Modified descent-projection method for solving variational inequalities

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Abstract

In this paper, we propose a modified descent-projection method for solving variational inequalities. The method makes use of a descent direction to produce the new iterate and can be viewed as an improvement of the descent-projection method by using a new step size. Under certain conditions, the global convergence of the proposed method is proved. In order to demonstrate the efficiency of the proposed method, we provide numerical results for a traffic equilibrium problems.

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Keywords: Variational inequalities; Traffic equilibrium problems; Projection method; Pseudomonotone operators

1. Introduction

A classical variational inequality problem, denoted by \( \text{VI}(T, K) \), is to find a vector \( u^* \in K \) such that

\[
(T(u^*), v - u^*) \geq 0 \quad \forall v \in K,
\]

where \( K \subset \mathbb{R}^n \) is a nonempty closed convex subset of \( \mathbb{R}^n \) and \( T \) is a mapping from \( \mathbb{R}^n \) into itself.

Variational inequality proved to be a very useful and powerful tool for investigation of many equilibrium type problems in economics, engineering, operation research and mathematical programming, see [1–32]. We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities and the related optimization problems. The fixed-point theory has played an important role in the development of various algorithms for solving variational inequalities. Using the projection operator technique, one usually establishes an equivalence between the variational inequalities and the fixed-point problem. This alternative equivalent formulation was used by Lions and Stampacchia [16] to study the existence of a

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1 This author was supported by MOEC Grant No. 20060284001 and by NSFC Grant Nos. 70571033 and 10571083.

2 This research is supported by the Higher Education Commission, Pakistan, through Grant No. 1-28/HEC/HRD/2005/90.
solution of the variational inequalities. Projection methods and its variants forms represent important tools for finding the approximate solution of variational inequalities. It is well known that the convergence of the projection method requires that the operator must be strongly monotone and Lipschitz continuous. Unfortunately these strict conditions rule out many applications of this method. To overcome the weakness of the projection method, Korpelevich [15] first proposed a modification of the method, which is called extragradient method, by performing an additional forward step and a projection at each iteration according to double projection. Its convergence requires only that a solution exists and the monotone operator is Lipschitz continuous. When the operator is not Lipschitz continuous or when the Lipschitz continuous is not known, Khobotov [14] removed the global Lipschitz and presented the idea of choosing \( q \) dynamically in a suitable way, and later the extragradient method and its variant forms [12,13,17,23,26,27,30] require an Armijo-like line search procedure to compute the step size with a new projection need for each trial, which leads to expensive computation. To overcome these difficulties, several modified projection and extragradient-type methods [2,9–11,13,20,21,24,25,31,28,32] have been suggested and developed for solving variational inequality problems. The main trend being toward the improvement of the projection method by appropriate choices of the step size parameters and removal the Lipschitz constant by using some line search.

In this paper, inspired and motivated by the above research, we suggest and analyze a new descent-projection method for solving variational inequality (1.1) by using the same descent direction as in [3] and we propose a new step size \( a_k \).

2. Preliminaries

In this section, we give some results which will be used in latter analysis.

**Lemma 2.1.** For a given \( u \in K, z \in \mathbb{R}^n \) satisfy the inequality
\[
\langle u - z, v - u \rangle \geq 0 \quad \forall v \in K
\]
holds if and only if \( u = P_K(z) \). It follows from Lemma 2.1 that
\[
\langle z - P_K(z), P_K(z) - v \rangle \geq 0 \quad \forall z \in \mathbb{R}^n, \ v \in K.
\] (2.1)

It follows that
\[
\|P_K(z) - v\| \leq \|z - v\| \quad \forall z \in \mathbb{R}^n, \ v \in K,
\] (2.2)
\[
\|P_K(z) - v\|^2 \leq \|z - v\|^2 - \|z - P_K(z)\|^2 \quad \forall z \in \mathbb{R}^n, \ v \in K.
\] (2.3)

It is well-known that the projection operator \( P_K \) is nonexpansive, that is,
\[
\|P_K(u) - P_K(v)\| \leq \|u - v\| \quad \forall u, \ v \in \mathbb{R}^n.
\] (2.4)

**Lemma 2.2.** \( u^* \) is a solution of problem (1.1) if and only if \( u^* \) satisfies the relation:
\[
u^* = P_K[u^* - \rho T(u^*)], \quad \text{where} \ \rho > 0.
\] (2.5)

From Lemma 2.2, it is clear that \( u \) is a solution of (1.1) if and only if \( u \) is a zero point of the function
\[
r(u, \rho) := u - P_K[u - \rho T(u)].
\]
The fixed-point formulation (2.5) was used to suggest and analyze the following algorithm for solving problem (1.1).

