Reliability analysis of the metal forming process

B. Radi$^a$, A. El Hami$^b, *$

$^a$ Mathématique’s Department, Faculté des Sciences et Techniques, Errachidia, Morocco
$^b$ LMR, INSA de Rouen, Avenue de l’Université, 76800 Saint Etienne de Rouvray, France

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Abstract

In this paper, we propose a reliability–mechanical study combination for treating the metal forming process. This combination is based on the augmented Lagrangian method for solving the deterministic case and the response surface method. Our goal is the computation of the failure probability of the frictionless contact problem. Normally, contact problems in mechanics are particularly complex and have to be solved numerically. There are several numerical techniques available for computing the solution. However, some design parameters are uncertain and the deterministic solutions could be unacceptable. Thus, a mechanical contact study is an important subject for reliability analysis: the augmented Lagrangian method coupled with the first order reliability method, and we use the Monte Carlo method to obtain the founding results. The metal forming process is treated numerically to validate the new approach.

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1. Introduction

In some mechanical structures, contacts occur between elements. These contacts constitute singularities which have to be considered as soon as possible in the design and throughout the life of structures. In fact, the contact phenomenon is the cause of frequent structural failures.

Unilateral contact conditions with or without friction are a frequent phenomenon in the metal forming process with plastic deformation. These conditions determine the geometric form of the metal obtained after deformation and give us an idea about the surface constraints.

Several numerical methods are available for solving this problem: the frictionless contact between a thin shell and rigid punch. Most of them use a specific contact algorithm and a finite element method (FEM) [1–5].

In fact, the validity of the founding solutions is conditioned by the choice of design parameter values. Physical tests or measures show that the mechanical properties, the geometrical characteristics or the load could be uncertain and follow statistical distributions. The reliability analysis allows us to take these hazards into account in the mechanical analysis. Their effect on the structure response and the risk of a failure can be evaluated when a contact between two or several solids occurs.
In the deterministic case, we treat the frictionless contact between the thin shell and rigid punch as an evolutive quasi-static problem in an implicit form with a Lagrangian formulation updated for each loading increment, taking into account the nonlinearities resulting from (i) the material (elastoplastic behaviour), (ii) the geometry (large displacement and large deformation) and (iii) the boundary conditions (friction in the contact zone).

The solution of this problem is accomplished by using an incremental process where the equilibrium state at the end of the increment step is the initial configuration for the next increment. Within the framework of the finite element approximation, we use the displacement approach.

In this paper, we propose a reliability–mechanical study combination of the augmented Lagrangian method presented in [5] and a response surface method [6,7] for computing the failure probability of such a nonlinear problem. One can find our first result in [8].

To validate the proposed method, some numerical results are given and compared with those from the classical Monte Carlo method for a concrete probabilistic contact application. Advantages of the reliability analysis are revealed and commented on.

2. Problem statement

2.1. Deterministic mechanical contact modelling

Suppose that the deformable solid (the thin shell) occupies a bounded open subset $\Omega^t$ of $\mathbb{R}^3$ with regular boundary $\Gamma^t$ at the time $t$ and the bounded open subset $\Omega^0$ at the time $t_0$, and is subjected to a body force $f$, surface traction $g$ applied to a portion $\Gamma^g$, imposed displacement at $\Gamma^u$ and a contact reaction $t_c$ to a portion of $\Gamma^c$ ($\Gamma^c$ is unknown a priori). We suppose that $\Gamma^g \cap \Gamma^u \cap \Gamma^c = \phi$ (see Fig. 1).

Let $M$ be a point of the domain; it has coordinates $(x, t)$ in the current configuration $C^t$ and $(X, t)$ in the initial configuration $C^0$. In the frame of the evolving quasi-static formulation, the local movement equation of the point $M$ in the current configuration is given by

\[
\begin{align*}
\text{(E)} & \quad \begin{cases}
\text{div} \sigma + f = 0 & \text{in } \Omega^t \\
u = \bar{u} & \text{on } \Gamma^u \\
 \sigma \cdot n = g & \text{on } \Gamma^g \\
 \sigma \cdot n = t_c & \text{on } \Gamma^c
\end{cases}
\end{align*}
\]

where $\sigma$ is the Cauchy stress tensor and $\Gamma^c$ is the candidate contact zone. In our context, the imposed displacement is equal to zero.

