First-Order Differential Inclusions with Nonlocal Initial Conditions

A. BOUCHERIF
King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
P.O. Box 5046, Dhahran 31261, Saudi Arabia
aboucher@kfupm.edu.sa

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Abstract—We investigate initial value problems for first-order differential inclusions with nonlocal conditions. We provide conditions on the right-hand side that are sufficient for obtaining a priori bounds on solutions. We then rely on a theorem of Bohnenblust and Karlin to prove existence of at least one solution. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Our objective in this paper is to investigate the existence of solutions of the following nonlocal initial value problem for first-order differential inclusions:

\[ x'(t) \in F(t, x(t)), \quad t \in (0, 1], \]
\[ x(0) + \sum_{k=1}^{m} a_k x(t_k) = x_0. \]  

(1)

Here \( F : J \times \mathbb{R} \to 2^\mathbb{R} \) is a set-valued map, \( J = [0, 1] \), \( x_0 \in \mathbb{R} \) is given, \( 0 < t_1 < t_2 < \cdots < t_m < 1 \), and \( a_k \neq 0 \) for all \( k = 1, 2, \ldots, m \). Nonlocal Cauchy problems for ordinary differential equations (single-valued \( F \) ) have been investigated by several authors, both for the scalar case and the abstract case (see, for instance, [1,2] and the references therein). Also, classical initial value problems for multivalued differential equations have been considered by many authors (see [3-5] and the references therein). A discussion about the importance of nonlocal conditions in different areas of applications can be found in [1] and the references therein. Also, reference [2] contains examples of problems with nonlocal conditions and references to other works dealing with nonlocal problems. For control problems with nonlocal conditions, we refer the interested reader to [6,7].

2. PRELIMINARIES

In this section, we introduce notations, definitions, and results that will be used in the remainder of the paper.
2.1. Function Spaces (see [8])

Let $J$ be a compact interval in $R$. $AC(J)$ is the Banach space of absolutely continuous real-valued functions defined on $J$, with the norm $\|x\|_0 = \sup\{|x(t)|; t \in J\}$ for $x \in AC(J)$; it is known that any $x \in AC(J)$ has a derivative almost everywhere and $x(t) - x(0) = \int_0^t x'(s) \, ds$.

$L^1(J) = \{x : J \to R \text{ measurable}; \int_J |x(t)| \, dt < +\infty\}$, and for $x \in L^1(J)$, define $\|x\|_{L^1} = \int_J |x(t)| \, dt$. $C_{ac}(J \times R_+, R_+)$ denotes the set of all functions $\phi : J \times R_+ \to R_+$ satisfying the local Carathéodory conditions, i.e.,

(i) $\phi(., x)$ is measurable for all $x \in R_+$,
(ii) $\phi(t, .)$ is continuous for almost all $t \in J$,
(iii) $\sup\{|\phi(., x)|; |x| \leq r\} \in L^1(J)$ for any $r \in (0, +\infty)$.

2.2. Set-Valued Maps

Let $X$ and $Y$ be Banach spaces. A set-valued map $G : X \to 2^Y$ is said to be compact if $G(X) = \bigcup \{G(x); x \in X\}$ is compact. $G$ has convex (closed, compact) values if $G(x)$ is convex (closed, compact) for every $x \in X$. $G$ is bounded on bounded subsets of $X$ if $G(B)$ is bounded in $Y$ for every bounded subset $B$ of $X$. A set-valued map $G$ is upper semicontinuous (usc, for short) at $z_0 \in X$ if for every open set $O$ containing $G(z_0)$, there exists a neighborhood $M$ of $z_0$ such that $G(M) \subset O$. $G$ is usc on $X$ if it is usc at every point of $X$. If $G$ is nonempty and compact-valued, then $G$ is usc if and only if $G$ has a closed graph. The set of all bounded closed convex and nonempty subsets of $X$ is denoted by $b(V)(X)$. A set-valued map $G : J \to b(V)(X)$ is measurable if for each $x \in X$, the function $t \mapsto \text{dist}(z, G(t))$ is measurable on $J$. For $X \subset Y$, $G$ has a fixed point if there exists $x \in X$ such that $x \in G(x)$. Also, $|G(x)| = \sup\{|y|; y \in G(x)\}$, and $\alpha G(x) := \{\alpha y; y \in G(x)\}$.

