On an iterative algorithm for general variational inequalities

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Abstract

In this paper, we suggest and analyze a new three-step iterative algorithm for solving the general variational inequalities. We also study the global convergence of the proposed method under some mild conditions. Since the general variational inequalities include variational inequalities, quasi variational inequalities and quasi complementarity problems as special cases, one can deduce similar results for these problems. Our results can be viewed as an improvement and refinement of the previously known and new results. Preliminary numerical experiments are included to illustrate the advantage and efficiency of the proposed method.

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1. Introduction

Variational inequality has become a rich of inspiration in pure and applied mathematics. In recent years, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences, see [1–25] and the references therein. A useful and important generation of variational inequalities is the general variational inequality involving two nonlinear operators, which was first introduced and studied by Noor [14] in 1988. It has been shown a wide class of nonsymmetric and odd-order moving, obstacle, free, equilibrium problems arising in various branches of pure and applied sciences can be studied via the general variational inequalities. In brief, this field is dynamic and is experiencing an explosive growth in both theory and applications. Consequently, several numerical techniques including projection, the Wiener–Hopf equations, auxiliary principle, decomposition and descent are being developed for solving various classes of variational inequalities and related optimization problems. To the best of our knowledge, very few methods have been developed for quasi variational inequalities and the general variational inequalities. Using essentially the
technique of Noor and Bnouhachem [22,23], we suggest and analyze a new three-step iterative method for solving the general variational inequalities. As a special case, we show that the method proposed in this paper can be used for solving a class of quasi variational inequalities. We also study the global convergence of the proposed method under the pseudomonotonicity of the operator, which is weaker condition than monotonicity. In this respect our method represents an improvement. Some preliminary numerical results for a traffic equilibrium problems are included to illustrate the efficiency of the proposed method. It is an interesting research problem to compare the proposed method for solving the general variational inequalities with the other known results.

2. Preliminaries

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $K$ be a closed convex set in $H$ and $g : H \to H$ be nonlinear operators. We now consider the problem of finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K.$$  \hspace{1cm} (2.1)

Problem (2.1) is called the general variational inequality, which was introduced and studied by Noor [14] in 1988. To convey an idea of the applications of the general variational inequalities, we consider the third-order obstacle boundary value problem of finding $u$ such that

\[
\begin{align*}
-u'' &\geq f(x) \quad \text{on } \Omega = [0, 1], \\
u &\geq \psi(x) \quad \text{on } \Omega = [0, 1], \\
[-u'' - f(x)][u - \psi(x)] &\equiv 0 \quad \text{on } \Omega = [0, 1], \\
u(0) &= 0, \quad u'(0) = 0, \quad u'(1) = 0,
\end{align*}
\]  \hspace{1cm} (2.2)

where $f(x)$ is a continuous function and $\psi(x)$ is the obstacle function. We study the problem (2.2) in the framework of variational inequality approach. To do so, we first define the set $K$ as

$$K = \{ v : v \in H_0^2(\Omega) : v \geq \psi \quad \text{on } \Omega \},$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev (Hilbert) space, see [1]. One can easily show that the energy functional associated with the problem (2.2) is

$$I[v] = -\int_0^1 \left( \frac{d^3 v}{dx^3} \right) \left( \frac{dv}{dx} \right) dx - 2 \int_0^1 f(x) \left( \frac{dv}{dx} \right) dx, \quad \text{for all } \frac{dv}{dx} \in K
$$

$$= \int_0^1 \left( \frac{d^2 v}{dx^2} \right)^2 dx - 2 \int_0^1 f(x) \left( \frac{dv}{dx} \right) dx = \langle Tu, g(v) \rangle - 2\langle f, g(v) \rangle,$$  \hspace{1cm} (2.3)

where

$$\langle Tu, g(v) \rangle = \int_0^1 \left( \frac{d^3 u}{dx^3} \right) \left( \frac{dv}{dx} \right) dx,$$

$$\langle f, g(v) \rangle = \int_0^1 f(x) \left( \frac{dv}{dx} \right) dx$$  \hspace{1cm} (2.4)

and $g = \frac{dv}{dx}$ is the linear operator.

It is clear that the operator $T$ defined by (2.4) is linear, $g$-symmetric and $g$-positive. Using the technique of Noor [17], one can easily show that the minimum $u \in H$ of the functional $I[v]$ defined by (2.3) associated with the problem (2.2) on the closed convex set $K$ can be characterized by the inequality of type

$$\langle Tu, g(v) - g(u) \rangle \geq \langle f, g(v) - g(u) \rangle, \quad \forall g(v) \in K,$$

which is exactly the general variational inequality (2.1). It is worth mentioning that a wide class of unrelated odd-order and nonsymmetric obstacle, unilateral, contact, free, moving, and equilibrium problems arising in
regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the general variational inequalities (2.1), see [15–21] and the references therein.