But in the initial configuration $C^0$, the movement equation of the point $M$ is

\[
\begin{align*}
\text{(E)} & \quad \begin{cases}
\text{div} T + f = 0 & \text{in } \Omega^0 \\
u = \bar{u}_0 & \text{on } \Gamma^u_0 \\
 T \cdot n = g & \text{on } \Gamma^g_0 \\
 T \cdot n = t_c & \text{on } \Gamma^c_0
\end{cases}
\end{align*}
\]

where $T$ is the first Piola–Kirchhoff stress tensor and $f_0$ is the external body force applied to the initial configuration.
To solve the mechanical problem in the large deformation elastoplastic area, the material behaviour equation must be integrated during the loading generated by the structure deformation. The numerical scheme used to integrate these behaviour equations must respect the material uniformity [9].

During the metal forming process, the boundary conditions relating to the contact and the friction are changing. From a geometrical point of view, this supposes that the metal nodes move as a function of the punch movement: whether or not they are in contact with the tool. On the contact surface between the deformable metal and the rigid tool, an orthonormal local reference \((n, t_1, t_2)\) is defined on each node, where \(n\) is the outward normal to the surface and the \((t_1, t_2)\) represents the plane tangent to the surface.

The displacement and the reaction are written with respect to this local reference configuration as

\[
\begin{align*}
    u_n &= u \cdot n \\
    u_t &= u - (u_n \cdot n) \\
    r_n &= r \cdot n \\
    r_t &= r - (r_n \cdot n).
\end{align*}
\]

And the unilateral conditions, written for on the part \(\Gamma_c\) of the boundary, following complementary conditions, are

\[
    u_n \leq 0, \quad r_n \leq 0 \quad u_n \cdot r_n = 0.
\]

Taking these conditions into consideration, the principle of virtual work gives the weak form of the problem \((E)\) in the usual manner as follows:

\[
    \int_{\Omega} \sigma : \nabla v d\Omega - \int_{\Omega} f v d\Omega - \int_{\Gamma_g} g v d\Gamma - \int_{\Gamma_c} t_c v d\Gamma = 0
\]

which must hold for all \(v\) with \(v = u\) on \(\Gamma_n\) satisfying \(v \cdot n \leq 0\) on \(\Gamma_n\).

\(\sigma : \nabla v\) and \(f \cdot v\) denote the usual dot products of tensors and vectors. One can obtain a similar weak form of the problem \((E_0)\).

We use here the classical Coulomb friction law. In spite of its elementary form, this friction law is capable of describing correctly numerous problems involving dry friction (especially for metal/metal contact) as long as the numerical treatment is carefully effected [10].

Following Duvaut [11], the friction law can be written as

\[
    \begin{cases}
        \|r_t\| \leq v_f |r_n| \\
        \|r_t\| < v_f |r_n| \Rightarrow \dot{u}_t = 0 \\
        \|r_t\| = v_f |r_n| \Rightarrow \exists \alpha \geq 0 \dot{u}_t = -\alpha r_t
    \end{cases}
\]

where \(v_f\) is the friction coefficient.

Remark. One can use other friction laws such as the nonlinear law proposed by Oden and Pires [12]. Our numerical experience has revealed that similar results can be also obtained using the local law (5) or nonlocal law in the deterministic case [13]. In [14], one can find a stochastic model for the nonlocal law and in [15], one can find a stochastic model for the local law.

2.2. Reliability contact analysis

Several parameters are uncertain in this contact problem. In fact, physical tests or measures show that the mechanical properties and the geometrical characteristics of structure elements or applied loads could be random and follow statistical distributions. This leads us to define a probabilistic model. In general, random variables give a good representation of structural stochastic parameters. Let

\[
    X = (X_1, X_2, \ldots, X_m)^T
\]

be the random vector of the probabilistic analysis.

To preserve the integrity of the structure, the failure mode must be defined and the corresponding limit state function \(G(X)\) established. During the unilateral contact problem with friction, the real contact surfaces and the
contact reactions depend on the values of design parameters which are uncertain. Thus, excessive stresses or strains could appear in an element of the structure. The contact surfaces could not conform to a standard of a geometrical rule and lead to a structural failure. So, problems involving contact are a suitable domain for a reliability analysis.

The structure is situated in its safe domain \( D_s \) if \( \{ G(X) > 0 \} \) and it is situated in its failure domain \( D_f \) if \( \{ G(X) \leq 0 \} \). Then, the failure probability is

\[
P_f = \text{Prob}(G(X) \leq 0)
\]  

or

\[
P_f = \int_{G(X) \leq 0} f_X(X) dX
\]

where \( f_X \) denotes the joint probability density function of the vector \( X \).

