Mutual Information and Bose-Einstein Condensation

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In the present work we are studying a bosonic quantum field system at finite temperature, and at zero and non-zero chemical potential. For a simple spatial partition we calculate the corresponding geometric entropy and we find that it contains terms originated from genuine quantum as well from classical thermal correlations. After the subtraction of the thermal entropy part, we derive the mutual information for the specific partition. In the case of non-zero chemical potential, we examine the influence of the underlying Bose-Einstein condensation on the behavior of the mutual information, and we find that its thermal derivative possesses a finite discontinuity at exactly the critical temperature.

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I. INTRODUCTION

The notion of the geometric entropy, in the framework of a quantum system, is quite old. One of the first calculations [1] was performed in the eighties for the case of a scalar field propagating in a black hole background. Some years later, a similar problem, in the framework of a quantum field theory, was addressed by several authors [2–5]. Geometric entropy, generally speaking, is a measure of the information loss after cutting out a spatial region of the system. It caught attention because of its characteristic behavior: for a system in its ground state, it grows like the boundary surface of the excluded sub-region, a property it shares with the black hole entropy. In fact, in the context of quantum field theory, pioneering work on the geometric entropy was driven in part by the suggested connection to the Bekenstein–Hawking black hole entropy [6]. From the very beginning, geometric entropy, named also entanglement entropy, has been tightly related with the presence of spatial entanglement in a quantum system. Entanglement is a fundamental ingredient of quantum mechanics leading to strong correlations between subsystems. From the early days of quantum mechanics up until now, it has been playing an increasingly important role in understanding and controlling quantum systems. The interest in it has been renewed [7] after the developments of the quantum information science in which it is viewed as a resource in quantum information processing. Geometric entropy has been considered as a measure of spatial entanglement when the system under consideration is in a pure quantum state with a density matrix of the form $\hat{\rho} = |\Psi\rangle\langle\Psi|$. By defining an “in” and an “out” spatial region and tracing out the “in” degrees of freedom one obtains the reduced density matrix for the “out” region: $\hat{\rho}_{\text{out}} = tr_{\text{in}}\hat{\rho}$. The entanglement entropy is then defined as the von Neumann entropy: $S_{\text{out}} = -tr\hat{\rho}_{\text{out}} \ln \hat{\rho}_{\text{out}}$.

When the system is in a thermal state, e.g. in a mixed state, the geometric entropy can be defined, following the von Neumann definition, in an analogous way. However, in this case, it does not have the same properties as the entanglement entropy in a pure state system, and it is no longer a good estimator of entanglement since it mixes correlations of different types [4, 5, 8], from genuine quantum to classical thermal correlations. Since it measures the thermal information loss, geometric entropy becomes an extensive quantity at the limit of an infinite system, and loses the “area law” behavior that characterizes a pure state system. As an alternative probe for the amount of quantum correlations in the case of thermal states, the notion of the so-called “mutual information” [9] has been proposed, which, roughly speaking, eliminates the thermal entropy contribution from the geometric entropy and can be considered as an upper limit for the entanglement entropy.

In any case, geometric entropy has been considered as a convenient construction, playing the role of an order parameter, for the investigation of finite temperature conformal quantum field systems [5] and in the context of the AdS/CFT correspondence, aiming at the physics of strongly coupled Quark-Gluon Plasma, the weakly coupled deconfined phase of Yang-Mills theories or the phase
structure of large N QCD at a finite density [10].

In the current work, we examine the geometric entropy in a free bosonic quantum field theory at finite temperature and at zero and non-zero chemical potential. Having found it, we subtract its extensive part, that is, the part related to the amount of information that is lost due to the mixed nature of the system. The resulting expression exhibits the known area law behavior and we adopt, for it, the term “mutual information”.

The underlying reason for the present study is connected to the Bose-Einstein condensation that characterizes the system, which has been in the center of theoretical and experimental investigations during the last fifteen years after the production of the condensate in the laboratory [11]. Bose-Einstein condensation has the characteristics of a phase transition albeit, theoretically, it can take place in an ideal system [12–15]. It is then natural to search for the interconnection between this phase transition and the spatial quantum correlations in the Bose system. Our findings indicate that, indeed, the Bose-Einstein condensation influences the behavior of the mutual information: We find that its derivative with respect to the temperature, $\partial m / \partial T$, has a finite discontinuity at the critical temperature both at the non-relativistic and the relativistic limit.