DEFINITION. A set-valued map $F : J \times R \to 2^R$ is said to be an $L^1$-Carathéodory if

(i) $t \mapsto F(t, y)$ is measurable for each $y \in R$;
(ii) $y \mapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$;
(iii) for each $R > 0$, there exists $h_R \in L^1(J, R_+)$ such that

$$|F(t, y)| = \sup\{|v| : v \in F(t, y)\} \leq h_R(t), \quad \text{for all } |y| \leq R \text{ and for almost all } t \in J.$$ 

$S^1_{F(., y(., .))} = \{v \in L^1(J, R) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}$ denotes the set of selectors of $F$ that belong to $L^1$. By a solution of (1) we mean an absolutely continuous function $x$ on $J$, such that $x' \in L^1$ and

$$x'(t) = f(t), \quad \text{a.e. } t \in (0, 1],$$

$$x(0) + \sum_{k=1}^m a_k x(t_k) = x_0,$$

where $f \in S^1_{F(., y(., .))}$. Note that for an $L^1$-Carathéodory multifunction $F : J \times R \to 2^R$, the set $S^1_{F(., y(., .))}$ is not empty (see [9]).

For more details on set-valued maps, we refer to [3].

2.3. A Linear Problem

For $f \in L^1(J)$, consider the following linear problem:

$$x'(t) = f(t), \quad \text{a.e. } t \in (0, 1],$$

$$x(0) + \sum_{k=1}^m a_k x(t_k) = x_0.$$

Simple computations show that the following result holds.
LEMMA 1. Suppose that $1 + \sum_{k=1}^{m} a_k \neq 0$. Then, the solution of problem (3) is given by

$$x(t) = a \left( x_0 - \sum_{k=1}^{m} a_k \int_{0}^{t} f(s) \, ds \right) + \int_{0}^{t} f(s) \, ds,$$

where $a = (1 + \sum_{k=1}^{m} a_k)^{-1}$.

This shows that the linear operator

$$L : AC(J) \to L^1(J)$$

defined by $Lx = x'$ has a bounded inverse, $L^{-1}$, given by

$$(L^{-1}f)(t) = a \left( x_0 - \sum_{k=1}^{m} a_k \int_{0}^{t} f(s) \, ds \right) + \int_{0}^{t} f(s) \, ds,$$

and

$$\|L^{-1}f\|_{L^1} \leq |a| |x_0| + A\|f\|_{L^1},$$

where $A = 1 + |a| \sum_{k=1}^{m} |a_k|$.

2.4. A Fixed-Point Theorem

In this paragraph, we state a fixed-point theorem for set-valued maps, which plays an important role in the proof of our main result.

THEOREM 2. Let $X$ be a Banach space, $D$ a nonempty subset of $X$, which is bounded, closed, and convex. Suppose $G : D \to 2^X \setminus \{\emptyset\}$ is upper semicontinuous, with closed, convex values, and such that $G(D) \subseteq D$ and $\bar{G}(D)$ compact. Then $G$ has a fixed point.

REMARK. This theorem is due to Bohnenblust and Karlin (see [10, Theorem 4; 3, Cor. 11.3(e)]).

3. MAIN RESULT

In this section, we state and prove our main result. We assume the following.

(H0) $a_k \neq 0$ for each $k = 1, 2, \ldots, m$ and $1 + \sum_{k=1}^{m} a_k \neq 0$.

(H1) $F : J \times R \to bcc(R)$, $(t, x) \mapsto F(t, x)$ is

(i) measurable in $t$, for each $x \in R$,

(ii) usc with respect to $x \in R$ for a.e. $t \in J$.

