On the global existence for the Kac model with external force

A. Bellouquid

University Cadi Ayyad, Ecole Nationale des Sciences Appliquées, Safi, Maroc, France

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ABSTRACT

The nonlinear Boltzmann Kac’s model under the influence of an external force field arising from a potential \( \phi = \phi(x) \) is considered, and the global existence and uniqueness for the general solution to the initial value problem is proved. The result is proved for small initial data in the case potential bounded from below. As an application, some results concerning the harmonic oscillator are obtained.

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1. Introduction to the Kac model and to the Cauchy Problem

A challenging problem of the mathematical kinetic theory is the qualitative analysis of the initial value problem for the Boltzmann equation in the whole space [1]. As known, the main difficulty consists in dealing with the quadratic localized nonlinearity.

Simpler models such as the discrete Boltzmann equation [2,3], or the models delivered to describe multicellular system in the kinetic theory for active particles [4,5], present the same or even additional technical difficulties.

In this paper we consider a simplified mathematical model of the non-linear Boltzmann equation which was introduced by Kac [6]. We give the local and global solution to the Kac’s model in one dimension (velocity \( v \) and space) when an external force term is present. Kac’s model can be written as follows:

\[
\begin{align*}
\partial_t F + v \partial_x F + \partial_v \phi \partial_v F &= Q(F, F), \\
F(t = 0, x, v) &= F_0(x, v).
\end{align*}
\] (1.1)

where \( F(t, x, v) \) is a nonnegative function representing the density of particles with position \( x \) and velocity \( v \) in the single-particle phase space \( \mathbb{R}_x \times \mathbb{R}_v \) at time \( t \). The interaction of particles through collisions is modeled by the operator \( Q(F, F) \), which acts only on the variable \( v \) and is generally nonlinear which is typically quadratic and is given by:

\[
Q(F, F) = \int_{[-\pi, \pi] \times \mathbb{R}} \left( F(v_1)F(v') - F(v)F(v_1) \right) \frac{I(\theta)}{2\pi} \, d\theta \, dv_1,
\] (1.2)

where \( v' \) et \( v'_1 \) are given by:

\[
v' = v \cos \theta - v_1 \sin \theta, \quad v'_1 = v \sin \theta + v_1 \cos \theta.
\] (1.3)

In particular, the following conservation equation:

\[
v^2 + v_1^2 = v'^2 + v_1'^2,
\] (1.4)
holds true, while the operator $Q$ satisfies the conservation properties of mass, and energy during collisions:

$$
\int_v Q(f, f) dv = \int_v v^2 Q(f, f) dv = 0, \quad (1.5)
$$

while the function $l(\theta)$ is supposed to be positive and integrable in $[-\pi, \pi]$. Mathematical aspects of model (1.1) have been studies by various mathematicians. Among others, the relation between rates of relaxation and convergence of Wild sums for the solution to the equation has been analyzed in [6], while the blow up of solutions for the equation with infinite energy is studied in [7,8]. A remarkable difficulty is the lack of entropy functionals to characterize the trend to equilibrium.

The contents of this paper are developed through two more sections that follow the above introduction. In details, Section 2 develops the statement of the initial value problem in the presence of a force field and reports the main results. Section 3 is focused on the proofs of the main results.

2. Main results

Due to the presence of a force field the characteristic lines are not straight. In particular, the trajectories of particles satisfy the following equation:

$$
\begin{aligned}
\frac{dX(t)}{dt} &= V(t), \\
\frac{dV(t)}{dt} &= \partial_x \phi, \\
X(0) &= x, \quad V(0) = v.
\end{aligned} \tag{2.1}
$$

The system is conservative, the hamiltonian $H(x, v) = -2\phi(x) + v^2$ is conservative along the characteristic:

$$
H(X(t), V(t)) = H(x, v). \tag{2.2}
$$

Indeed:

$$
\frac{d}{dt} H(X(t), V(t)) = -2X'(t)\partial_x \phi(X(t)) + 2V'V = -2V'V + 2V'V = 0. \tag{2.3}
$$