The paper is structured as follows: In section II we present the calculation of the geometric entropy at finite temperature and zero chemical potential and we define the resulting mutual information, setting the stage for our main goal which is the topic of the section that follows. This is section III, in which we apply our results in an environment with a finite charge density, we calculate the explicit form of the mutual information, and we derive the discontinuity of its temperature derivative. Finally, in the last section we summarize our findings. In Appendix A we present some technical details of our calculations.

II. GEOMETRIC ENTROPY AT FINITE TEMPERATURE

Our starting point is the thermal density matrix $\hat{\rho} = e^{-\beta H} / Z$ of a quantum field system and its Fock space representation:

$$\rho[\Phi', \Phi] = \frac{1}{Z} \langle \Phi' | e^{-\beta H} | \Phi \rangle. \quad (1)$$

Here $\Phi$ denotes a single scalar field or a collection of fields. The matrix element (1) can be written as a functional integral:

$$\rho[\Phi', \Phi] = \frac{1}{Z(\beta)} \int D\Phi(\tau, x) e^{-\int_0^\infty d\tau \int d^Dx L[\Phi]} \delta(\Phi(0, x) - \Phi(0)) \delta(\Phi(\beta, x) - \Phi'(\beta)). \quad (2)$$

It worth noting that at the zero temperature limit, $\beta \to \infty$, Eq. (2) is just an interpretation of the ground state density matrix [4]. It is then natural to expect that our result will reproduce, at this limit, the known [3–5] entanglement entropy. To derive the geometric entropy we follow the usual line of reasoning and we divide the $D$ dimensional space on which our system is defined, into two regions $A$ : $(x_1 > 0, \vec{x}_\perp)$ and $B$ : $(x_1 < 0, \vec{x}_\perp)$. Tracing out the “in” region, and gluing along the axis $x_1 > 0$, $n$ copies of the resulting reduced density matrix, we find [3–5]:

$$\text{tr}(\hat{\rho}_R)^n = \frac{1}{Z^n(\beta)} \int_{M_n} D\Phi e^{-S[\Phi]} = \frac{Z_n(\beta)}{Z^n(\beta)}. \quad (3)$$

In $Z_n$ the fields are defined on a $D + 1$ dimensional space $M_n = R_{D-1} \times C_n$. The subspace $R_{D-1}$ is an Euclidean space with metric $ds^2 = dx_1^2 + \cdots + dx_D^2$, while $C_n$ is a two dimensional Riemann space consisting of $n$ sheets glued together along the positive $x_1$ axis. This $n$ folded structure turns eventually [4] the $(\tau, x_1)$ plane into a flat cone with a deficit angle $\delta = 2\pi(1 - n)$ at the origin. Having found $\text{tr}(\hat{\rho}_R)^n$ the geometric entropy is defined through the relation:

$$-\lim_{n \to 1} \frac{\text{tr}(\hat{\rho}_R)^n - 1}{n - 1} = -\text{tr}(\hat{\rho}_R \ln \hat{\rho}_R)$$

$$\equiv S_g = -\left(\frac{\partial}{\partial n} - 1\right) \ln Z_n \bigg|_{n=1}. \quad (4)$$

For a free bosonic theory, the partition function can be deduced by following standard steps:

$$\ln Z_n(\beta) = \frac{1}{2} \int_{0^+}^{\infty} \frac{dT}{T} e^{-T m^2} \text{tr}_{M_n} e^{-T(\partial^2_\tau + \partial^2_\vec{x})}$$

$$ \times V_{D-1} \int_{0^+}^{\infty} \frac{dT}{T(D+1)/2} e^{-T m^2} \text{tr}_{C_n} e^{-T(\partial^2_\tau + \partial^2_\vec{x} + m^2)} \quad (5)$$

where $\partial^2_\vec{x} = \partial^2_\tau + \partial^2_\vec{x} + m^2$. Due to the locality of the action, the partition function in Eq. (5) is not expected to depend explicitly on the details of the Riemann surface. Thus, in order to calculate the non-trivial trace appearing in (5), we start with the finite temperature propagator of a free particle in cartesian coordinates:

$$A_{\beta n}(\vec{x}, \vec{x}') = \langle \vec{x}' | e^{-T(\partial^2_\vec{x})} | \vec{x} \rangle_{\beta n}. \quad (6)$$

The subscript $\beta_n = \beta n$ indicates the periodic boundary conditions imposed on the thermal Green’s function. As it is obvious, they are dictated by the $n$ folded structure.
of the Riemann space $C_n$. The next step is to transfer the result onto a two dimensional cone with deficit angle $2\pi(1 - n)$:

\[
ds^2 = dp^2 + \rho^2 n^2 d\theta^2, \quad 0 \leq \theta \leq 2\pi.
\] (7)