For \( g \equiv I \), where \( I \) is the identity operator, problem (2.1) is equivalent to finding \( u \in K \) such that
\[
\langle Tu, v - u \rangle \geq 0 \quad \forall v \in K,
\]
which is known as the classical variational inequality introduced and studied by Stampacchia [24] in 1964. For the recent state-of-the-art, see [1–25] and the references therein.

If \( K^* = \{u \in H : \langle u, v \rangle \geq 0, \forall v \in K \} \) is a polar (dual) cone of a convex cone \( K \) in \( H \), then problem (2.1) is equivalent to finding \( u \in H \) such that
\[
g(u) \in K, \quad Tu \in K^* \quad \text{and} \quad \langle Tu, g(u) \rangle = 0,
\]
which is known as the general complementarity problem. For \( g(u) = m(u) + K \), where \( m \) is a point-to-point mapping, problem (2.6) is called the implicit (quasi) complementarity problem. If \( g \equiv I \), then problem (2.6) is known as the generalized complementarity problem. Such problems have been studied extensively in the literature, see [1–25] and the references therein. For suitable and appropriate choice of the operators and spaces, one can obtain several classes of variational inequalities and related optimization problems.

We now recall the following well known result and concepts.

**Lemma 2.1.** For a given \( z \in H \), \( u \in K \) satisfies the inequality
\[
\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,
\]
if and only if
\[
u = P_K[z],
\]
where \( P_K \) is the projection of \( H \) onto \( K \). Also, the projection operator \( P_K \) is nonexpansive.

**Definition 2.1.** \( \forall u, v \in H \), the operator \( T : H \rightarrow H \) is said to be

(a) \( g \)-monotone, iff,
\[
\langle Tu - Tv, g(u) - g(v) \rangle \geq 0.
\]

(b) \( g \)-pseudomonotone, iff,
\[
\langle Tu, g(v) - g(u) \rangle \geq 0 \Rightarrow \langle Tv, g(v) - g(u) \rangle \geq 0.
\]

For \( g = I \), the identity operator, Definition 2.1 reduces to the well-known definition of monotonicity and pseudomonotonicity. It is well-known [4] that monotonicity implies pseudomonotonicity, but the converse is not true.

Using Lemma 2.1, one can easily show that the problem (2.1) is equivalent to the fixed-point problem, see, for example, [14,17].

**Lemma 2.2.** \( u^* \in H : g(u^*) \in K \) is solution of the variational inequality (2.1) if and only if \( u^* \in H : g(u^*) \in K \) satisfies the relation:
\[
g(u^*) = P_K[g(u^*) - \rho Tu^*],
\]
where \( \rho > 0 \) is a constant. From Lemma 2.2, it is clear that \( u \) is a solution of (2.1) if and only if \( u \) is a zero of the function
\[
r(u, \rho) = g(u) - P_K[g(u) - \rho Tu].
\]
Throughout this paper, we make following assumptions.

Assumptions

- $H$ is a finite dimension space.
- $g$ is homeomorphism on $H$, i.e., $g$ is bijective, continuous and $g^{-1}$ is continuous.
- $T$ is continuous and $g$-pseudomonotone operator.
- The solution set of problem (2.1), denoted by $\Omega^*$, is nonempty.

3. Basic results

In this section, we prove some basic properties, which will be used to establish the sufficient and necessary conditions for the convergence of the proposed method. The following results can be proved by similar arguments as those in [3, 22]. Hence the proofs will be omitted.

Lemma 3.1 [3]. For all $u \in H : g(u) \in K$ and $\rho' \geqslant \rho > 0$, it holds that

$$
\|r(u, \rho')\| \geqslant \|r(u, \rho)\|
$$

(3.1)

and

$$
\frac{\|r(u, \rho')\|}{\rho'} \leqslant \frac{\|r(u, \rho)\|}{\rho}.
$$

(3.2)

Lemma 3.2 [3]. If $u$ is not a solution of problem (2.1), then there exist $\delta \in (0,1)$ and $\epsilon' > 0$, such that for all $\rho \in (0, \epsilon')$,

$$
\rho\|T(u) - T(P_K[g(u) - \rho T(u)])\| \leqslant \delta\|r(u, \rho)\|.
$$

(3.3)

Lemma 3.3 [22]. For all $u \in H : g(u) \in K, g(u^*) \in \Omega^*$ and $\rho > 0$, we have

$$
\langle g(u) - g(u^*), d(u, \rho) \rangle \geqslant \phi(u, \rho),
$$

(3.4)

where

$$
d(u, \rho) := r(u, \rho) - \rho T(u) + \rho Tg^{-1}(P_K[g(u) - \rho T(u)])
$$

(3.5)

and

$$
\phi(u, \rho) := \|r(u, \rho)\|^2 - \rho \langle r(u, \rho), T(u) - Tg^{-1}(P_K[g(u) - \rho T(u)]) \rangle.
$$

(3.6)

From Lemmas 3.2 and 3.3 we have

$$
\langle g(u) - g(u^*), d(u, \rho) \rangle \geqslant \phi(u, \rho) \geqslant (1 - \delta)\|r(u, \rho)\|^2.
$$

(3.7)

Taking the above inequality into consideration, we suggest and consider a new three-step iterative method for solving variational inequality (2.1).

