega_0 \tau},
\]

(8)

\[
f(\eta, \tau) \to 0, \quad \text{as } \eta \to \infty,
\]

where

\[
y = \frac{\nu}{W} \eta, \quad u = U_0 f(\eta, \tau), \quad t = \frac{\nu}{W^2} \tau, \quad k = \frac{k_0 W^2}{\rho \nu^2}, \quad \alpha = \frac{W^2}{\nu} \omega_0,
\]

(9)

and \( U_0 \) is a reference velocity.
2.1. Solution of the Problem

Let us suppose that $f(\eta, \tau)$ is an arbitrary function that has a Fourier transform in the variable $\tau$. That is,

$$ F(\eta, \omega) = \int_{-\infty}^{\infty} f(\eta, \tau) e^{-i\omega \tau} \, d\tau \quad (10) $$

By applying the Fourier transform termwise to the partial differential equation (7) and conditions (8), using (10), we obtain

$$ kF''' + (1 - ik\omega)F'' + F' - i\omega F = 0, \quad (11) $$

$$ F(0, \omega) = 2\pi \sum_{n=-\infty}^{\infty} m_n \delta (\omega - n\omega_0), \quad (12) $$

$$ F(\eta, \omega) \to 0, \quad \text{as} \ \eta \to \infty, $$

where $F(\eta, \omega)$ denotes the Fourier transform of $f(\eta, \tau)$ and prime denotes differentiation with respect to $\eta$.

We observe that the mathematical problem in the transformed plane $\omega$ is the same as in [7] except that boundary conditions (12) replace (6). Thus, in order to avoid repetition, details are omitted and the general solution of equation (11) satisfying the boundary conditions (12) is

$$ F(\eta, \omega) = 2\pi \sum_{n=-\infty}^{\infty} m_n e^{-an^2 + bn^2 \delta (\omega - n\omega_0)} \quad (13) $$

The inversion of the above equation will provide us with the solution for the problem. Hence, using a property of the delta function after Fourier inversion, we have

$$ u(\eta, \tau) = U_0 \sum_{n=-\infty}^{\infty} m_n e^{-a_n \eta + i(\omega n \eta + b_n \eta)}, \quad (14) $$

where

$$ a_n = (a)_{\omega=n\omega_0} = \frac{1}{2} (1 + \alpha_{0n}) - \frac{A_n k}{2\sqrt{1 + 16n^2 \omega_0^2}} + \frac{C_n k^2}{\sqrt{1 + 16n^2 \omega_0^2}}, $$

$$ b_n = (b)_{\omega=n\omega_0} = \frac{1}{2} \beta_{0n} + \frac{B_n k}{2\sqrt{1 + 16n^2 \omega_0^2}} + \frac{D_n k^2}{\sqrt{1 + 16n^2 \omega_0^2}}, $$

$$ \alpha_{0n} = \left[ \frac{1}{2} \left( 1 + 16n^2 \omega_0^2 + 1 \right) \right]^{1/2}, $$

$$ \beta_{0n} = \left[ \frac{1}{2} \left( 1 + 16n^2 \omega_0^2 - 1 \right) \right]^{1/2}, $$

$$ A_n = \alpha_{0n} + \alpha_{0n}^2 + \beta_{0n}^2 - 2\omega_0^2 n^2 \alpha_{0n} + 4\omega_0 n \beta_{0n}, $$

$$ B_n = 4\omega_0 n \alpha_{0n} + 2\omega_0^2 n^2 \alpha_{0n} + 2\omega_0^2 n^2 \beta_{0n} + 2\omega_0 n \beta_{0n}^2 - \beta_{0n}, $$

$$ C_n = P_n \alpha_{0n} + q_n \beta_{0n}, $$

$$ D_n = q_n \alpha_{0n} - P_n \beta_{0n}, $$

$$ P_n = \left[ \frac{(2\omega_0 n \beta_{0n} - 3(1 + \alpha_{0n})) A_n + (8\omega_0^2 + 2\omega_0 n \alpha_{0n} + 3\beta_{0n}) B_n}{(A_n^2 - B_n^2)} \right]^{-1} \sqrt{1 + 16n^2 \omega_0^2}, $$