**Algorithm 2.1** [3]. For given \( u^k \in K \) and \( \rho_k > 0 \), compute
\[
\tilde{u}^k = P_K[u^k - \rho_k T(u^k)],
\] (2.6)
such that
\[
\|\rho_k(T(u^k) - T(\tilde{u}^k))\| \leq \delta\|u^k - \tilde{u}^k\|, \quad 0 < \delta < 1.
\] (2.7)
and the new iterate is defined by:

$$u^{k+1}(x_k) = P_k[u^k - x_k d(u^k, \rho_k)]$$

where

$$\varepsilon^k := \rho_k(T(\tilde{u}^k) - T(u^k)),$$

$$d(u^k, \rho_k) := u^k - \tilde{u}^k + \varepsilon^k,$$

$$\phi(u^k, \rho_k) := \langle u^k - \tilde{u}^k, d(u^k, \rho_k) \rangle$$

(2.8)

(2.9)

(2.10)

and the step size

$$\alpha_k := \gamma \sigma_k, \text{ where } 1 \leq \gamma < 2 \text{ and } \sigma_k := \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}.$$ (2.11)

where \(\rho_k\) is choosing according to some self-adaptive rule.

In what follows, we always assume that the underlying function is continuously differentiable on \(\mathbb{R}^n\) and pseudomonotone i.e.,

$$\langle T(u), u' - u \rangle \geq 0 \Rightarrow \langle T(u'), u' - u \rangle \geq 0 \quad \forall u', \: u \in \mathbb{R}^n,$$

and the solution set of problem (1.1), denoted by \(S^*\), is nonempty.

**Remark 2.1.** (2.7) implies that

$$||u^k - \tilde{u}^k, \varepsilon^k|| \leq \delta ||u^k - \tilde{u}^k||^2, \quad 0 < \delta < 1.$$ (2.12)

For the convergence analysis of the proposed method, we need the following results.

**Lemma 2.3.** For given \(u^k \in \mathbb{R}^n\) and \(\rho_k > 0\), let \(\tilde{u}^k\) and \(\varepsilon^k\) satisfy (2.6) and (2.8), then

$$\phi(u^k, \rho_k) \geq (1 - \delta) ||u^k - \tilde{u}^k||^2$$

(2.13)

and

$$\sigma_k \geq \frac{1}{2}.$$ (2.14)

**Proof.** It follows from (2.9) and (2.12) that

$$\phi(u^k, \rho_k) := \langle u^k - \tilde{u}^k, d(u^k, \rho_k) \rangle = ||u^k - \tilde{u}^k||^2 + \langle u^k - \tilde{u}^k, \varepsilon^k \rangle \geq (1 - \delta) ||u^k - \tilde{u}^k||^2.$$ (2.15)

Otherwise, we have

$$\langle u^k - \tilde{u}^k, d(u^k, \rho_k) \rangle = ||u^k - \tilde{u}^k||^2 + \langle u^k - \tilde{u}^k, \varepsilon^k \rangle \geq \frac{1}{2} ||u^k - \tilde{u}^k||^2 + \langle u^k - \tilde{u}^k, \varepsilon^k \rangle + \frac{1}{2} ||\varepsilon^k||^2 = \frac{1}{2} ||d(u^k, \rho_k)||^2,$$

where the inequality follows from (2.7) and

$$\sigma_k \geq \frac{1}{2},$$ (2.16)

we can get the assertion of this Lemma. \(\square\)

**Lemma 2.4.** \(\forall u^k \in \mathbb{R}^n, u^* \in S^* \text{ and } \rho_k > 0, \text{ we have}\)

$$\langle u^k - u^*, d(u^k, \rho_k) \rangle \geq \langle u^k - \tilde{u}^k, d(u^k, \rho_k) \rangle.$$ (2.17)