Algorithm 3.1. For a given $u^k \in K$, compute the approximate solution $u^{k+1}$ by the iterative schemes:

**Step 1**

$$
g(\tilde{u}^k) = P_K[g(u^k) - \rho_k T(u^k)],
$$

(3.8)

where $\rho_k$ satisfies

$$
\|\rho_k(T(u^k) - T(\tilde{u}^k))\| \leqslant \delta\|g(u^k) - g(\tilde{u}^k)\|, \quad 0 < \delta < 1.
$$

(3.9)

**Step 2**

$$
g(\tilde{u}^k) = P_K[g(u^k) - \alpha_k d(u^k, \rho_k)],
$$
where
\[
\hat{u}^k = \rho_k(T(\hat{u}^k) - T(u^k)),
\]
(3.10)
\[
d(u^k, \rho_k) := g(u^k) - g(\hat{u}^k) + \varepsilon^k,
\]
(3.11)
\[
\phi(u^k, \rho_k) := \langle g(u^k) - g(\hat{u}^k), d(u^k, \rho_k) \rangle
\]
(3.12)
and
\[
\alpha_k := \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}.
\]
(3.13)

**Step 3.** For \( \tau > 0 \), the new iterate \( u^{k+1}(\tau) \) is defined by
\[
g(u^{k+1}(\tau)) = P_K[g(u^k) - \tau(g(u^k) - g(\hat{u}^k))].
\]
(3.14)

How to choose a suitable step length \( \tau > 0 \) to force convergence will be discussed in Section 3.

**Remark 3.1.** (3.9) implies that
\[
|\langle g(u^k) - g(\hat{u}^k), \varepsilon^k \rangle| \leq \delta\|g(u^k) - g(\hat{u}^k)\|^2, \quad 0 < \delta < 1.
\]
(3.15)
The next result shows that \( \alpha_k \) and \( \phi(u^k, \rho_k) \) are lower bounded away from zero.

**Lemma 3.4.** For a given \( u^k \in H \) and \( \rho_k > 0 \), let \( \{\tilde{u}^k\} \) and \( \tilde{e}^k \) satisfy (3.8) and (3.10), then
\[
\phi(u^k, \rho_k) \geq (1 - \delta)\|g(u^k) - g(\tilde{u}^k)\|^2
\]
(3.16)
and
\[
\alpha_k \geq \frac{1}{2}.
\]
(3.17)

**Proof.** It follows from (3.11) and (3.15) that
\[
\phi(u^k, \rho_k) := \langle g(u^k) - g(\tilde{u}^k), d(u^k, \rho_k) \rangle = \|g(u^k) - g(\tilde{u}^k)\|^2 + \langle g(u^k) - g(\tilde{u}^k), \varepsilon^k \rangle \geq (1 - \delta)\|g(u^k) - g(\tilde{u}^k)\|^2.
\]

Otherwise, we have
\[
\langle g(u^k) - g(\tilde{u}^k), d(u^k, \rho_k) \rangle = \|g(u^k) - g(\tilde{u}^k)\|^2 + \langle g(u^k) - g(\tilde{u}^k), \varepsilon^k \rangle
\]
\[
\geq \frac{1}{2}\|g(u^k) - g(\tilde{u}^k)\|^2 + \langle g(u^k) - g(\tilde{u}^k), \varepsilon^k \rangle + \frac{1}{2}\|\varepsilon^k\|^2 = \frac{1}{2}\|d(u^k, \rho_k)\|^2,
\]

where the inequality follows from (3.9) and
\[
\alpha_k \geq \frac{1}{2},
\]
the required result. \( \square \)

We now consider the criteria of \( \tau \), which ensures that \( u^{k+1}(\tau) \) is closer to the solution set than \( u^k \). For this purpose, we define
\[
\Gamma(\tau) := \|g(u^k) - g(u^*)\|^2 - \|g(u^{k+1}(\tau)) - g(u^*)\|^2.
\]
(3.18)

**Theorem 3.1.** Let \( u^* \in \Omega^* \). Then we have
\[
\Gamma(\tau) \geq \tau\left(\|g(u^k) - g(\tilde{u}^k)\|^2 + \Theta_k(\alpha_k)\right) - \tau^2\|g(u^k) - g(\tilde{u}^k)\|^2,
\]
(3.19)
where
\[
\Theta_k(\alpha_k) := \|g(u^k) - g(u^*)\|^2 - \|g(u^k) - g(u^*)\|^2.
\]
(3.20)
Proof. It follows from (3.14) that
\[
\Gamma(\tau) \geq \|g(u^k) - g(u^*)\|^2 - \|g(u^k) - \tau(g(u^k) - g(\bar{u}^k)) - g(u^*)\|^2
\]
\[
= 2\tau\|g(u^k) - g(u^*)\|^2 - 2\|g(u^k) - g(\bar{u}^k)\|^2 - \|g(u^k) - g(\bar{u}^k)\|^2
\]
\[
= 2\tau\{\|g(u^k) - g(\bar{u}^k)\|^2 - \|g(u^*) - g(\bar{u}^k)\|^2\} - \|g(u^k) - g(\bar{u}^k)\|^2.
\] (3.21)