$$ \beta_{0n} = 4\sqrt{1 + 16n^2 \omega_0^2}.$$
Equation (14) describes the velocity field over the porous plate oscillating periodically in its own plane. As a special case of this oscillation, the flow for different plate oscillations is obtained by an appropriate choice of the Fourier coefficients which give rise to different plate oscillations. Hence, it is necessary to prescribe some physically realistic form for \( g(t) \).

2.2. Flow Fields for Some Special Oscillations

In this section, we shall consider some flows due to the special oscillations. The periodic oscillations and their corresponding coefficients are given in Table 1.

<table>
<thead>
<tr>
<th>Oscillations ( { g(t) } )</th>
<th>Fourier Coefficients ( { m_n } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( e^{\omega_0 t} )</td>
<td>( m_1 = 1 ) and ( m_n = 0 ) (( n \neq 1 ))</td>
</tr>
<tr>
<td>2 ( \cos \omega_0 t )</td>
<td>( m_1 = m_{-1} = \frac{1}{2} ) and ( m_n = 0 ), otherwise</td>
</tr>
<tr>
<td>3 ( \sin \omega_0 t )</td>
<td>( m_1 = -m_{-1} = \frac{1}{2i} ) and ( m_n = 0 ), otherwise</td>
</tr>
<tr>
<td>4 ( {1, \ T_1 &lt; \lvert t \rvert &lt; T_0} )</td>
<td>( m_0 = \frac{2T_1}{T_0} ) and ( m_n = \frac{\sin n\omega_0 T_1}{n\pi} ), otherwise</td>
</tr>
<tr>
<td>5 ( \sum_{n=-\infty}^{\infty} \delta (t - nT_0) )</td>
<td>( m_n = \frac{1}{T_0} ), for all ( n )</td>
</tr>
</tbody>
</table>

Using the appropriate Fourier Coefficients in equation (14), the flows in the above five cases are, respectively, given by:

\[
\begin{align*}
  u_1(\eta, \tau) &= U_0 e^{-a_1 \eta + i(\omega_0 \tau + b_1 \eta)}, \\
  u_2(\eta, \tau) &= \frac{U_0}{2} \left[ e^{-a_1 \eta + i(\omega_0 \tau + b_1 \eta)} + e^{-a_{-1} \eta - i(\omega_0 \tau + b_{-1} \eta)} \right], \\
  u_3(\eta, \tau) &= \frac{U_0}{2i} \left[ e^{-a_1 \eta + i(\omega_0 \tau + b_1 \eta)} - e^{-a_{-1} \eta - i(\omega_0 \tau + b_{-1} \eta)} \right], \\
  u_4(\eta, \tau) &= U_0 \sum_{n=-\infty}^{\infty} \frac{\sin (n\omega_0 T_1)}{n\pi} e^{a_n \eta + i(\omega_0 n \tau + b_n \eta)}, \quad n \neq 0, \\
  u_5(\eta, \tau) &= \frac{U_0}{T_0} \sum_{n=-\infty}^{\infty} e^{-a_n \eta + i(\omega_0 n \tau + b_n \eta)}
\end{align*}
\]