Our purpose is a reliability analysis of a structure where a frictionless contact occurs between two solids. In this situation, the analytical expressions of the limit state function \( G \) and its derivatives are often not available as functions of the physical random variables \( X_1, X_2, \ldots, X_m \). It is only possible to obtain the failure probability in an implicit numerical form. So, to solve our reliability contact problem, a combination of the augmented Lagrangian method coupled with the finite element method and the response surface method \([7,16]\) is proposed.

3. Reliability–mechanical contact method

3.1. Deterministic approach

3.1.1. Finite element approximation

Within the framework of the finite element approximation, the shell finite element of Ahmad type (3D degenerate) is used, taking into account the transverse shear by using a mixed interpolation of the transverse distortion.

These finite elements were developed by Gelin and Lochegnies \([17]\) for the quadrangular element and by Boisse et al. \([18]\) for the triangular elements. These elements are isoparametric with linear interpolation. To avoid the locking problem associated with the plastic incompressibility, a mixed formulation is used.

3.1.2. Augmented Lagrangian method

We have extended the approach proposed by Simo and Laursen (in 2D) \([19]\) to the metal forming process: the contact between a thin shell and a rigid punch (3D and elastoplastic behaviour).

The augmented Lagrangian approach relating to the frictionless contact problem is given by the weak form of the equilibrium state:

\[
G^*(u, \delta u) = G(u, \delta u) + \int_{\Gamma_c^e} (r_n \delta u_t + r_t \cdot \delta u_t) d\Gamma^c.
\]  

Taking into consideration the contact condition \([3]\), the admissible displacement must verify \( \delta u \cdot n \geq 0 \) along the contact zone \( \Gamma_c \). The nonlinear equation \((9)\) must be solved at each time step, but this equation can’t be solved directly, so \( r_t \) and \( r_n \) are considered as known via an iterative process and the Newton–Raphson method is used.

With the aim of using the prediction–correction scheme, Coulomb’s friction law is introduced in the following form:

\[
\Phi(r_n, r_t) = \|r_t\| - \nu_f |r_n| \leq 0
\]  

\[
\dot{u}_t = -\zeta \frac{\partial \Phi}{\partial r_t}
\]  

\[
\zeta \geq 0
\]  

\[
\zeta \Phi = 0.
\]

It is easy to see that this form \((10)–(13)\) is equivalent to \((5)\) above.
Quantities \( r_n \) and \( r_t \) are treated as the sums of the penalization part and the Lagrange multipliers. Eq. (11) is regularized taking this sum into account:

\[
\dot{u}_t - \varsigma \frac{\partial \Phi}{\partial r_t} = \frac{1}{\epsilon_t} (r_t - \lambda_t).
\]  (14)

The treatment of the contact and the friction is done inside the iterative process, relating to the resolution of the equilibrium equation as follows:

**For the contact:**

\[
r_n^{k+1} = \lambda_n^{\text{old}} + \epsilon_n u_n^k
\]  (15)

where \( \lambda_n^{\text{old}} \) is the Lagrange multiplier obtained on the old equilibrium state.

**For the friction:**

We use the analogy with the plasticity model to compute the tangential reaction. During the current iteration, the tangential reaction is computed as follows:

\[
r_t^{k+1} = \lambda_t^k + (r_t^0)^k
\]  (16)

where \( (r_t^0)^k \) is the penalized term and given by

\[
(r_t^0)^k = r_t^k + \Delta \lambda_t^k + \epsilon_t \Delta u_t^k.
\]  (17)

At this level, the contact of the Coulomb model is obtained:

If \( \Phi(r_n^{k+1}, r_t^{k+1}) \leq 0 \) then the friction law is verified and \( \lambda_t^{k+1} = r_t^{k+1} \).

Otherwise, the correction of the reaction \( r_t^{k+1} \) is done as follows:

\[
r_t^{k+1} = \nu f_n r_n^{k+1} \frac{(r_t^0)^k}{\|(r_t^0)^k\|}
\]  (18)

and the equilibrium equation is solved using these reactions, \( r_t^{k+1} \) and \( r_n^{k+1} \).