In order to study problem (1.1) with potential $\phi \neq 0$, we suppose in the following that the potential $\phi$ satisfies the following properties:

$$
\begin{aligned}
\forall (t, x, v) \in [0, \infty) \times \mathbb{R}^2 \quad &\text{there exist a unique solution} \\
(X, V) = (X(t, x, v), V(t, x, v)) \quad &\text{of problem (2.1). The solution is such that} \\
\forall (t, x, v) \in [0, \infty) \times \mathbb{R}^2 \quad &\text{there exists } T \text{ such that } X(t + T, x, v) = X(t, x, v). \tag{2.4}
\end{aligned}
$$

For $\phi = 0$, Shizuta and Niskiyama [9] have proved the global existence of the Cauchy problem (1.1) near an absolute Maxwellian. There is no proof concerning the existence of (1.1) with $\phi \neq 0$. Several authors (among others Asano [10,11], Bellomo et al. [12], Glikson [13,14], Hamdache [15], Tabata [16,17], and Tabata, Eshima [18], Ukai, Yang, Zhao [19]) have considered the existence problem of the Boltzmann equation with external force. In details, Asano [10] and Glikson [13,14] have considered the local existence problem, while Asano [11] the global existence with some decreasing conditions at infinity for the potential $\phi(x)$. This condition is not satisfied in the case of the harmonic oscillator. In [16,17] Tabata has considered the case of the harmonic oscillator for the linear Boltzmann equation, Hamdache [15] has considered the existence problem of the harmonic oscillator for the Boltzmann equation. Tabata, Eshima [18] have considered the decay of solution of the linearized Boltzmann equation with an external potential in a polyhedral bonded domain. Bellomo et al. [12] have proved the global existence of the Enskog Boltzmann equation with some conditions on the potential $\phi(t, x, v)$, and finally Ukai, Zhao [19] have proved the global classical solutions to the Boltzmann equation with external force and nontrivial large time behavior in the whole space. The development of asymptotic methods for the derivation of hydrodynamic models is proposed in [20].

In this paper, we consider the Cauchy problem (1.1) for $\phi \neq 0$. The potential is assumed to depend on the space variable. We show the locally existence of (1.1) by fixed point Theorem. By assuming that the characteristics are T-periodic, we solve the problem in each period, and obtain the global existence. We show that our assumption (2.4) is satisfied for a harmonic oscillator. The technique follows the arguments of [15] for Boltzmann equation in the case of the harmonic oscillator, which we extend the method to the Kac’s model with potential satisfying (2.4).

Let us introduce a class of functional spaces suitable for the problem, which we consider in the following. Moreover, let

$$
H_\beta = \{ F \in C^0_b(\mathbb{R}_x \times \mathbb{R}_v) : \sup_{x, v} \exp \beta (-2\phi(x) + v^2)|F(x, v)| < \infty \}.
$$

The norm of $H_\beta$ is given by:

$$
\|F\|_\beta = \sup_{x, v} \exp \beta (-2\phi(x) + v^2)|F(x, v)|.
$$
Let us now define the following space
\[ X_\beta([0, T]) = \{ F(t, X(t), V(t)) \in C^0([0, T] \times \mathbb{R} \times \mathbb{R}) : \sup_{t \in [0, T]} \| F(t, X(t), V(t)) \|_\beta < \infty \} \]
with norms:
\[ \| F(t) \|_\beta = \sup_{t \in [0, T]} \| F(t, X(t), V(t)) \|_\beta. \]

Using Assumption (2.4) the following results are obtained:

**Theorem 2.1.** Let \( a \in ]0, 1[ \), then there exists \( a_0 \) such that, for any \( F_0 \) satisfying \( \| F_0 \|_\beta \leq a_0 \) there exists a unique solution \( F \in X_\beta \) of the problem (1.1) in \([0, T] \) satisfying:
\[ \| F(t) \|_\beta \leq a_0 (1 + a), \quad \forall t \in [0, T]. \] (2.5)