2.3. Shear Stresses

The more interesting part of the solution is, however, the prediction of the behavior of the shear stresses. The expression for the shear stress on the plate \( y = 0 \) is given by the real part of

\[
P_{xy} = \eta_0 \frac{\partial u}{\partial y} - k_0 \left[ \frac{\partial^2 u}{\partial y^2} - W \frac{\partial^2 u}{\partial y^2} \right],
\]

which in terms of the variable \( \eta \) becomes

\[
P_{xy} = U_0 \rho W \Re \left\{ \frac{\partial f}{\partial \eta} - k \left[ \frac{\partial^2 f}{\partial \eta \partial \tau} - \frac{\partial^2 f}{\partial \eta^2} \right] \right\},
\]
where \( \text{Re} \) denotes the real part. Substituting the expression for \( u \), given by (14)–(19), we find that

\[
(P_{xy}) = U_0 \rho W \text{Re} \left[ \sum_{n=-\infty}^{\infty} m_n \{(-a_n + i b_n) ((1 + a_n) - i (b_n + n k w_0)) e^{i w_0 n \tau} \} \right],
\]

(22)

\[
(P_{xy})_1 = U_0 \rho W \text{Re} \left[ \{ (-a_1 (1 + a_1) + b_1 (b_1 + k w_0)) \} e^{i w_0 \tau} \right]
\]

(23)

\[
(P_{xy})_2 = \frac{U_0 \rho W}{2} \text{Re} \left[ \{ (-a_1 + i b_1) ((1 + a_1) - i (b_1 + k w_0)) e^{i w_0 \tau} \right],
\]

(24)

\[
(P_{xy})_3 = -\frac{U_0 \rho W}{2} \text{Re} \left[ \{((-a_1 + i b_1) ((1 + a_1) - i (b_1 + k w_0)) e^{i w_0 \tau} \right),
\]

(25)

\[
(P_{xy})_4 = \frac{U_0 \rho W}{\pi} \text{Re} \left[ \sum_{n=-\infty}^{\infty} \frac{\sin (n w_0 T_1)}{n} (-a_n + i b_n) ((1 + a_n) - i (b_n + n k w_0)) e^{i w_0 \tau}, n \neq 0 \right],
\]

(26)

\[
(P_{xy})_5 = \frac{U_0 \rho W}{T_0} \text{Re} \left[ \sum_{n=-\infty}^{\infty} (-a_n + i b_n) ((1 + a_n) - i (b_n + n k w_0)) e^{i w_0 \tau} \right]
\]

(27)

From the unsteady boundary layers (14), we notice that the fluctuations of the motion die out exponentially as we move away from the plate. In fact, a typical length that characterizes the amplitude decrease is the quantity \( l/a_n \). In other words, the length scale of variation of the amplitude in the normal direction is \( l/a_n \). Smaller values of viscosity or larger values of frequency imply smaller thickness of the extent to which the fluid is disturbed by the plate oscillation. Mathematically, of course, the fluid is disturbed all the way to infinity, regardless of the value of viscosity, frequency, or material parameters. The amplitude has its maximum value at \( \eta = 0 \). Also, equation (14) shows that there is a phase shift in the motion of the fluid and that this phase shift is proportional to \( \eta \). Further, it is observed that shearing stress decreases with increasing frequency.

### 3. PERIODIC FLOWS ON A NONPOROUS PLATE

The problem for the flow \( W = 0 \) of an elastico-viscous fluid due to an oscillating \( (e^{i \omega_0 t}) \) infinite plate have been solved by Foote et al [7]. In this section, we shall derive the flow due to general periodic oscillations \( g(t) \). On introducing the nondimensional quantities

\[
y = \frac{\nu}{U_0} \tilde{\eta}, \quad \tilde{u} = U_0 f(\tilde{\eta}, \tilde{\tau}), \quad t = \frac{\nu}{U_0^2} \tilde{\tau}, \quad \tilde{k} = \frac{k_0 U_0^2}{\rho \nu^2}, \quad \alpha = \frac{U_0^2}{\nu} \tilde{\omega}_0
\]

in equations (3) and (6) and then using the same method of solution as in Sections 2 1–2 3, we arrive at

\[
\tilde{u}(\tilde{\eta}, \tilde{\tau}) = U_0 \sum_{n=-\infty}^{\infty} m_n e^{-\tilde{\alpha}_n \tilde{\eta} + i (\tilde{\omega}_0 \tilde{\tau} - \tilde{b}_n \tilde{\eta})},
\]