**Proof.** Let \(u^* \in S^*\) be a solution of problem (1.1) and \(\tilde{u}^k = P_k[u^k - \rho_k T(u^k)]\). Then we have

$$\langle \rho_k T(u^*), \tilde{u}^k - u^* \rangle \geq 0 \quad \forall \rho_k > 0.$$ (2.18)
Using the pseudomonotonicity of $T$, we obtain
\[ \langle p_k T(\bar{u}^k), \bar{u}^k - u^* \rangle \geq 0. \] (2.17)
Substituting $z = u^k - p_k T(u^k)$ and $v = u^*$ into (2.1), we get
\[ \langle u^k - \bar{u}^k - p_k T(u^k), \bar{u}^k - u^* \rangle \geq 0. \] (2.18)
Adding (2.17) and (2.18), we have
\[ \langle u^k - \bar{u}^k - p_k [T(u^k) - T(\bar{u}^k)], \bar{u}^k - u^* \rangle \geq 0, \]
which can be rewritten as
\[ \langle d(u^k, \rho_k), u^k - u^* - \bar{u}^k \rangle \geq 0, \]
then
\[ \langle u^k - u^*, d(u^k, \rho_k) \rangle \geq \langle u^k - \bar{u}^k, d(u^k, \rho_k) \rangle, \]
and the conclusion of Lemma 2.4 is proved. \qed

3. Main results

In this section, we prove some useful results which will be used in the consequent analysis and then investigate the strategy of how chose the step size $\alpha_k$.

**Theorem 3.1.** For given $u^k \in K, u^* \in S^*$ and $\rho_k > 0$, let $\bar{u}^k$ and $v^k$ satisfy (2.6) and (2.8), and $d(u^k, \rho_k)$ is defined in (2.9). Then
\[ \Theta(z_k) := \|u^k - u^*\|^2 - \|u^{k+1}(z_k) - u^*\|^2 \geq \Phi(z_k) \geq \Psi(z_k), \] (3.1)
where
\[ \Phi(z_k) := \|u^k - u^{k+1}(z_k)\|^2 + 2\alpha_k \langle u^{k+1}(z_k) - \bar{u}^k, d(u^k, \rho_k) \rangle, \] (3.2)
\[ \Psi(z_k) := 2\alpha_k \langle u^k - \bar{u}^k, d(u^k, \rho_k) \rangle - \alpha_k^2 \|d(u^k, \rho_k)\|^2. \] (3.3)
**Proof.** Since $u^* \in K$ and $u^{k+1}(z_k) = P_K[u^k - \alpha_k d(u^k, \rho_k)]$, it follows from (2.3) that
\[ \|u^{k+1}(z_k) - u^*\|^2 \leq \|u^k - \alpha_k d(u^k, \rho_k) - u^*\|^2 - \|u^k - \alpha_k d(u^k, \rho_k) - u^{k+1}(z_k)\|^2. \] (3.4)
Consequently, using the definition of $\Theta(z_k)$, we get
\[ \Theta(z_k) \geq \|u^k - u^*\|^2 - \|u^k - u^* - \alpha_k d(u^k, \rho_k)\|^2 + \|u^k - u^{k+1}(z_k) - \alpha_k d(u^k, \rho_k)\|^2 \]
\[ = \|u^k - u^{k+1}(z_k)\|^2 + 2\alpha_k \langle u^{k+1}(z_k) - \bar{u}^k, d(u^k, \rho_k) \rangle - 2\alpha_k \langle u^{k+1}(z_k) - \bar{u}^k, d(u^k, \rho_k) \rangle. \]
Applying Lemma 2.4 to the last term in the right side of the above inequality, we obtain
\[ \Theta(z_k) \geq \|u^k - u^{k+1}(z_k)\|^2 + 2\alpha_k \langle u^{k+1}(z_k) - u^k, d(u^k, \rho_k) \rangle + 2\alpha_k \langle u^k - \bar{u}^k, d(u^k, \rho_k) \rangle, \]
\[ = \|u^k - u^{k+1}(z_k)\|^2 + 2\alpha_k \langle u^{k+1}(z_k) - \bar{u}^k, d(u^k, \rho_k) \rangle, \]
\[ = \Phi(z_k), \]
\[ = \|u^k - u^{k+1}(z_k) - \alpha_k d(u^k, \rho_k)\|^2 + 2\alpha_k \langle u^k - \bar{u}^k - u^{k+1}(z_k), d(u^k, \rho_k) \rangle, \]
\[ \geq \Psi(z_k), \] (3.6)
and the assertion of this theorem is proved. \qed
Proposition 3.1. Assume that $T$ is continuously differentiable, then we have