**Theorem 2.2.** Let \( a \in ]0, 1[ \) be fixed, then there exists \( a_0 \) such that for any \( F_0 \in H_\beta \) satisfying \( \| F_0 \|_\beta \leq a_0 \) there exists a unique solution \( F \in X_\beta([0, nT]) \) of (1.1) satisfying for any \( r \leq n - 1 \) the following estimate:
\[ \| F(t + rT) \|_\beta \leq a_0 \left( \sum_{i=0}^{r-1} a^i \right), \quad \forall t \in [0, T]. \] (2.6)

**Application: Harmonic oscillator.** In this case the potential \( \phi \) is given by:
\[ \phi(x) = \frac{x^2}{2} + \phi_0. \] (2.7)
with \( \phi_0 > 0 \). Then, the potential \( \phi \) satisfies the estimate \( \phi(x) \geq \phi_0 \), moreover, in this case problem (1.1) can be written as follows:
\[ \begin{cases} \dot{F} + v \dot{F} + x \dot{F} = Q(F, F), \\ F(t = 0, x, v) = F_0(x, v). \end{cases} \] (2.8)

The trajectory of the system is given by:
\[ \begin{align*}
\frac{dX(t)}{dt} &= V(t), \quad X(0) = x, \\
\frac{dV(t)}{dt} &= -X(t), \quad V(0) = v,
\end{align*} \] (2.9)
which gives the equation for harmonic oscillator
\[ \begin{cases} X''(t) + X(t) = 0, \\ X(0) = x, \quad X'(0) = v. \end{cases} \] (2.10)
The trajectory has the following solution:
\[ \begin{align*}
X(t) &= x \cos t + v \sin t, \\
V(t) &= -x \sin t + v \cos t,
\end{align*} \] (2.11)
which are \( 2\pi \) periodic. Finally, the following result concerning the harmonic oscillator is obtained:

**Theorem 2.3.** Let \( a \in ]0, 1[ \) be fixed, then there exists \( a_0 \) such that for any \( F_0 \in H_\beta \) satisfying \( \| F_0 \|_\beta \leq a_0 \) there exists a unique solution \( F \in C([0, 2\pi]), H_\beta \) of (2.8) satisfying for any \( r \leq n - 1 \) the following estimate:
\[ |F(t + 2r\pi, x \cos t + v \sin t, -x \sin t + v \cos t)| \leq a_0 \left( \sum_{i=0}^{r+1} a^i \right) e^{-\beta(x^2 + v^2)}, \quad \forall t \in [0, 2\pi]. \] (2.9)

3. Proofs

This section reports the technical proofs of the main theorems announced in Section 2. Local and global existence are proved in the following subsections.

3.1. Local existence: Proof of Theorem 2.1

The solution of problem (1.1) is given for any \( t \in \mathbb{R} \) by:
\[ F(t, X(t), V(t)) = F_0(x, v) + \int_0^t Q(F, F)(s, X(s), V(s))ds. \] (3.1)
Let us consider the map $N$ given by:

$$N(G)(t, X(t), V(t)) = f_0(x, v) + S(G)(t, X(t), V(t)),$$

where $S(G)$ is given by:

$$S(G)(t, X(t), V(t)) = \int_0^t Q(G, G)(s, X(s), V(s))ds.$$

(3.2)

The following estimate for the operator $Q$ holds true:

**Lemma 3.1.** Let $F, G \in H_\beta$, then one has:

$$|Q(G, G)(s, X(s), V(s))| \leq \frac{\|G\|_\beta^2}{\sqrt{\pi \beta}} \exp(2\beta \phi_0) \left( \int_{-\pi}^{\pi} |I(\theta)|d\theta \right) \exp(\beta(2\phi(x) - \nu^2)),\quad (3.4)$$

$$|Q(F, F)(s, X(s), V(s)) - Q(G, G)(s, X(s), V(s))| \leq \frac{\|F\|_\beta + \|G\|_\beta}{\sqrt{\pi \beta}} \exp(2\beta \phi_0) \left( \int_{-\pi}^{\pi} |I(\theta)|d\theta \right) \exp(\beta(2\phi(x) - \nu^2)) \|G - H\|_\beta.\quad (3.5)$$

**Proof.** Taking into account (1.2) yields:

$$Q(G, G)(s, X(s), V(s)) = \int_{[-\pi, \pi] \times R} \left\{ G(v'_1(s))G(v'(s)) - G(V(s))G(v_1) \right\} \frac{|I(\theta)|}{2\pi} d\theta dv_1.$$