(28)

\[
\tilde{u}_1(\tilde{\eta}, \tilde{\tau}) = U_0 e^{-\tilde{\alpha}_1 \tilde{\eta} + i (\tilde{\omega}_0 \tilde{\tau} - \tilde{b}_1 \tilde{\eta})},
\]

(29)
\[ \tilde{u}_2 (\eta, \varphi) = \frac{U_0}{2} \left[ e^{-\tilde{a}_1 \eta + i(\tilde{\omega}_0 \varphi - \tilde{b}_1 \eta)} + e^{-\tilde{a}_1 \eta - i(\tilde{\omega}_0 \varphi + \tilde{b}_1 \eta)} \right], \]  
\[ \tilde{u}_3 (\eta, \varphi) = \frac{U_0}{2i} \left[ e^{-\tilde{a}_1 \eta + i(\tilde{\omega}_0 \varphi - \tilde{b}_1 \eta)} - e^{-\tilde{a}_1 \eta - i(\tilde{\omega}_0 \varphi + \tilde{b}_1 \eta)} \right], \]  
\[ \tilde{u}_4 (\eta, \varphi) = U_0 \sum_{n=-\infty}^{\infty} \frac{\sin(n\tilde{\omega}_0 T_1)}{n\pi} e^{-\tilde{a}_n \eta + i(\tilde{\omega}_0 n \varphi - \tilde{b}_n \eta)}, \quad n \neq 0, \]  
\[ \tilde{u}_5 (\eta, \varphi) = \frac{U_0}{T_0} \sum_{n=-\infty}^{\infty} e^{-\tilde{a}_n \eta + i(\tilde{\omega}_0 n \varphi - \tilde{b}_n \eta)}, \]  
\[ (\tilde{P}_{xy}) = U_0^2 \rho \Re \left[ \sum_{n=-\infty}^{\infty} m_n (\tilde{a}_n + i\tilde{b}_n) \left\{ (1 + \tilde{a}_n) - i \left( \tilde{b}_n + n\tilde{k}\tilde{\omega}_0 \right) \right\} e^{i\tilde{\omega}_0 n \varphi} \right], \]  
\[ (\tilde{P}_{xy})_1 = U_0^2 \rho \Re \left[ \left\{ \frac{1}{2} \left( \tilde{a}_1 (1 + \tilde{a}_1) + \tilde{b}_1 (\tilde{b}_1 + \tilde{k}\tilde{\omega}_0) \right) \right\} e^{i\tilde{\omega}_0 \varphi}, \right] \]  
\[ (\tilde{P}_{xy})_2 = \frac{U_0^2 \rho}{2} \Re \left[ \left\{ \frac{1}{2} \left( \tilde{a}_1 + i\tilde{b}_1 \right) \left( 1 + \tilde{a}_1 + i\tilde{b}_1 \right) \right\} e^{i\tilde{\omega}_0 \varphi} \right], \]  
\[ (\tilde{P}_{xy})_3 = \frac{-U_0^2 \rho}{2} \Re \left[ \left\{ \frac{1}{2} \left( \tilde{a}_1 + i\tilde{b}_1 \right) \left( 1 + \tilde{a}_1 + i\tilde{b}_1 \right) \right\} e^{i\tilde{\omega}_0 \varphi} \right], \]  
\[ (\tilde{P}_{xy})_4 = \frac{U_0^2 \rho}{\pi} \Re \left[ \sum_{n=-\infty}^{\infty} \frac{\sin(n\tilde{\omega}_0 T_1)}{n} (\tilde{a}_n + i\tilde{b}_n) \left\{ (1 + \tilde{a}_n) - i \left( \tilde{b}_n + n\tilde{k}\tilde{\omega}_0 \right) \right\} e^{i\tilde{\omega}_0 n \varphi}, \right] \]  
\[ \left. \times e^{i\tilde{\omega}_0 n \varphi}, \quad n \neq 0, \right\}, \]  
\[ (\tilde{P}_{xy})_5 = \frac{U_0^2 \rho}{T_0} \Re \left[ \sum_{n=-\infty}^{\infty} (\tilde{a}_n + i\tilde{b}_n) \left\{ (1 + \tilde{a}_n) - i \left( \tilde{b}_n + n\tilde{k}\tilde{\omega}_0 \right) \right\} e^{i\tilde{\omega}_0 n \varphi} \right], \]  
where
\[ \tilde{a}_n = \left[ \frac{\tilde{n}\tilde{\omega}_0}{1 + \tilde{k}^2 n^2 \tilde{\omega}_0^2} \left\{ \sqrt{1 + \tilde{k}^2 n^2 \tilde{\omega}_0^2} + \tilde{k}\tilde{n}\tilde{\omega}_0 \right\} \right]^{1/2}, \]
\[ \tilde{b}_n = \left[ \frac{\tilde{n}\tilde{\omega}_0}{1 + \tilde{k}^2 n^2 \tilde{\omega}_0^2} \left\{ \sqrt{1 + \tilde{k}^2 n^2 \tilde{\omega}_0^2} - \tilde{k}\tilde{n}\tilde{\omega}_0 \right\} \right]^{1/2}. \]