(H2) $|F(t, x)| \leq \omega(t, |x|)$ for a.e. $t \in R$, all $x \in R$, where $\omega \in Car_{loc}(J \times R_+, R_+)$ is nondecreasing in its second argument and such that $\limsup_{\rho \to \infty} (1/\rho) \int_{0}^{t} \omega(t, \rho) \, dt < 1/A$, where $A = 1 + |a| \sum_{k=1}^{m} |a_k|$.

Our main result reads as follows.

THEOREM 3. If Assumptions (H0), (H1), and (H2) are satisfied, then the initial value problem (1) has at least one solution.

PROOF. This proof will be given in several steps, and uses some ideas from [5] and [11].

STEP 1. Consider the set-valued operator $\Phi : AC(J) \to L^1(J)$ defined by

$$(\Phi x)(t) = F(t, x(t)).$$

$\Phi$ is well defined, usc, with convex values and sends bounded subsets of $AC(J)$ into bounded subsets of $L^1(J)$. In fact, we have

$$\Phi x := \{ u : J \to R \text{ measurable; } u(t) \in F(t, x(t)) \text{ a.e. } t \in J \}.$$
Let \( z \in AC(J) \). If \( u \in \Phi z \), then
\[
|u(t) - \omega(t, |z(t)|)| \leq \omega(t, \|z\|_0).
\]

Hence, \( \|u\|_{L^1} \leq C_0 := \int_0^1 \sup\{\omega(t, u); 0 \leq u \leq \|z\|_0\} \, dt \). This shows that \( \Phi \) is well defined. It is clear that \( \Phi \) is convex valued.

Now, let \( B \) be a bounded subset of \( AC(J) \). Then, there exists \( K > 0 \) such that \( \|u\|_0 \leq K \) for \( u \in B \). So, for \( w \in \Phi u \) we have \( \|w\|_{L^1} \leq C_1 \), where \( C_1 = \int_0^1 \omega(t, K) \, dt \).

Also, we can argue as in [5, p. 161] to show that \( \Phi \) is usc.

**STEP 2. A PRIORI ESTIMATES.** Let \( x \) be a possible solution of (1). Then there exists a positive constant \( R^* \), not depending on \( x \), such that
\[
|x(t)| \leq R^*, \quad \text{for } t \in J, \tag{5}
\]
for it follows from the definition of solutions of (1) that
\[
x'(t) = f(t), \quad \text{a.e. } t \in (0,1], \quad x(0) + \sum_{k=1}^m a_k x(t_k) = x_0,
\]
where \( f \in S^1_{F(.,x(.))} \). It follows from Section 2.3 that
\[
x(t) = a \left( x_0 - \sum_{k=1}^m a_k \int_0^{t_k} f(s) \, ds \right) + \int_0^t f(s) \, ds.
\]

Hence,
\[
|x(t)| \leq |a| \left( |x_0| + \sum_{k=1}^m |a_k| \int_0^{t_k} |f(s)| \, ds \right) + \int_0^t |f(s)| \, ds.
\]

Assumption (H2) yields
\[
|x(t)| \leq |a| \left( |x_0| + \sum_{k=1}^m |a_k| \int_0^{t_k} \omega(s, |x(s)|) \, ds \right) + \int_0^t \omega(s, |x(s)|) \, ds.
\]

Let
\[
R_0 = \max\{|x(t)|; t \in J\}.
\]

Then, since \( \omega \) is nondecreasing in its second argument,
\[
R_0 \leq |a| \left( |x_0| + \sum_{k=1}^m |a_k| \int_0^{t_k} \omega(s, R_0) \, ds \right) + \int_0^t \omega(s, R_0) \, ds
\]
or
\[
R_0 \leq |a||x_0| + \left[ |a| \sum_{k=1}^m |a_k| + 1 \right] \int_0^1 \omega(s, R_0) \, ds.
\]

The last inequality implies that (recall that \( A = |a| \sum_{k=1}^m |a_k| + 1 \))
\[
1 \leq \frac{|a||x_0|}{R_0} + \frac{A}{R_0} \int_0^1 \omega(s, R_0) \, ds.
\]

Now, the condition on \( \omega \) in (H2) shows that there exists \( R^* > 0 \) such that for all \( R > R^* \),
\[
\frac{|a||x_0|}{R} + \frac{A}{R} \int_0^1 \omega(s, R) \, ds < 1.
\]

Comparing these last two inequalities, we see that \( R_0 \leq R^* \).
Consequently, we obtain \(|x(t)| \leq R^*\) for all \(t \in J\).