By using the inequalities

$$|G(s, X(s), v'_1(s))| \leq \|G\|_\beta \exp(2\beta(X(s)) - \nu_1^2(s)),\quad (3.6)$$

and

$$|G(s, X(s), v'(s))| \leq \|G\|_\beta \exp(2\beta(X(s)) - \nu^2(s)).\quad (3.7)$$

yields:

$$|G(s, X(s), v'_1(s))G(s, X(s), v'(s))| \leq \|G\|_\beta^2 \exp(4\beta(X(s)) - \nu_1^2(s) - \nu^2(s)).$$

In the same way, one obtains:

$$|G(s, X(s), V(s))G(s, X(s), v_1)| \leq \|G\|_\beta^2 \exp(4\beta(X(s)) - V^2(s) - v_1^2).\quad (3.8)$$

Using the energy conservation equation (1.4), one gets

$$\nu_1^2(s) + \nu^2(s) = V^2(s) + v_1^2.\quad (3.9)$$

Combining (3.6), (3.7) and (3.8) yields:

$$|Q(G, G)(s, X(s), V(s))| \leq 2\|G\|_\beta^2 \int_{[-\pi, \pi] \times R} \exp(4\beta(X(s)) - V^2(s) - v_1^2) \frac{|I(\theta)|}{2\pi} d\theta dv_1.$$

Finally (3.9) and the conservation equation of Hamiltonian $H(2.2)$, give:

$$|Q(G, G)(s, X(s), V(s))| \leq 2\|G\|_\beta^2 \exp(2\beta(\phi(x) - \nu^2)) \int_{[-\pi, \pi] \times R} \exp(2\beta(X(s)) - v_1^2) \frac{|I(\theta)|}{2\pi} d\theta dv_1 \leq 2\|G\|_\beta^2 \exp(2\beta(\phi(x) - \nu^2)) \exp(-2\beta \nu_1^2) \int_{[-\pi, \pi] \times R} \exp(-2\beta \nu_1^2) \frac{|I(\theta)|}{2\pi} d\theta dv_1 \leq \|G\|_\beta^2 \exp(2\beta \phi_0) \exp(2\beta(\phi(x) - \nu^2)) \left( \int_{-\pi}^{\pi} |I(\theta)|d\theta \right).$$

This completes the proof of (3.4).

In the same way, in order to prove (3.5), we use the same technique of the proof of (2 = 3.4) and the following relation

$$Q(G, G) - Q(F, F) = Q(G - F, G) + Q(F, G - F),$$

where $Q(F, G)$ is given by:

$$Q(F, G) = \frac{1}{2} \int_{[-\pi, \pi] \times R} \left\{ F(v'_1)G(v') + G(v'_1)F(v') - F(v)G(v_1) - G(v)F(v_1) \right\} \frac{|I(\theta)|}{2\pi} d\theta dv_1.$$

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Indeed, using the invariance of the Hamiltonian $H (3.7)$ yields
\[
|Q(G - F, G)(s, X(s), V(s))| \leq \|G - F\|_\beta \int_{-\pi}^{\pi} \left| \exp \beta (4\phi (X(s)) - V^2(s) - v^2) \frac{1(\theta)}{2\pi} \right| d\theta dv. \\
\leq \sup_{\theta < \pi} \int_{-\pi}^{\pi} \left| \exp \beta (2\phi_0) \exp \beta (2\phi(x) - v^2) \right| d\theta dv. \\
\leq \|G - F\|_\beta \frac{\exp(2\beta\phi_0)}{\sqrt{\pi\beta}} \exp \beta (2\phi(x) - v^2) \int_{-\pi}^{\pi} l(\theta) d\theta.
\]
Moreover,
\[
|Q(F, G - F)(s, X(s), V(s))| \leq \|G - F\|_\beta \frac{\exp(2\beta\phi_0)}{\sqrt{\pi\beta}} \exp \beta (2\phi(x) - v^2) \int_{-\pi}^{\pi} l(\theta) d\theta.
\]
This completes the proof. \(\Box\)