### 4. DISK PROBLEM

In this section, we investigate the effects of a porous disk executing rotary oscillations of small amplitude about its axes in an elastico-viscous fluid. We consider the unbounded volume of the fluid on the side \( z > 0 \) and the disk is rotating about the \( z \)-axis with angular velocity \( \Omega \). Assuming uniform suction, the velocity components are taken as
\[ V_r = 0, \quad V_\theta = r\Omega (z, t), \quad V_z = -W. \]
Now, using equation (42) into the balance of linear momentum, we, respectively, obtain the $r$-, $\Theta$-, and $z$-components of the velocity field as

\begin{align}
-\frac{\rho \Omega^2}{r} &= -\frac{\partial p}{\partial r} + 2k_0 \frac{r}{\Omega^2} \frac{\partial \Omega}{\partial z}, \\
\rho \left[ \frac{\partial \Omega}{\partial t} - W \frac{\partial \Omega}{\partial z} \right] &= -\frac{1}{r^2} \frac{\partial p}{\partial \Theta} + \eta_0 \frac{\partial^2 \Omega}{\partial z^2} - k_0 \left[ \frac{\partial^3 \Omega}{\partial z^2 \partial t} - W \frac{\partial^3 \Omega}{\partial z^3} \right], \\
0 &= -\frac{\partial p}{\partial z} + k_0 \left[ \frac{4 \Omega \partial^2 \Omega}{\partial z^2} + 2r^2 \frac{\partial}{\partial z} \left( \frac{\partial \Omega}{\partial z} \right)^2 \right].
\end{align}

We note from the above three equations that suction only contributes in the $z$-component (i.e., equation (44)). Thus, we shall seek a solution of equation (44) in which $\frac{\partial p}{\partial \Theta}$ is zero. This would follow from the assumption of rotational symmetry. Thus, equation (44) can be written as

\begin{align}
\rho \left[ \frac{\partial \Omega}{\partial t} - W \frac{\partial \Omega}{\partial z} \right] &= \eta_0 \frac{\partial^2 \Omega}{\partial z^2} - k_0 \left[ \frac{\partial^3 \Omega}{\partial z^2 \partial t} - W \frac{\partial^3 \Omega}{\partial z^3} \right].
\end{align}

Since we are interested in a porous disk of large radius $R$ executing rotary oscillations of small amplitude about its axes, the angle of rotation is

$$\Theta = \Theta_0 \cos \omega t,$$

where $\Theta_0 \ll 1$ and $\alpha$ is the frequency. For the problem in question, the boundary conditions are

\begin{align}
\Omega(0, t) &= -\alpha \Theta_0 \sin \omega t, \\
\Omega(z, t) &\to 0, \quad \text{as } z \to \infty.
\end{align}

We note that for $k_0 = 0 = W$, equation (46) corresponds to the differential equation for Newtonian fluid [9].