Using the following identity
\[
\langle g(u^*) - g(\bar{u}^k), g(u^k) - g(\bar{u}^k) \rangle = \frac{1}{2}\{\|g(u^*) - g(\bar{u}^k)\|^2 - 2\|g(u^k) - g(\bar{u}^k)\|^2 + \|g(u^k) - g(u^*)\|^2\},
\]
we have
\[
\|g(u^k) - g(\bar{u}^k)\|^2 - 2\|g(u^*) - g(\bar{u}^k), g(u^k) - g(\bar{u}^k)\| = (\|g(u^*) - g(\bar{u}^k)\|^2 - \|g(u^k) - g(u^*)\|^2).
\] (3.22)

Substituting (3.22) into (3.21) and using the notation of \(\Theta_k(z_k)\), we obtain (3.19), the required result. \(\blacksquare\)

Theorem 3.2. Let \(u^* \in H: g(u^*) \in K\) be a solution of problem (2.1) and let \(\phi(u^k, \rho_k)\) and \(\Theta_k(z_k)\) be defined by (3.10) and (3.20), respectively. Then we get
\[
\Theta_k(z_k) \geq z_k \phi(u^k, \rho_k).
\] (3.23)

Proof. Let \(u^* \in H: g(u^*) \in K\) be a solution of problem (2.1). Then from (3.7) and (3.13) we obtain
\[
\Theta_k(z_k) = \|g(u^k) - g(u^*)\|^2 - \|g(u^k) - g(\bar{u}^k)\|^2
\]
\[
\geq \|g(u^k) - g(u^*)\|^2 - \|g(u^k) - g(u^*) - z_k d(u^k, \rho_k)\|^2
\]
\[
= 2z_k \langle g(u^k) - g(u^*), d(u^k, \rho_k) \rangle - z_k^2 \|d(u^k, \rho_k)\|^2 \geq z_k \phi(u^k, \rho_k). \quad \blacksquare \] (3.24)

Using Theorems 3.1 and 3.2, we get
\[
\Gamma(\tau) \geq A(\tau),
\] (3.25)

where
\[
A(\tau) = \tau\{\|g(u^k) - g(\bar{u}^k)\|^2 + z_k \phi(u^k, \rho_k)\} - \tau^2\|g(u^k) - g(\bar{u}^k)\|^2.
\] (3.26)

The above inequality tells us how to choose a suitable \(\tau_k\). Since \(A(\tau_k)\) is a quadratic function of \(\tau_k\) and it reaches its maximum at
\[
\tau_k^* = \frac{\|g(u^k) - g(\bar{u}^k)\|^2 + z_k \phi(u^k, \rho_k)}{2\|g(u^k) - g(\bar{u}^k)\|^2}
\]
and
\[
A(\tau_k^*) = \frac{\tau_k^*\{\|g(u^k) - g(\bar{u}^k)\|^2 + z_k \phi(u^k, \rho_k)\}}{2}.
\] (3.27)

Then, from Lemma 3.4, we get
\[
\tau_k^* \geq \frac{1 - \delta}{2} \left( \frac{\|g(u^k) - g(\bar{u}^k)\|^2 + \|g(u^k) - g(\bar{u}^k)\|^2}{2\|g(u^k) - g(\bar{u}^k)\|^2} \right) \geq \frac{1 - \delta}{4}, \quad \blacksquare \] (3.28)

and
\[
A(\tau_k^*) \geq \frac{\tau_k^*(1 - \delta)}{4} \|g(u^k) - g(\bar{u}^k)\|^2 \geq \frac{(1 - \delta)^2}{16} \|g(u^k) - g(\bar{u}^k)\|^2.
\] (3.29)
For the fast convergence, we take a relaxation factor $\gamma \in [1, 2)$ and the step-size $\tau_k$ by $\tau_k = \gamma \tau_k^*$. Simple calculations show that
\[
A(\gamma \tau_k^*) = \gamma \tau_k^* \{ \| g(u^k) - g(\bar{u}^k) \|^2 + \alpha_k \phi(u^k, \rho_k) \} - (\gamma^2 \tau_k^*)^\gamma \| g(u^k) - g(\bar{u}^k) \|^2 = \gamma(2 - \gamma)A(\tau_k^*).
\] (3.30)

4. Convergence analysis

In this section, we consider the convergence analysis of Algorithm 3.1 and this is the main motivation.