**Lemma 3.2.** Let $F, G \in X_\beta$, then there exists $C > 0$ such that:

\[
S(G)(t) \in X_\beta, \\
\|S(G)\|_\beta \leq C\|G\|_\beta^2, \\
\|S(G) - S(F)\|_\beta \leq C\|G - F\|_\beta (\|G\|_\beta + \|F\|_\beta).
\]

**Proof.** Using (3.3) and the estimate (3.4), yields
\[
|S(G)(t, X(t), V(t))| \leq t \frac{\|G\|_\beta^2}{\sqrt{\pi\beta}} \exp(2\beta\phi_0) \exp \beta (2\phi(x) - v^2) \left( \int_{-\pi}^{\pi} l(\theta) d\theta \right).
\]
which gives (3.11) by taking
\[
C = \frac{T}{\sqrt{\pi\beta}} \exp(2\beta\phi_0) \left( \int_{-\pi}^{\pi} l(\theta) d\theta \right).
\]
In the same way, by using (3.3) and the estimate (3.5), the estimate (3.12) is obtained with the same constant $C$ given by (3.13).

The continuity of $S(G)$ follows from the estimate:
\[
|S(G)(t) - S(G)(s)| \leq \int_t^s \left| Q(G, G)(w, X(w), V(w)) \right| dw \\
\leq |t - s| \frac{\|G\|_\beta^2}{\sqrt{\pi\beta}} \exp(2\beta\phi_0) \exp \beta (2\phi(x) - v^2) \left( \int_{-\pi}^{\pi} l(\theta) d\theta \right).
\]
This completes the proof of the Lemma. \(\Box\)

**Proof of Theorem 2.1.** Let $a \in ]0, 1[, n \geq 1$ be fixed, we suppose that a constant $a_0 > 0$ satisfies the hypothesis:
\[
2Ca_0 \leq a^n(1 - a)^2,
\]
where $C$ is the same constant as in the Lemma 3.2 given by (3.13). In particular, one has
\[
2Ca_0 \left( \sum_{i=0}^{n} a^i \right)^2 \leq a^n.
\]
Let $B$ the closed ball in $X_\beta$: $B = \{ F \in X_\beta : \|F\|_\beta \leq k \}$. Moreover, let $G, H \in B$, and using Lemma 3.2, yields:
\[
\|N(G)(t)\|_\beta \leq (a_0 + Ck^2), \\
\|N(G)(t) - N(H)(t)\|_\beta \leq 2Ck\|G - H\|_\beta.
\]
In order to prove that $N$ is a contraction in a ball $B$, one needs
\[
a_0 + Ck^2 \leq k.
\]
If
\[
k = a_0(1 + a).
\]
is chosen, Eq. (3.16) is equivalent to
\[ Ca_0(1 + a)^2 \leq a, \]
which is satisfied for \( n = 1 \) by using (3.15).
Moreover, by (3.15), one has also \( 2Ck = 2Ca_0(1 + a) < a < 1 \). This proves that the maps \( N \) is a contraction in a ball \( B \).
This completes the proof of Theorem 2.1 by using the fixed point Theorem. \( \square \)

3.2. Global existence: Proof of Theorem 2.2

The aim of this subsection consists in proving the global existence of solutions to problem (1.1). It will be proved that the solution can be extended in each interval \([0, nT]\), for \( n \in \mathbb{N} \).
Bearing this in mind, let us suppose that the characteristics \((X(t), V(t))\) are \( T \)-periodic. The following Lemma holds true:

**Lemma 3.3.** Let \( F_0 \) satisfy the conditions of Theorem 2.1. Consequently, the solution to (1.1) can be extended in the interval \([T, 2T]\) and satisfies the estimate:
\[ \|F(t + T)\|_\beta \leq a_0(1 + a + a^2), \quad \forall t \in [0, T]. \] (3.18)