(i) $\Phi'(x_k) = 2(u^{k+1}(x_k) - \tilde{u}^k, d(u^k, \rho_k))$.
(ii) $\Phi'(x_k)$ is decreasing function with respect to $z_k \geq 0$, i.e., $\Phi(x_k)$ is concave.

Proof. For given $u^k, \tilde{u}^k \in \mathbb{R}^n$, let

$$h(x_k, y) = ||y - [u^k - z_k d(u^k, \rho_k)]||^2 - z_k^2 ||d(u^k, \rho_k)||^2 - 2z_k \langle \tilde{u}^k - u^k, d(u^k, \rho_k) \rangle. \quad (3.7)$$

It easy to see that the solution of the following problem:

$$\min_y \{h(x_k, y) | y \in K \}$$

is $y^* = P_K [u^k - z_k d(u^k, \rho_k)]$. Substituting $y^*$ into (3.7) and simplifying it, we have

$$\Phi(x_k) = h(x_k, y_k) = \min_y \{h(x_k, y) | y \in K \}.$$

It follows from [1] (Chapter 4, Theorem 1.7) that $\Phi(x_k)$ is differentiable and its derivative is given by:

$$\Phi'(x_k) = \frac{\partial h(x_k, y)}{\partial x_k} \bigg|_{y = P_K [u^k - z_k d(u^k, \rho_k)]}$$

$$= 2(u^{k+1}(x_k) - u^k + z_k d(u^k, \rho_k), d(u^k, \rho_k)) - 2z_k ||d(u^k, \rho_k)||^2 - 2\langle \tilde{u}^k - u^k, d(u^k, \rho_k) \rangle$$

$$= 2(u^{k+1}(x_k) - \tilde{u}^k, d(u^k, \rho_k)),$$

and the first conclusion is proved. We now establish the proof of the second assertion. Let $\bar{x}_k > x_k > 0$, we will prove that

$$\Phi'(x_k) \leq \Phi'(x_k)$$

i.e.,

$$\langle u^{k+1}(\bar{x}_k) - u^{k+1}(x_k), d(u^k, \rho_k) \rangle \leq 0. \quad (3.8)$$

Setting $z := u^k - \bar{x}_k d(u^k, \rho_k), v := u^{k+1}(x_k)$ and $z := u^k - x_k d(u^k, \rho_k), v := u^{k+1}(\bar{x}_k)$ in (2.1), respectively, we have

$$\langle u^k - \bar{x}_k d(u^k, \rho_k) - u^{k+1}(x_k) - u^{k+1}(\bar{x}_k) \rangle \leq 0 \quad (3.9)$$

and

$$\langle u^k - x_k d(u^k, \rho_k) - u^{k+1}(x_k) - u^{k+1}(\bar{x}_k) \rangle \leq 0. \quad (3.10)$$

Adding (3.9) and (3.10), we obtain

$$\langle u^{k+1}(\bar{x}_k) - u^{k+1}(x_k), u^{k+1}(\bar{x}_k) - u^{k+1}(x_k) + (\bar{x}_k - x_k) d(u^k, \rho_k) \rangle \leq 0,$$

i.e.,

$$\|u^{k+1}(\bar{x}_k) - u^{k+1}(x_k)\|^2 + (\bar{x}_k - x_k) \langle u^{k+1}(\bar{x}_k) - u^{k+1}(x_k), d(u^k, \rho_k) \rangle \leq 0.$$

It follows that

$$\langle u^{k+1}(\bar{x}_k) - u^{k+1}(x_k), d(u^k, \rho_k) \rangle \leq \frac{-1}{(\bar{x}_k - x_k)} ||u^{k+1}(\bar{x}_k) - u^{k+1}(x_k)||^2 \leq 0.$$