**Theorem 4.1.** Let $u^* \in H : g(u^*) \in K$ be a solution of problem (2.1) and let $u^{k+1}$ be the sequence obtained from Algorithm 3.1. Then
\[
\| g(u^{k+1}) - g(u^*) \|^2 \leq \| g(u^k) - g(u^*) \|^2 - \frac{1}{16}(2 - \gamma)(1 - \delta)^2 \| g(u^k) - g(\bar{u}^k) \|^2
\] (4.1)
and $u^k$ is bounded if $g$ is onto and $g^{-1}$ exists.

**Proof.** Let $u^*$ be a solution of the variational inequality (2.1). Then, from (3.24), (3.29) and (3.28), we have
\[
\| g(u^{k+1}) - g(u^*) \|^2 = \| g(u^k) - g(u^*) \|^2 - \Gamma(\gamma \tau_k^*) \leq \| g(u^k) - g(u^*) \|^2 - (2 - \gamma)A(\tau_k^*)
\]
\[
\leq \| g(u^k) - g(u^*) \|^2 - \frac{1}{16}(2 - \gamma)(1 - \delta)^2 \| g(u^k) - g(\bar{u}^k) \|^2.
\]
Since $\gamma \in [1, 2)$ and $\delta \in (0, 1)$, we have
\[
\| g(u^{k+1}) - g(u^*) \| \leq \| g(u^k) - g(u^*) \| \leq \cdots \leq \| g(\bar{u}^k) - g(u^*) \|.
\]
This shows that the sequence $u^k$ is bounded. \hfill \Box

We now prove the convergence of Algorithm 3.1 and this is the main motivation of our next result.

**Theorem 4.2.** The sequence $\{u^k\}$ generated by Algorithm 3.1 converges to a solution of the variational inequality (2.1), if $g^{-1}$ exists.

**Proof.** It follows from (4.1) that
\[
\sum_{k=0}^{+\infty} \| g(u^k) - g(\bar{u}^k) \|^2 < +\infty,
\]
which means that
\[
\lim_{k \to +\infty} \| g(u^k) - g(\bar{u}^k) \| = 0,
\] (4.2)
since the operator $g$ is injective.

Consequently $\{\bar{u}^k\}$ is also bounded. Since $\| r(u^k, \rho) \|$ is a nondecreasing function of $\rho$, it follows from $\rho_k \geq \rho_{\min}$ that
\[
\| r(\bar{u}^k, \rho_{\min}) \| \leq \| r(\bar{u}^k, \rho_k) \| = \| g(\bar{u}^k) - P_k[g(\bar{u}^k) - \rho_k T(\bar{u}^k)] \|
\]
\[
= \| P_k[g(\bar{u}^k) - \rho_k T(\bar{u}^k) + \bar{e}^k] - P_k[g(\bar{u}^k) - \rho_k T(\bar{u}^k)] \| \quad \text{using (3.8) and (3.10)}
\]
\[
\leq \| g(\bar{u}^k) - g(\bar{u}^k) + \bar{e}^k \|
\]
\[
\leq (1 + \delta)\| g(\bar{u}^k) - g(\bar{u}^k) \| \quad \text{using (3.9)}
\]
and from (4.2), we get
\[
\lim_{k \to +\infty} r(\bar{u}^k, \rho_{\min}) = 0.
\] (4.3)
Let \( \bar{u} \) be a cluster point of \( \{ \tilde{u}^k \} \) and the subsequence \( \{ w^k \} \) converges to \( \bar{u} \). Since \( r(u, \rho) \) is a continuous function of \( u \), it follows from (4.3) that
\[
\lim_{j \to +\infty} r(w^j, \rho_{\min}) = 0.
\]

According to Lemma 2.2, \( \bar{u} \) is a solution of (2.1). Note that inequality (4.1) is true for all solutions of problem (2.1). Hence we have
\[
\| g(u^{k+1}) - g(\bar{u}) \| \leq \| g(u^k) - g(\bar{u}) \|, \quad \forall k \geq 0.
\]

Since \( \{ g(w^k) \} \to g(\bar{u}) \) and \( g(u^k) - g(\bar{u}) \to 0 \), for any given \( \varepsilon > 0 \), there is an \( l > 0 \), such that
\[
\| g(u^k) - g(\bar{u}) \| < \varepsilon/2 \quad \text{and} \quad \| g(u^k) - g(w^k) \| < \varepsilon/2.
\]

Therefore, for any \( k \geq k_1 \), it follows from (4.4) and (4.5) that
\[
\| g(u^k) - g(\bar{u}) \| \leq \| g(u^k) - g(\bar{u}) \| \leq \| g(u^k) - g(w^k) \| + \| g(w^k) - g(\bar{u}) \| < \varepsilon
\]
and thus the sequence \( \{ u^k \} \) converges to \( \bar{u} \).