The updated value of the \( \Delta \lambda_t^{k+1} \) is given by

\[
\Delta \lambda_t^{k+1} = \Delta \lambda_t^k + \epsilon_t \left( \Delta u_t^k - \Delta \varsigma^k \frac{(r_t^0)^k}{\|(r_t^0)^k\|} \right).
\]  (19)

When the equilibrium state is obtained, the test relating to the contact condition at the contact zone is done. If this condition is verified then the iterative process is stopped and the new increment is begun. Otherwise, the Lagrange multiplier \( \lambda_n \) is updated as follows:

\[
\lambda_n^{\text{new}} = \lambda_n^{\text{old}} + \epsilon_n u_n^\text{equi}.
\]  (20)

The iterative process goes on until the contact condition is verified.

The equilibrium equations are completely solved before the update of the Lagrange multiplier (\( \lambda_n \) and \( \lambda_t \)). With this strategy, the contact conditions are respected and the global system obtained is symmetric.

**Remark.**

1. The advantage of this method is that it does not need large penalization parameters \( \epsilon_n \) and \( \epsilon_t \) to verify the contact condition with friction. Such parameters would increase the possibility of conditioning problems in the system.

2. It is difficult to deal with the relative slide in three dimensions when the slide direction is unknown. Nevertheless, good results were obtained via the proposed regularization (14).
3.2. Reliability numerical method

The response surface methods have been widely developed in nonlinear reliability analysis. Several authors have proposed solutions for improving the accuracy of results, decreasing the number of necessary numerical calculations on FEM codes and increasing the robustness of algorithms [7,16].

In our nonlinear study, we propose an adaptive surface method coupled with the first order reliability method (FORM). Sets of design points and response surfaces are generated in the space of standard Gaussian variables. The scheme of the adaptive process is given as follows:

1. $k = 1$; the generated set of points is a central composite design. Its centre coordinates are the mean values of random variables. $d^{(1)}$ is a fixed real number and the distance from the central point to a ‘corner’ in the design is equal to $\sqrt{md^{(1)}}$. So

$$
\begin{align*}
  u^{(k,1)} &= (0, 0, \ldots, 0)^T \\
  u^{(k,r)} &= (0, \ldots, \pm d^{(k)}, \ldots, 0)^T, \quad r = 2, 2m + 1 \\
  u^{(k,r)} &= (\pm d^{(k)}, \pm d^{(k)}, \ldots, \pm d^{(k)})^T, \quad r = 2m + 2, 2m + 2m + 1.
\end{align*}
$$

2. The response surface $\tilde{h}^{(k)}(u)$ is a second order polynomial with crossed terms:

$$
\tilde{h}^{(k)}(u) = a_0 + \sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} u_i u_j.
$$

3. The polynomial coefficient identification is done by the least squares method:

$$
E^{(k)} = \sum_{j=1}^{p} w_j [\tilde{h}^{(k)}(u^{(k,r)}) - h(u^{(k,r)})]^2
$$

$$
\frac{\partial E^{(k)}}{\partial a_i} = 0 \quad i = 0, N_h.
$$

$p = 2m + 2m + 1$ and $N_h = (m + 1)(m + 2)/2$ is the number of coefficients of the function $\tilde{h}^{(k)}(u) \cdot w_i = 1$.

4. The SQP optimization algorithm [20] is used to compute the reliability index $\beta_{HL}^{(k)}$ and the design point $u^{(k,*)}$, solutions of the following minimization problem:

$$
\beta_{HL}^{(k)} = \min \sqrt{u^T \cdot u} \quad \text{subjected to:} \; \tilde{h}^{(k)}(u) = 0.
$$

5. $k = k + 1$; generation of a new set of points. Its centre is the point $u^{(k-1,*)}$ and the distance from the central point to a ‘corner’ in the design is equal to $\sqrt{md^{(k)}}$ with

$$
d^{(k)} = d^{(k-1)}/q
$$

where $q > 1$ is a real number which plays the role of a zoom factor,

$$
\begin{align*}
  u^{(k,1)} &= u^{(k-1,*)} \\
  u^{(k,r)} &= (u_1^{(k-1,*)}, \ldots, u_i^{(k-1,*)} \pm d^{(k)}, \ldots, u_m^{(k-1,*)})^T, \quad r = 2, 2m + 1 \\
  u^{(k,r)} &= (u_1^{(k-1,*)} \pm d^{(k)}, u_2^{(k-1,*)} \pm d^{(k)}, \ldots, u_m^{(k-1,*)} \pm d^{(k)}), \quad r = 2m + 2, 2m + 2m + 1.
\end{align*}
$$