Then, we obtain the inequality (3.8) and complete the proof. □

Now for the same $k$th approximate solution $u^k$, let

$$z_k^* = \arg \max_z \{ \Psi(z) | z > 0 \} \quad (3.11)$$

and

$$z_k^* = \arg \max_z \{ \Phi(z) | 0 < z \leq m_1 z_k^* \}, \quad (3.12)$$

where $m_1 \geq 1$. 
From \(d(u^k, p_k)\neq 0\) and (2.11), it is clear that \(x_{k_1}^*\) and \(x_{k_2}^*\) are both defined. Note that \(x_{k_2}^* = u_k (2.11)\) which is used in [3].

Based on the Theorem 3.1 and Proposition 3.1, the following convergence results can be proved easily.

**Proposition 3.2.** Let \(x_{k_1}^*\) and \(x_{k_2}^*\) be defined by (3.12) and (3.11) respectively, \(T\) be pseudomonotone and continuously differentiable, then we have

\[
\begin{align*}
\text{(i)} \quad & \|u^k - u^*\|^2 - \|u^{k+1}(x_{k_1}^*) - u^*\|^2 \geq \Phi(x_{k_1}^*), \\
\text{(ii)} \quad & \|u^k - u^*\|^2 - \|u^{k+1}(x_{k_2}^*) - u^*\|^2 \geq \Psi(x_{k_2}^*), \\
\text{(iii)} \quad & \Phi(x_{k_1}^*) \geq \Psi(x_{k_2}^*), \\
\text{(iv)} \quad & \|u^k - u^*\|^2 - \|u^{k+1}(x_{k_1}^*) - u^*\|^2 \geq \|u^k - u^{k+1}(x_{k_1}^*)\|^2.
\end{align*}
\]

By using Lemma 2.3 and Proposition 3.2 it is easy to prove the following results.

**Theorem 3.2.** Let \(u^* \in S^*\) be a solution of problem (1.1) and let \(\{u^{k+1}\}\) be the sequence obtained from the proposed method. Then \(u^k\) is bounded and

\[
\|u^{k+1}(x_{k_1}^*) - u^*\|^2 \leq \|u^k - u^*\|^2 - \frac{1}{2} \gamma(2 - \gamma)(1 - \delta)\|r(u^k, p_k)\|^2. \tag{3.13}
\]

The convergence of the proposed method can be proved by similar arguments as in [3]. Hence the proof is omitted.

The Proposition 3.2 motivates us that we can improve the descent method by choosing a more proper step size \(\alpha\) based on finding \(x_{k_1}^*\) instead of \(x_{k_2}^*\) which is used in [3] and extending the step size by solving the following subproblem:

\[
x_k = \max_x \{x_{k_1}^* \leq x \leq m_2 x_{k_1}^* \mid \Phi(x) \geq \eta \Phi(x_{k_1}^*)\}, \tag{3.14}
\]

where \(\eta \in (0, 1)\) and \(m_2 \geq 2\).

We now describe the new algorithm as follows.

**Algorithm 3.1**

Step 0. Let \(\rho_0 > 0, \varepsilon > 0, 0 < \mu < \delta < 1, 0 < \eta < 1, m_1 \geq 1, m_2 \geq 2, u^0 \in K\) and set \(k := 0\).

Step 1. If \(\|r(u^k, p_k)\|_\infty \leq \varepsilon\), then stop. Otherwise, go to Step 2.

Step 2.

\[
\begin{align*}
\tilde{u}^k &= P_K[u^k - \rho_k T(u^k)], \\
\bar{v}^k &= \rho_k (T(\tilde{u}^k) - T(u^k)), \\
r &= \|\bar{v}^k\|_\infty.
\end{align*}
\]

While \((r > \delta)\)

\[
\begin{align*}
\rho_k := 0.7 \times \rho_k \times \min \{1, 1/r\}, \\
\tilde{u}^k &= P_K[u^k - \rho_k T(u^k)], \\
\bar{v}^k &= \rho_k (T(\tilde{u}^k) - T(u^k)), \\
r &= \|\bar{v}^k\|_\infty.
\end{align*}
\]

end While

Step 3. Searching step size \(\tilde{x}\):

Let \(\tilde{x}_k = \arg \max_x \{\Psi(x) \mid x > 0\}\), where \(\Psi(x)\) is defined by (3.3).