### 4.1. Solution of the Problem

The boundary value problem (equations (46) and (48)) can be rewritten in the following dimensionless form

\begin{align}
\frac{\partial f}{\partial \tau} - \frac{\partial f}{\partial \eta} &= \frac{\partial^2 f}{\partial \eta^2} - k \left( \frac{\partial^3 f}{\partial \eta^3} - \frac{\partial^3 f}{\partial \eta^2 \partial \tau} \right), \\
\eta = -\alpha \Theta_0 f(\eta, \tau), \\
f(0, \tau) &= \sin \omega \tau, \\
f(\eta, \tau) &\to 0, \quad \text{as } \eta \to \infty.
\end{align}

where

\begin{align}
z &= \frac{\nu}{W} \eta, \\
\Omega &= -\alpha \Theta_0 f(\eta, \tau), \\
\tau &= \frac{\nu}{W^2} \tau, \\
k &= \frac{k_0 W^2}{\mu \nu^2}, \\
\alpha &= \frac{W^2}{\nu} \omega.
\end{align}

We observe that the mathematical problem in the dimensionless form is the same as that of Foote et al. [7] except that $\sin \omega \tau$ in equation (50) replaces $\exp(i \omega \tau)$ in equation (5) of [7]. Thus, using the same method of solution as in [7], we have

$$\Omega = -\alpha \Theta_0 e^{-a \eta} \sin(b \eta + \omega \tau),$$

\(52\)
where
\[ a = \frac{1}{2} (1 + \alpha_0) - \frac{A_k}{2\sqrt{1 + \omega^2}} - \frac{C_k^2}{\sqrt{1 + \omega^2}}, \]
\[ b = -\frac{1}{2} \beta_0 + \frac{B_k}{2\sqrt{1 + \omega^2}} + \frac{D_k^2}{\sqrt{1 + \omega^2}}, \]
\[ \alpha_0 = \left[ \frac{1}{2} \left\{ \sqrt{1 + 16\omega^2} + 1 \right\} \right]^{1/2}, \]
\[ \beta_0 = \left[ \frac{1}{2} \left\{ \sqrt{1 + 16\omega^2} - 1 \right\} \right]^{1/2}, \]
\[ A = \alpha_0 + \alpha_0^2 + \beta_0^2 - 2\omega^2\alpha_0 + 4\omega\beta_0, \]
\[ B = 4\omega\alpha_0 + 2\omega\alpha_0^2 + 2\omega^2\beta_0 + 2\omega\beta_0^2 - \beta_0, \]
\[ C = Pa_0 + q\beta_0, \]
\[ D = q\alpha_0 - P\beta_0, \]
\[ P = \begin{bmatrix} (2\omega\beta_0 - 3(1 + \alpha_0)) A + (8\omega + 2\omega\alpha_0 + 3\beta_0) B \\ - \frac{(A^2 - B^2)}{\sqrt{1 + 16\omega^2}} \end{bmatrix} \times \left\{ \frac{4}{\sqrt{1 + 16\omega^2}} \right\}^{-1}, \]
\[ q = \begin{bmatrix} (2\omega\beta_0 - 3(1 + \alpha_0)) B - (2\omega\alpha_0 - 4\omega - 3\beta_0) A \\ - \frac{2AB}{\sqrt{1 + 16\omega^2}} \end{bmatrix} \times \left\{ \frac{4}{\sqrt{1 + 16\omega^2}} \right\}^{-1}. \]

With the help of equations (42), (51), and (52), we have
\[ V_\Theta = -r\alpha \Theta_0 e^{(a W z)/\nu} \sin \left( \frac{b W z}{\nu} + \alpha t \right) \] (53)