One can easily find that the free thermal propagator in Eq. (6) assumes the form [16]:

\[
A_{\beta_n}(\vec{x}, \vec{y}) = \frac{1}{4\pi T} \sum_{\nu=\infty}^{\infty} e^{-\frac{\pi}{4\nu} \left( (x_1'-x_1)^2 + (x_0'-x_0+\nu\beta_n)^2 \right)}
\]

\[
= \frac{1}{4\pi T} \sum_{\nu=\infty}^{\infty} e^{-\frac{\pi}{4\nu} (x'^2 - x^2)^2 + \frac{\nu \pi}{4\nu} (x_0' - x_0) + \nu \pi \beta_n^2}.
\] (8)

The above expression can be written in the conical metric (7) by making the replacements $x_0' = \rho \sin(n\theta)$, $x_1 = \rho \cos(n\theta)$, and using the expansion [17]:

\[
e^{-iz \cos(n\theta)} = \sum_{m=-\infty}^{\infty} c_m J_{|m|} (z) e^{im\theta}, \quad c_m = i^{\frac{1}{n}} m.
\] (9)

Thus, the thermal propagator on the surface (7) reads:

\[
A_{\beta_n}(\vec{x}': \vec{y}; \rho; \beta, \nu, n) = \frac{1}{4\pi T} \sum_{\nu, m, m_1, m_2} e^{\frac{(\rho')^2}{4\nu} + \frac{(\rho^2 - \rho')^2}{4\nu}} \times
\]

\[
\times e^{im(\theta' - \theta)} e^{im_1 \theta} e^{im_2 \theta} e^{-\nu \pi (m_1 + m_2)} \times
\]

\[
\times J_{|m|} (\frac{\rho'}{2\nu}) J_{|m_1|} (\frac{\nu \beta_n}{2\nu}) J_{|m_2|} (\frac{\nu \beta_n}{2\nu}) \times
\]

\[
\times i^{\frac{1}{n}} m_1 \frac{1}{m} i^{\frac{1}{n}} m_2 \frac{1}{m} (-1)^{\frac{1}{n}} m.
\] (10)

In the last expression the rotation $T \rightarrow i T$ has been adopted in order to secure convergence of all our intermediate steps.

Tracing out Eq. (10) we find:

\[
A_{\beta_n} = tr_{C_n} e^{-T(-\rho^2)} = \frac{1}{2it} \sum_{\nu, m, m_1} e^{-\frac{(\nu \theta)^2}{4\nu} + \frac{1}{n} \rho^2} \times
\]

\[
\times \int_0^{\infty} dp e^{-\frac{\rho^2}{4\nu} J_{|m|} (\frac{\rho}{2\nu}) J_{|m_1|} (\frac{\nu \beta_n}{2\nu})}. \quad \text{ (11)}
\]

At this point we stress the fact that for $n \neq 1$, the trace over the conical metric (7), that is the integration over $\rho$, must be performed before the summations over $m$ or $m_1$. The relevant calculations can be facilitated by using the fact that we are only interested in the derivative of Eq. (11) with respect to $n$:

\[
(\partial_n A_{\beta_n})_{n=1} = \frac{1}{2it} \sum_{\nu} e^{-\frac{(\nu \theta)^2}{4\nu} + \frac{1}{n} \rho^2} \partial_n \left[ \int_0^{\infty} dp e^{-\frac{\rho^2}{4\nu} J_{|m|} (\frac{\rho}{2\nu}) J_{|m_1|} (\frac{\nu \beta_n}{2\nu})} \right].
\]

\[
+ \frac{1}{2it} \partial_n \left[ \sum_{\nu, m} e^{-\frac{\nu \beta_n^2}{4\nu} + \frac{1}{n} \rho^2} d\rho, J_{|m|} (\frac{\nu \beta_n}{2\nu}) \right]. \quad \text{ (12)}
\]

In obtaining the last expression we have used the identities:

\[
\sum_{m} i^{-m} J_m(z) = e^{-iz}, \quad \sum_{m} J_m^2(z) = 1.
\] (13)