6. Repeat (2)–(5) until a test of convergence on $\beta_{HL}^{(k)}$ stops the iterative algorithm.

7. Then the failure probability is evaluated by the first order reliability method:

$$
P_f \approx \Phi(-\beta_{HL}).
$$

$u = (u_1, u_2, \ldots, u_m)^T$ is a realization of the random vector $U \cdot \tilde{h}^{(k)}(u)$ is the approximated limit state function in the space of standard Gaussian variables. $U$ is the image of $X$ under the probabilistic transformation. $\Phi$ is the standard normal distribution function.
Table 1
Laws of random variables

<table>
<thead>
<tr>
<th>Variables</th>
<th>Laws of probability</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Normal</td>
<td>2100 MPa</td>
<td>200 MPa</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>Normal</td>
<td>50 MPa</td>
<td>5 MPa</td>
</tr>
<tr>
<td>$F$</td>
<td>Normal</td>
<td>500 N</td>
<td>50 N</td>
</tr>
</tbody>
</table>

This iterative scheme is particularly efficient. The adaptive central composite designs give a very good representation of the random variables domain. The second order polynomial and the least squares method assure a good compromise between the computational effort and the approximation accuracy of the real limit state function $h(u)$. The number of necessary calculations is reasonable and depends on the number of variables. The SQP algorithm is robust and efficient for this application in nonlinear finite element reliability analysis.

4. Numerical results

4.1. Position of the deterministic problem

We have applied the proposed method to solve for the forming of the deformable solid under the influence of one rigid punch. The sheet is spherical with the following mean values of the mechanical characteristics: $E = 2.10$ GPa and the Poisson ratio equal to 0.33 and fixed on the external part; the punch is hemispherical. The friction coefficient is equal to 0.2 and the behaviour of the material is supposed elastoplastic with isotropic hardening. In this problem, we study the stress field in the deformable solid.

4.2. Random variables

However, some parameters are uncertain, particularly the mechanical properties and the load. Thus the Young’s modulus $E$, the yield limit stress of the material $\sigma_y$ and the load $F$ follow statistical distributions. So $E$, $\sigma_y$ and $F$ are random variables and their laws of probability are given in Table 1. These random variables are independent.

4.3. Failure criterion

In this context, we now estimate the risk of the failure in the deformable solid. We use the following criterion:

$$G(E, \sigma_y, F) = a_c - a_n$$

where $a_c$ is the critical depth of the centre point of the deformable solid and $a_n$ is the computed depth of the same point at each increment. This function is an implicit form of the random variables.

4.4. Finite element modelling

Before the reliability analysis, we validate our finite element modelling and the contact method via deterministic computations with the random variables mean values. A quarter of the sheet is discretized into 149 triangular elements. We use the shell finite element. The descent of the punch is 40 mm. Fig. 2 gives the results found by the augmented Lagrangian method and Fig. 3 gives the distribution of the Von Mises constraints.

4.5. Reliability–mechanical study results

The failure probability is evaluated by the numerical method of Section 3. The results are given in Table 2. To validate the proposed combination, we perform Monte Carlo simulations and compare the results. We observe that the call’s number of the limit state is very high using the augmented Lagrangian coupled with the Monte Carlo method.

In this mechanical application, we have studied a classical case of reliability: the risk evaluation of failure occurring in the structure where some uncertainties exist. We want just to show the interest of using a reliability analysis to characterize the stress field in a unilateral contact problem. Thus the failure probability is of the order of $10^{-2}$. The design point components in the space of physical random variables are given in Table 3.
A sensitivity analysis gives the importance of each random variable in the risk of failure. The direction cosines are presented in Table 4. The absolute values of the direction cosines are elevated and show the stochastic importance of the three random variables.

The uncertainties in the problem of unilateral contact influence the structure response. The stress field in the deformable solid is highly sensitive to the random characters of parameters $E$ and $F$. So, the reliability analysis is essential in this mechanical context.
5. Conclusion

In a metal forming process, the real contact surfaces and the contact reactions are unknown and could be unacceptable for a particular set of design parameters. This consideration of uncertainties on the mechanical properties, on the load . . . increases the failure risk of the structures and the reliability approach is essential.

In this paper, we have proposed a numerical method for solving a reliability contact problem: an augmented Lagrangian method coupled with a response surface method. This numerical combination is efficient and gives accurate results in this nonlinear mechanical case.

A concrete problem of reliability contact has been described and validated on a numerical example. We have shown that the response of the structure, the stress field, is influenced by the hazards of the design parameters. So the interest of the reliability analysis in the metal forming is really high.

References