Solve the following optimization problem

\[
x_k = \arg \max_x \{\Phi(x) \mid 0 < x \leq m_1 \tilde{x}_k\}, \quad \text{where } \Phi(x) \text{ is defined by (3.2)}.
\]

Step 4. Extending the step size:

\[
x_k = \max_x \{x_{k_1}^* \leq x \leq m_2 x_{k_1}^* \mid \Phi(x) \geq \eta \Phi(x_{k_1}^*)\}
\]

and

\[
u^{k+1} = P_K[u^k - x_k d(u^k, p_k)].
\]

Step 5.\[\rho_{k+1} = \begin{cases} \rho_k \times 0.9 & \text{if } r \leq \mu; \\
\rho_k & \text{otherwise.}\end{cases}\]

Step 6.\(k = k + 1;\) go to Step 1.
4. Preliminary computational results

In this section, we apply the proposed method to the traffic equilibrium problems and present corresponding numerical results.

Consider a network \([N, L]\) of nodes \(N\) and directed links \(L\), which consists of a finite sequence of connecting links with a certain orientation. Let \(a, b, \text{ etc.}\), denote the links, and let \(p, q, \text{ etc.}\), denote the paths. We let \(\omega\) denote an origin/destination (O/D) pair of nodes of the network and \(P_\omega\) denotes the set of all paths connecting O/D pair \(\omega\). Note that the path-arc incidence matrix and the path-O/D pair incidence matrix, denoted by \(A\) and \(B\), respectively, are determined by the given network and O/D pairs. To see how to convert a traffic equilibrium problem into a variational inequality, we take into account a simple example depicted in Fig. 1.

For the given example in Fig. 1, the path-arc incidence matrix \(A\) and the path-O/D pair incidence matrix \(B\) have the following forms:

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}.
\]

Let \(u_p\) represent the traffic flow on path \(p\) and \(f_a\) denote the link load on link \(a\), then the arc-flow vector \(f\) is given by:

\[f = A^T u.\]

Let \(d_\omega\) denote the traffic amount between O/D pair \(\omega\), which must satisfy

\[d_\omega = \sum_{p \in P_\omega} u_p.\]

Thus, the O/D pair-traffic amount vector \(d\) is given by:

\[d = B^T u.\]
Let \( t(f) = \{ t_a, a \in L \} \) be the vector of link travel costs, which is a function of the link flow. A user travelling on path \( p \) incurs a (path) travel cost \( \theta_p \). For given link travel cost vector \( t \), the path travel cost vector \( \theta \) is given by:

\[

\theta = \sum_{a \in p} t_a \\

\]