**STEP 3. EXISTENCE OF SOLUTIONS.** Let \(R^*\) be the constant obtained in Step 2. Define a truncation function \(h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\) by

\[
h(R^*, u) = \begin{cases} 
1, & 0 \leq u \leq R^*, \\
2 - \frac{u}{R^*}, & R^* \leq u \leq 2R^*, \\
0, & u \geq 2R^*.
\end{cases}
\]

Then, we can easily see that \(h\) is continuous and \(0 \leq h(R^*, u) \leq 1\) for all \(u \in \mathbb{R}_+\).

Let \(\Gamma(t, x) := h(R^*, |x|)F(t, x)\). The assumptions on \(F\) and the properties of \(h\) imply that the set-valued map \(\Gamma\) satisfies the global Carathéodory conditions, so that there exists \(g \in L^1(J)\) such that \(|f(t)| \leq g(t)\) for almost all \(t \in J\) and all \(f \in S_{\Gamma(x(t))}^1\).

In fact, \(g(t) := \sup\{|\Gamma(t, x)|; |x| \leq 2R^*\}\).

Hence, \(\Gamma\) is a bounded set-valued map.

Consider the following modified problem:

\[
x'(t) \in \Gamma(t, x(t)), \quad t \in (0, 1], \\
x(0) + \sum_{k=1}^m a_k x(t_k) = x_0.
\]

We will show that this problem has at least one solution that satisfies estimate (5).

Consider \(D := \{x \in AC(J); \|x\|_0 \leq \|a\|_0 + A\|g\|_1\}\). Then \(D\) is a nonempty bounded, closed and convex subset of \(AC(J)\). Define a set-valued operator \(G\) by

\[
G(x) := \{z \in AC(J); z(t) = L^{-1}f(t), f(t) \in \Gamma(t, x(t))\text{ a.e. } t \in J\}.
\]

We have that \(G = L^{-1}\Phi\), where \(\Phi\) is the operator defined in Step 1. Also, it is clear that the solutions of the modified problem (6) are fixed points of \(G\) and vice versa. It follows from the properties of \(L^{-1}\), \(\Phi\), and \(\Gamma\) that \(G\) is a compact, convex, upper semicontinuous set-valued operator. Hence, \(G(D)\) is compact. Also, it is readily seen that \(G\) maps \(D\) into itself. Therefore, by Theorem 2, \(G\) has a fixed point \(y\), which is a solution of the modified problem (6), i.e.,

\[
y'(t) \in \Gamma(t, y(t)), \quad t \in (0, 1], \\
y(0) + \sum_{k=1}^m a_k y(t_k) = x_0.
\]

**STEP 4.** This solution \(y\) satisfies estimate (5), for the definition of \(\Gamma\) and Assumption (H2) imply that \(|\Gamma(t, x)| \leq \Omega(t, |x|)\) with \(\Omega(t, |x|) := h(R^*, |x|)\omega(t, |x|)\).

This shows that \(\Omega \in \text{Car}(J \times \mathbb{R}_+, \mathbb{R}_+)\) is nondecreasing in its second argument and

\[
\limsup_{\rho \to \infty} \left(\frac{1}{\rho}\right) \int_0^1 \Omega(t, \rho) \, dt < \frac{1}{A},
\]

so that \(\Gamma\) satisfies condition (H2) with \(\omega(t, |x|)\) replaced by \(\Omega(t, |x|)\). It follows from Step 2 that

\[
|y(t)| \leq R^*, \quad \text{for all } t \in J.
\]

But, for all \(x\) such that \(|x| \leq R^*\), the set-valued maps \(\Gamma\) and \(F\) coincide. Therefore, \(y\) is a solution of problem (1). This completes the proof of our main result.
REFERENCES