We now prove that the sequence \( \{ u^k \} \) has exactly one cluster point. Assume that \( \bar{u} \) is another cluster point and satisfies
\[
\delta := \| g(\bar{u}) - g(\bar{u}) \| > 0.
\]

Since \( \bar{u} \) is a cluster point of the sequence \( \{ u^k \} \), there is a \( k_0 > 0 \) such that
\[
\| g(u^{k_0}) - g(\bar{u}) \| \leq \frac{\delta}{2}.
\]

On the other hand, since \( \bar{u} \in S^* \) and from (4.1), we have
\[
\| g(u^k) - g(\bar{u}) \| \leq \| g(u^{k_0}) - g(\bar{u}) \| \quad \text{for all} \quad k \geq k_0,
\]

it follows that
\[
\| g(u^k) - g(\bar{u}) \| \geq \| g(\bar{u}) - g(\bar{u}) \| - \| g(u^k) - g(\bar{u}) \| \geq \frac{\delta}{2} \quad \forall k \geq k_0.
\]

This contradicts the assumption that \( \bar{u} \) is cluster point of \( \{ u^k \} \), thus the sequence \( \{ u^k \} \) converges to \( \bar{u} \in \Omega^* \).

The details of Algorithm 3.1 are as follows.

Algorithm 4.1

**Step 0.** Let \( \rho_0 = 1, \delta := 0.95 < 1, \gamma = 1.95, \epsilon > 0, k = 0 \) and \( u^0 \in H \).
**Step 1.** If \( \| r(u^k, \rho_k) \|_{\infty} \leq \epsilon \), then stop. Otherwise, go to Step 2.
**Step 2**
\[
g(\tilde{u}^k) = P_h[g(u^k) - \rho_k T(u^k)], \quad \tilde{v}^k = \rho_k (T(\tilde{u}^k) - T(u^k)),
\]
\[
r = \frac{\| \tilde{v}^k \|}{\| g(\tilde{u}^k) - g(u^k) \|}. \quad \text{While} \ (r \geq \delta)
\]
\[
\rho_k = 0.8 \cdot \rho_k, \quad g(\tilde{u}^k) = P_h[g(u^k) - \rho_k T(u^k)],
\]
\[
\tilde{v}^k = \rho_k (T(\tilde{u}^k) - T(u^k)), \quad r = \frac{\| \tilde{v}^k \|}{\| g(\tilde{u}^k) - g(u^k) \|}.
\]
**end While**
Step 3. Set
\[
\begin{align*}
    d(u^k, \rho_k) & := g(u^k) - g(\bar{u}^k) + v^k, \\
    \phi(u^k, \rho_k) & := (g(u^k) - g(\bar{u}^k), d(u^k, \rho_k)), \quad \alpha_k = \frac{\phi(u^k, \rho_k)}{||d(u^k, \rho_k)||^2}, \\
    g(\bar{u}^k) & = P_K[g(u^k) - \alpha_k d(u^k, \rho_k)], \\
    \tau_k & = \frac{||g(u^k) - g(\bar{u}^k)||^2 + \alpha_k \phi(u^k, \rho_k)}{2||d(u^k, \rho_k)||^2}, \quad \tau_k = \gamma \tau_k, \\
    g(u^{k+1}) & = P_K[g(u^k) - \tau_k (g(u^k) - g(\bar{u}^k))].
\end{align*}
\]

Step 4
\[
\rho_{k+1} = \begin{cases} 
    \frac{\rho_k^{0.7}}{r} & \text{if } r \leq 0.5; \\
    \rho_k & \text{otherwise.}
\end{cases}
\]

Step 5. \( k := k + 1; \) go to Step 1.

If \( g = I \), the identity operator, then Algorithm 4.1 reduces to the following Algorithm, which is due to Noor and Bnouhachem [23] for solving the classical variational inequalities (2.5).

Algorithm 4.2

Step 0. Let \( \rho_0 = 1, \delta := 0.95 < 1, \gamma = 1.95, \epsilon > 0, k = 0 \) and \( u^0 \in H \).

Step 1. If \( ||r(u^k, \rho_k)||_\infty \leq \epsilon \), then stop. Otherwise, go to Step 2.

Step 2
\[
\begin{align*}
    \bar{u}^k & = P_K[u^k - \rho_k T(u^k)], \quad v^k = \rho_k(T(\bar{u}^k) - T(u^k)), \\
    r & = \frac{||v^k||}{||u^k - \bar{u}^k||}. \quad \text{While } (r > \delta) \\
    \rho_k & = 0.8 \frac{r}{r}, \quad \bar{u}^k = P_K[u^k - \rho_k T(u^k)], \\
    \varepsilon^k & = \rho_k(T(\bar{u}^k) - T(u^k)), \quad r = \frac{||v^k||}{||u^k - \bar{u}^k||}.
\end{align*}
\]