The shear stress \( F \) acting on a unit area of the disk \( z = 0 \) perpendicularly to the radius is given by
\[ F = \eta_0 \frac{\partial V_\Theta}{\partial z} - k_0 \left[ \frac{\partial^2 V_\Theta}{\partial z \partial t} - W \frac{\partial^2 V_\Theta}{\partial z^2} \right] \] (54)

Substitution of equation (53) in equation (54) yields
\[ F = \frac{r\alpha \Theta_0 W}{\nu} e^{-a W z/\nu} \left[ F_1 \sin \left( \frac{b W z}{\nu} + \alpha t \right) + F_2 \cos \left( \frac{b W z}{\nu} + \alpha t \right) \right], \] (55)

where
\[ F_1 = a\eta_0 - k_0 b\alpha + \frac{k_0 W^2}{\nu} (b^2 - \alpha^2), \]
\[ F_2 = -b\eta_0 - k_0 a\alpha + \frac{2ab k_0 W^2}{\nu}. \]

The moment \( M \) of the shear stress acting on a disk of large but finite radius \( R \) is
\[ M = 4\pi \int_0^R r^2 F \, dr \] (56)

Finally, on using equation (55) in equation (56), the moment of the shear stress is completely expressed as
\[ M = \frac{\pi R^4 \alpha \Theta_0 W}{\nu} \left[ F_1 \sin \alpha t + F_2 \cos \alpha t \right] \] (57)
We note, from equation (57), that, in the limiting case of $k_0 = 0 = W$, the result reduces to the classical result of Landau and Lifshitz [9].

Equation (53) represents a transverse wave since its velocity component $V_\Theta$ is perpendicular to the direction of propagation $z$. Also, the wave is rapidly damped in the interior of the fluid, i.e., the amplitude decreases exponentially with $z$. Thus, transverse waves can occur in an elastico-viscous fluid, but they are rapidly damped as they move away from the solid surface whose motion generates the waves.

The depth of penetration of the vorticity can be calculated from equation (53) and is $\approx \nu/aW$. This depth increases with viscosity and decreases with fundamental frequency. This is not surprising, if we slowly oscillate a disk in a sticky fluid, we expect to drag large masses of fluid along with the disk, on the other hand, if we move the disk rapidly in a fluid of low viscosity, we expect the fluid essentially to ignore the disk, except in a thin boundary layer.

The phase velocity $V_\Phi^*$ can be obtained from equation (53) and is written as

$$V_\Phi^* \left( \frac{dz}{dt} = 0 \right) = -\frac{\alpha \nu}{bW}$$

This shows that $V_\Phi^*$ increases with the increase of $\omega$ and $W$.

Our next job is to see the effects of elasticity on the physical parameter of the fluid flow in order to discuss the difference between viscous and elastico-viscous fluids. Thus, we try to explain the effect of $k$ on (a) the velocity and (b) the depth of penetration.

(a) From equation (53), we observe that as $k$ increases the damping of the velocity increases.

(b) The depth of penetration decreases more rapidly with the increase of $k$ in an elastico-viscous fluid.

5. CONCLUDING REMARKS

We obtained exact solutions for the Stokes problem and the corresponding problem on a porous plate for an elastico-viscous fluid, taking into account the general periodic oscillation of a plate. The velocity fields due to certain special values of oscillations in both problems are then derived as a special case of periodic oscillations including the one in Sections 2 2 and 3 considered by Foote et al [7]. We have also studied the flow of a porous oscillating disk.

We remark that the results of the second grade fluid in all the presented problems can be obtained by taking $-k_0 = \alpha_1 = \alpha_2$ (where $\alpha_1$ and $\alpha_2$ are the material moduli in the second grade fluid). Further, the results of viscous fluid [9] can also be obtained as a special case by taking $W = 0 = k_0$. This provides a useful check.

REFERENCES

3. P N Kaloni, Fluctuating flow of an elastico-viscous fluid past a porous plate, Phys Fluids 10, 1344 (1967)