Table 1

<table>
<thead>
<tr>
<th>The link traversing cost functions ( t_a(f) ) in the example</th>
</tr>
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<tbody>
<tr>
<td>( t_1(f) = 5 \times 10^{-5} f_1^4 + 5 f_1 + 2 f_2 + 500 )</td>
</tr>
<tr>
<td>( t_2(f) = 3 \times 10^{-5} f_2^3 + 4 f_2 + 4 f_1 + 200 )</td>
</tr>
<tr>
<td>( t_3(f) = 5 \times 10^{-5} f_3^3 + 3 f_3 + f_4 + 350 )</td>
</tr>
<tr>
<td>( t_4(f) = 3 \times 10^{-5} f_4^3 + 6 f_4 + 3 f_5 + 400 )</td>
</tr>
<tr>
<td>( t_5(f) = 6 \times 10^{-5} f_5^3 + 6 f_5 + 4 f_6 + 600 )</td>
</tr>
<tr>
<td>( t_6(f) = 7 f_6 + 3 f_7 + 500 )</td>
</tr>
<tr>
<td>( t_7(f) = 8 \times 10^{-5} f_7^3 + 8 f_7 + 2 f_8 + 400 )</td>
</tr>
<tr>
<td>( t_8(f) = 4 \times 10^{-5} f_8^3 + 5 f_8 + 2 f_9 + 650 )</td>
</tr>
<tr>
<td>( t_9(f) = 10^{-5} f_9^3 + 6 f_9 + 2 f_9 + 700 )</td>
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<tr>
<td>( t_{10}(f) = 4 f_{10} + f_{12} + 800 )</td>
</tr>
<tr>
<td>( t_{11}(f) = 7 \times 10^{-5} f_{11}^3 + 7 f_{11} + 4 f_{12} + 650 )</td>
</tr>
<tr>
<td>( t_{12}(f) = 8 f_{12} + 2 f_{13} + 700 )</td>
</tr>
<tr>
<td>( t_{13}(f) = 10^{-5} f_{13}^3 + 7 f_{13} + 3 f_{14} + 600 )</td>
</tr>
<tr>
<td>( t_{14}(f) = 8 f_{14} + 3 f_{15} + 500 )</td>
</tr>
<tr>
<td>( t_{15}(f) = 3 \times 10^{-5} f_{15}^3 + 9 f_{15} + 2 f_{16} + 200 )</td>
</tr>
<tr>
<td>( t_{16}(f) = 8 f_{16} + 5 f_{12} + 300 )</td>
</tr>
<tr>
<td>( t_{17}(f) = 3 \times 10^{-5} f_{17}^3 + 7 f_{17} + 2 f_{15} + 450 )</td>
</tr>
<tr>
<td>( t_{18}(f) = 5 f_{18} + f_{16} + 300 )</td>
</tr>
<tr>
<td>( t_{19}(f) = 8 f_{19} + 3 f_{17} + 600 )</td>
</tr>
<tr>
<td>( t_{20}(f) = 3 \times 10^{-5} f_{20}^3 + 6 f_{20} + f_{21} + 300 )</td>
</tr>
<tr>
<td>( t_{21}(f) = 4 \times 10^{-5} f_{21}^3 + 4 f_{21} + f_{22} + 400 )</td>
</tr>
<tr>
<td>( t_{22}(f) = 2 \times 10^{-5} f_{22}^3 + 6 f_{22} + f_{23} + 500 )</td>
</tr>
<tr>
<td>( t_{23}(f) = 3 \times 10^{-5} f_{23}^3 + 9 f_{23} + 2 f_{24} + 350 )</td>
</tr>
<tr>
<td>( t_{24}(f) = 2 \times 10^{-5} f_{24}^3 + 8 f_{24} + f_{25} + 400 )</td>
</tr>
<tr>
<td>( t_{25}(f) = 3 \times 10^{-5} f_{25}^3 + 9 f_{25} + 3 f_{26} + 450 )</td>
</tr>
<tr>
<td>( t_{26}(f) = 6 \times 10^{-5} f_{26}^3 + 7 f_{26} + 8 f_{27} + 300 )</td>
</tr>
<tr>
<td>( t_{27}(f) = 3 \times 10^{-5} f_{27}^3 + 8 f_{27} + 3 f_{28} + 500 )</td>
</tr>
<tr>
<td>( t_{28}(f) = 3 \times 10^{-5} f_{28}^3 + 7 f_{28} + 650 )</td>
</tr>
<tr>
<td>( t_{29}(f) = 3 \times 10^{-5} f_{29}^3 + 3 f_{29} + f_{30} + 450 )</td>
</tr>
<tr>
<td>( t_{30}(f) = 4 \times 10^{-5} f_{30}^3 + 7 f_{30} + 2 f_{31} + 600 )</td>
</tr>
<tr>
<td>( t_{31}(f) = 3 \times 10^{-5} f_{31}^3 + 8 f_{31} + f_{32} + 750 )</td>
</tr>
<tr>
<td>( t_{32}(f) = 6 \times 10^{-5} f_{32}^3 + 8 f_{32} + 3 f_{33} + 650 )</td>
</tr>
<tr>
<td>( t_{33}(f) = 4 \times 10^{-5} f_{33}^3 + 9 f_{33} + 2 f_{31} + 750 )</td>
</tr>
<tr>
<td>( t_{34}(f) = 6 \times 10^{-5} f_{34}^3 + 7 f_{34} + 3 f_{35} + 550 )</td>
</tr>
<tr>
<td>( t_{35}(f) = 3 \times 10^{-5} f_{35}^3 + 8 f_{35} + 3 f_{32} + 600 )</td>
</tr>
<tr>
<td>( t_{36}(f) = 2 \times 10^{-5} f_{36}^3 + 8 f_{36} + 4 f_{31} + 750 )</td>
</tr>
<tr>
<td>( t_{37}(f) = 6 \times 10^{-5} f_{37}^3 + 5 f_{37} + f_{36} + 350 )</td>
</tr>
</tbody>
</table>

Fig. 2. A directed network with 25 nodes and 37 links.
\( \theta = At(f) \) and thus, \( \theta(u) = At(A^T u) \).