\textbf{end While}

Step 3. Set
\[
\begin{align*}
    d(u^k, \rho_k) & := u^k - \bar{u}^k + \varepsilon^k, \\
    \phi(u^k, \rho_k) & := (u^k - \bar{u}^k, d(u^k, \rho_k)), \quad \alpha_k = \frac{\phi(u^k, \rho_k)}{||d(u^k, \rho_k)||^2}, \\
    \bar{u}^k & = P_K[u^k - \alpha_k d(u^k, \rho_k)], \\
    \tau_k & = \frac{||u^k - \bar{u}^k||^2 + \alpha_k \phi(u^k, \rho_k)}{2||u^k - \bar{u}^k||^2}, \quad \tau_k = \gamma \tau_k, \\
    u^{k+1} & = P_K[u^k - \tau_k (u^k - \bar{u}^k)].
\end{align*}
\]

Step 4
\[
\rho_{k+1} = \begin{cases} 
    \frac{\rho_k^{0.7}}{r} & \text{if } r \leq 0.5; \\
    \rho_k & \text{otherwise.}
\end{cases}
\]

Step 5. \( k := k + 1; \) go to Step 1.
5. Application to quasi variational inequalities

In this section we show that the results obtained in Section 3 can be extended for a class of quasi variational inequalities. If the convex set \( K \) depends upon the solution explicitly or implicitly, then variational inequality problem is known as the quasi variational inequality. For a given operator \( T : H \to H \), and a point-to-set mapping \( K : u \to K(u) \), which associates a closed convex-valued set \( K(u) \) with any element \( u \) of \( H \), we consider the problem of finding \( u \in K(u) \) such that

\[
\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K(u).
\] (5.1)

Inequality of type (5.1) is called the quasi variational inequality. To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding \( u \) such that

\[
\begin{align*}
-u'' &\geq f(x) \quad \text{on } \Omega = [a, b], \\
u &\geq M(u) \quad \text{on } \Omega = [a, b], \\
[-u'' - f(x)][u - M(u)] &= 0 \quad \text{on } \Omega = [a, b], \\
u(a) &= 0, \quad u(b) = 0.
\end{align*}
\] (5.2)

where \( f(x) \) is a continuous function and \( M(u) \) is the cost (obstacle) function. The prototype encountered is

\[
M(u) = k + \inf \{u'\}.
\] (5.3)

In (5.3), \( k \) represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator \( M \) provides the coupling between the unknowns \( u = (u^1, u^2, \ldots, u^l) \), see [2]. We study the problem (5.2) in the framework of variational inequality approach. To do so, we first define the set \( K \) as

\[
K(u) = \{v : v \in H^1_0(\Omega) : v \geq M(u), \text{ on } \Omega\},
\]

which is a closed convex-valued set in \( H^1_0(\Omega) \), where \( H^1_0(\Omega) \) is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (5.2) is

\[
I[v] = - \int_a^b \left( \frac{d^2v}{dx^2} \right) v \, dx - 2 \int_a^b f(x) v \, dx, \quad \forall v \in K(u) = \int_a^b \left( \frac{dv}{dx} \right)^2 \, dx - 2 \int_a^b f(x) v \, dx
\]

\[
= \langle Tv, v \rangle - 2\langle f, v \rangle,
\] (5.4)

where

\[
\langle Tu, v \rangle = - \int_a^b \left( \frac{d^2u}{dx^2} \right) (v) \, dx = \int_a^b \frac{du}{dx} \frac{dv}{dx} \, dx,
\]

\[
\langle f, v \rangle = \int_a^b f(x) (v) \, dx.
\] (5.5)

It is clear that the operator \( T \) defined by (5.5) is linear, symmetric and positive. Using the technique of Noor [17], one can show that the minimum of the functional \( I[v] \) defined by (5.4) associated with the problem (5.2) on the closed convex-valued set \( K(u) \) can be characterized by the inequality of type

\[
\langle Tu, v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K(u),
\] (5.6)

which is exactly the quasi variational inequality (5.1). See also [3, 6–20] for the formulation, applications, numerical methods and sensitivity analysis of the quasi variational inequalities.

Using Lemma 2.1, one can show that the quasi variational inequality (5.1) is equivalent to finding \( u \in K(u) \) such that

\[
u = P_{K(u)}[u - \rho Tu].
\] (5.7)
In many important applications [2], the convex-valued set \( K(u) \) is of the form

\[
K(u) = m(u) + K,
\]

where \( m \) is a point-to-point mapping and \( K \) is a closed convex set.

From (5.7) and (5.8), we see that problem (5.1) is equivalent to

\[
\begin{align*}
   u &= P_{K(u)}[u - \rho Tu] = P_{m(u) + K}[u - \rho Tu] = m(u) + P_K[u - m(u) - \rho Tu], \\
   g(u) &= P_K[g(u) - \rho Tu] \quad \text{with } g(u) = u - m(u),
\end{align*}
\]

which implies that

\[
g(u) = P_K[g(u) - \rho Tu]
\]

which is equivalent to the general variational inequality (2.1) by an application of Lemma 3.1. We have shown that the quasi variational inequalities (5.1) with the convex-valued set \( K(u) \) defined by (5.8) are equivalent to the general variational inequalities (2.1). Thus all the results obtained in this paper continue to hold for quasi variational inequalities (5.1) with \( K(u) \) defined by (5.8).