**Proof.** Theorem 2.1 provides the following estimate
\[ \|F(t + T)\|_\beta \leq a_0(1 + a). \]
We solve the problem (1.1) in \([T, 2T]\) with initial condition given by \( F(T, x, v) \). Then, for any \( t \in [0, T] \), one has:
\[
F(t + T, X(t + T), V(t + T)) = F(T, X(T), V(T)) + \int_0^{t} Q(F, F(s + T, X(s + T), V(s + T)))ds
\]
\[
= F(t, x, v) + \int_0^{t} Q(F, F(s + T, X(s + T), V(s + T)))ds.
\]
Let now \( F \in B \), using the technique as in the proof of Theorem 2.1 yields:
\[ \|F(t + T)\|_\beta \leq (a_0(1 + a) + Ck^2). \]
The constant \( C \) is the same as in the Lemma 3.2.
Let us now choose \( k \) as the follows:
\[ k = a_0(1 + a + a^2). \]
Application of the fixed point Theorem needs
\[ a_0(1 + a) + Ca_0^2(1 + a + a^2)^2 \leq a_0(1 + a + a^2), \]
which is equivalent to
\[ Ca_0(1 + a + a^2)^2 \leq a^2. \] (3.19)
Considering that
\[ 1 + a + a^2 \leq \frac{1}{1 - a}, \]
the estimate (3.19) is satisfied by using (3.14) for \( n = 2 \). Moreover
\[ 2Ca_0(1 + a + a^2) \leq a^2(1 - a) < 1. \]
Therefore, the fixed point Theorem give the existence of solution in \([T, 2T]\) satisfying (3.18). This solution is continued in \([0, 2T]\) and in particular, satisfies the estimate
\[ \|F(2T, x, v)\|_\beta \leq a_0(1 + a + a^2). \]
This completes the proof of Lemma 3.3. \( \square \)

The iteration process is applied to prove global existence in \([0, \infty)\). Suppose that the solution exists and is continued \([0, (n - 1)T]\) satisfying:
\[
\begin{cases}
\|F(rT, x, v)\|_\beta \leq a_0 \left( \sum_{i=0}^{r-1} a^i \right), & r = 0, 1, \ldots n - 1, \\
\|F((r - 1)T)\|_\beta \leq a_0 \left( \sum_{i=0}^{r-1} a^i \right), & r = 1, 2, \ldots n - 1, t \in [0, T].
\end{cases}
\] (3.20)
It can be proved that we can extend the solution in \([(n-1)T, nT]\), satisfying for any \(t \in [0, T]\) the following:

\[
\begin{align*}
\|F(t + (n-1)T)\|_\beta &\leq a_0 \left( \sum_{i=0}^{n} a^i \right), \\
\|F(nT, x, v)\|_\beta &\leq a_0 \left( \sum_{i=0}^{n} a^i \right).
\end{align*}
\tag{3.21}
\]

Let \(F\) be the solution of the following problem:

\[
F(t + (n-1)T, X(t), V(t)) = F((n-1)T, x, v) + \int_0^t Q(F, F)(s + (n-1)T, X(s + (n-1)T), V(s + (n-1)T)) \, ds.
\tag{3.22}
\]

Suppose now that \(F \in X_\beta\), consequently by using the technique of Lemma 3.1 yields:

\[
\int_0^t \|Q(F, F)(s + (n-1)T, X(s + (n-1)T), V(s + (n-1)T))\|_\beta \leq C \|F\|_{\beta}^2
\]

with the same constant \(C\) as in the Lemma 3.2.

The solution \(F\) given by (3.22) satisfies the estimate:

\[
\|F(t + (n-1)T)\|_\beta \leq \left( a_0 \sum_{i=0}^{n-1} a^i + Ck \right).
\]

Choose \(k\) such that:

\[
k = a_0 \sum_{i=0}^{n} a^i.
\]

The application of the fixed point Theorem needs

\[
a_0 \sum_{i=0}^{n-1} a^i + C a_0^2 \left( \sum_{i=0}^{n} a^i \right)^2 \leq a_0 \sum_{i=0}^{n} a^i,
\]

which is equivalent to choose \(a_0\) satisfying (3.14).

Moreover

\[
2Ck = 2Ca_0 \sum_{i=0}^{n} a^i < 1.
\]

This completes the proof of Theorem 2.2. \(\square\)

References