Associated with every O/D pair \( \omega \), there is a travel disutility \( \lambda_\omega(d) \). Since both the path costs and the travel disutilities are functions of the flow pattern \( u \), the traffic network equilibrium problem is to seek the path flow pattern \( u^* \) such that

\[
\begin{align*}
 u^* \geq 0 \quad & (u - u^*)^T T(u^*) \geq 0 \quad \forall u \geq 0 \\
\text{where} \quad & T_p(u) = \theta_p(u) - \lambda_\omega(d(u)) \quad \forall \omega, \ p \in P_\omega \\
\text{and thus} \quad & T(u) = At(A^T u) - B\lambda(B^T u).
\end{align*}
\]

We apply the proposed method to the example taken from [19] (Example 7.5 in [19]), which consisted of 25 nodes, 37 links and 6 O/D pairs. The network is depicted in Fig. 2. For this example, there are together 55 paths for the given O/D pairs and hence the dimension of the variable \( u \) is 55. Therefore, the path-arc incidence matrix \( A \) is a \( 55 \times 37 \) matrix and the path-O/D pair incidence matrix \( B \) is a \( 55 \times 6 \) matrix. The user cost of traversing link \( a \) is given in Table 1. The disutility function is given by:

\[
\lambda_\omega(d) = -m_\omega d_\omega + q_\omega
\]

and the coefficients \( m_\omega \) and \( q_\omega \) in the disutility function of different O/D pairs for this example are given in Table 2.

In all tests we take \( \eta = 0.05, m_1 = 3, \mu = 0.3, m_2 = 4, \delta = 0.9, \gamma = 1.8, u^0 = (0.1, \ldots, 0.1)^T \) as starting point, \( \rho_0 = 1 \) and the stop criterion is

\[
\left\| \frac{\min\{u, T(u)\}}{\max\{u^0, T(u^0)\}} \right\|_\infty \leq \varepsilon.
\]

The test results for problems (4.1) for different \( \varepsilon \) are reported in Table 3, \( k \) is the number of iterations and \( l \) denotes the number of evaluations of mapping \( T \).

Table 2
The O/D pairs and the parameters in (4.2) of the example

<table>
<thead>
<tr>
<th>(O,D) Pair ( \omega )</th>
<th>(1,20)</th>
<th>(1,25)</th>
<th>(2,20)</th>
<th>(3,25)</th>
<th>(1,24)</th>
<th>(11,25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_{\omega} )</td>
<td>1</td>
<td>6</td>
<td>10</td>
<td>5</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>( q_{\omega} )</td>
<td>1000</td>
<td>800</td>
<td>2000</td>
<td>6000</td>
<td>8000</td>
<td>7000</td>
</tr>
<tr>
<td>(</td>
<td>P_{\omega}</td>
<td>)</td>
<td>10</td>
<td>15</td>
<td>9</td>
<td>6</td>
</tr>
</tbody>
</table>

Table 3
Numerical results for different \( \varepsilon \)

<table>
<thead>
<tr>
<th>Different ( \varepsilon )</th>
<th>Algorithm 2.1</th>
<th>Algorithm 3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( k )</td>
<td>( l )</td>
</tr>
<tr>
<td>( 10^{-5} )</td>
<td>293</td>
<td>644</td>
</tr>
<tr>
<td>( 10^{-6} )</td>
<td>381</td>
<td>836</td>
</tr>
<tr>
<td>( 10^{-7} )</td>
<td>469</td>
<td>1028</td>
</tr>
<tr>
<td>( 10^{-8} )</td>
<td>557</td>
<td>1220</td>
</tr>
<tr>
<td>( 10^{-9} )</td>
<td>648</td>
<td>1418</td>
</tr>
</tbody>
</table>
References