6. Computational results

In this section, we apply Algorithm 4.2 for solving variational inequalities (2.5) to a traffic equilibrium problem, which is a classical and important problem in transportation science, see [11,25]. The numerical results show that the new method is attractive in practice.

Consider a network \([N, L]\) of nodes \( N \) and directed links \( L \), which consists of a finite sequence of connecting links with a certain orientation. Let \( a, b, \) etc., denote the links; \( p, q, \) etc., denote the paths; \( \omega \) denote an origin/destination (O/D) pair of nodes of the network; \( P_\omega \) denotes the set of all paths connecting O/D pair \( \omega \); \( u_p \) represent the traffic flow on path \( p \); \( d_\omega \) denote the traffic demand between O/D pair \( \omega \), which must satisfy

\[
d_\omega = \sum_{p \in P_\omega} u_p,
\]

where \( u_p \geq 0, \forall p; \) and \( f_a \) denote the link load on link \( a \), which must satisfy the following conservation of flow equation

\[
f_a = \sum_{p \ni a} \delta_{ap} u_p,
\]

where

\[
\delta_{ap} = \begin{cases} 
1, & \text{if } a \text{ is contained in path } p; \\
0, & \text{otherwise}. 
\end{cases}
\]

Let \( A \) be the path-arc incidence matrix of the given problem and \( f = \{f_a, a \in L\} \) be the vector of the link load. Since \( u \) is the path-flow, \( f \) is given by

\[
f = A^T u.
\]

In addition, let \( t = \{t_a, a \in L\} \) be the row vector of link costs, with \( t_a \) denoting the user cost of traveling link \( a \) which is given by

\[
t_a(f_a) = t^0_a \left[ 1 + 0.15 \left( \frac{f_a}{C_a} \right)^4 \right],
\]

where \( t^0_a \) is the free-flow travel cost on link \( a \) and \( C_a \) is designed capacity of link \( a \). Then \( t \) is a mapping of the path-flow \( u \) and its mathematical form is

\[
t(u) := t(f) = t(A^T u).
\]

Note that the travel cost on the path \( p \) denoted by \( \theta_p \) is

\[
\theta_p = \sum_{a \in L} \delta_{ap} t_a(f_a).
\]
Let $P$ denote the set of all the paths concerned. Let $\theta = \{\theta_p, p \in P\}$ be the vector of (path) travel cost. For given link travel cost vector $t$, $\theta$ is a mapping of the path-flow $u$, which is given by

$$\theta(u) = At(u) = At(A^T u).$$

Associated with every O/D pair $\omega$, there is a travel disutility $\lambda_\omega(d)$, which is defined as follows:

$$\lambda_\omega(d) = -m_\omega \log(d_\omega) + q_\omega. \quad (6.2)$$

Note that both the path costs and the travel disutilities are functions of the flow pattern $u$. The traffic network equilibrium problem is to seek the path flow pattern $u^*$, which induces a demand pattern $d^* = d(u^*)$, for every O/D pair $\omega$ and each path $p \in P_\omega$,

$$T_p(u) = \theta_p(u) - \lambda_\omega(d(u)).$$

The problem can be reduced to a variational inequality in the space of path-flow pattern $u$:

Find $u^* \geq 0$ such that $(u - u^*)^T T(u^*) \geq 0, \quad \forall u \geq 0, \quad (6.3)$

which is exactly the variational inequality (2.5).

In particular, we test the example studied in [11,23,25]. The network is depicted in Fig. 1. The free-flow travel cost and the designed capacity of links (6.1) are given in Table 1, the O/D pairs and the coefficient $m$ and $q$ in the disutility function (6.2) are given in Table 2. For this example, there are together 12 paths for the four given O/D pairs listed in Table 4.

In all tests we take $\rho = 0.95$ and $\gamma = 1.95$. All iterations start with $u^0 = (1, \ldots, 1)^T$ and $\rho_0 = 1$, and stopped whenever $\|r(u, \rho)\|_\infty \leq \epsilon$. All codes are written in Matlab and run on a P4-2.00G note book computer. The

![Fig. 1. The network used for the numerical test.](image)

---

### Table 1

<table>
<thead>
<tr>
<th>Link</th>
<th>Free-flow travel time $t^0_{\omega}$</th>
<th>Capacity $C_{\omega}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>200</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>200</td>
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<td>200</td>
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<td>5</td>
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<td>6</td>
<td>1</td>
<td>100</td>
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<td>11</td>
<td>200</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>200</td>
</tr>
<tr>
<td>11</td>
<td>15</td>
<td>200</td>
</tr>
</tbody>
</table>
iteration numbers and the computational time for Algorithm 4.2 and the method in [3] for different $\varepsilon$ are reported in Table 3. For the case $\varepsilon = 10^{-8}$, the optimal path flow and link flow are given in Tables 4 and 5, respectively.
The numerical experiments show that the new method is more flexible and efficient to solve the traffic equilibrium problem.

References