In Appendix A we prove that:

\[
\frac{1}{2it} \sum_{\nu} \int_0^{\infty} dp e^{-\frac{\rho^2}{4\nu} + \frac{1}{n} \rho^2} J_{|m|} (\frac{\nu \beta_n}{2\nu}) = \frac{V_2}{4\pi T} + \frac{1}{2} \left( \frac{1}{n} - n \right) + O\left( \frac{T}{V_2} \right). \quad \text{ (14)}
\]

and

\[
\frac{1}{2it} \sum_{\nu} \int_0^{\infty} dp e^{-\frac{\rho^2}{4\nu} + \frac{1}{n} \rho^2} d\rho, J_{|m|} (\frac{\nu \beta_n}{2\nu}) = \frac{V_2}{4\pi T} + O\left( \frac{T}{V_2} \right). \quad \text{ (15)}
\]

In the above equations we introduced an upper cutoff $R$ in the $\rho$-integrals and we have written as $V_2 = \pi R^2$ the volume of the two dimensional subspace. Substituting the first term in the rhs of Eq. (14) into Eq. (12) and feeding with the result Eq. (12) we find (see Appendix A) that it leads to the logarithm of the partition function:

\[
\frac{1}{2} \frac{V_{D-1} V_2}{(4\pi)^{\frac{D+1}{2}}} \int_0^{\infty} d\omega \omega^{\frac{D-1}{2}} e^{-\omega m^2} \sum_{\nu} e^{-\frac{1}{4\nu} \omega^2} = \ln Z_1(\beta).
\] (16)

Following the same steps for the first term in the rhs of Eq. (15) we can prove that it is connected to the thermal entropy of the system:

\[
\frac{1}{2} \frac{V_{D-1} V_2}{(4\pi)^{\frac{D+1}{2}}} \int_0^{\infty} d\omega \omega^{\frac{D-1}{2}} e^{-\omega m^2} \sum_{\nu} e^{-\frac{1}{4\nu} \omega^2} = \ln Z_1(\beta).
\] (17)

where $\omega^2 = p^2 + m^2$. The contribution to Eq. (12) of the second term in the rhs of Eq. (14) assumes the form:

\[
\frac{1}{12} \frac{1}{(4\pi)^{\frac{D+1}{2}}} V_{D-1} \int_0^{\infty} d\omega \omega^{\frac{D-1}{2}} e^{-\omega m^2} \sum_{\nu} e^{-\frac{1}{4\nu} \omega^2}
\]
Collecting everything together and using Eq. (5) we get the geometrical entropy:

\[ S_g = \frac{\pi}{6} V_{D-1} \int \frac{d^p p}{(2\pi)^D \omega} \frac{1}{\tanh(\frac{\omega}{2})} \]

(18)

At the limit \( p \to \infty \), \( \tanh(\sqrt{\beta^2 + m^2} \beta) \to 1 \) and consequently the fist integral in Eq. (19) diverges. The same divergence appears in the case of zero temperature:

\[ S_g(\beta = \infty) = \frac{\pi}{6} V_{D-1} \int \frac{d^p p}{(2\pi)^D \omega} \frac{1}{\sqrt{\beta^2 + m^2}} \]

(20)

After this observation we are led to write:

\[ S_g(\beta) = S_g(\beta = \infty) \]

(21)

Some comments are in order at this point. The first term in the last expression represents the well known [3–5] entanglement entropy at zero temperature. This is a quantity that diverges in the absence of an ultraviolet cutoff, while it grows like the boundary surface of the excluded subregion. This fact clearly indicates the existence of very strong quantum correlations between fields defined at neighboring points, a direct consequence of a local quantum field theory. A quantitative explanation of such a behavior can be traced back to the uncertainty relations. Even at zero temperature, the notion of a sharp, well defined, boundary surface is more classical than quantum. The divergences appearing in \( S_g(T = 0) \) are connected to the fact that in Eq. (20) we integrate down to zero distance, driving to infinity the density of the reduced density matrix eigenvalues. The second term is finite and well-defined for \( m^2 > 0 \). It is also proportional to the boundary surface and it is an increasing function of the temperature. We can consider it as a measure of the number of degrees of freedom that have been excited on the boundary surface due to the non-zero temperature. The last term is the thermal entropy of the system, an obviously extensive quantity. Subtracting this term from the geometric entropy we are led to define the mutual information:

\[ I_m(\beta) = S_g(\beta = \infty) + \frac{\pi}{3} V_{D-1} \int \frac{d^p p}{(2\pi)^D \omega} \frac{1}{\sqrt{\beta^2 + m^2}} \]

(22)
where the arrow underlines the fact that we have kept only the terms that are relevant for determining the mutual information.

In Appendix A we show that:

\[ I_m = \frac{\pi}{6} V_{D-1} \times \]

\[ \int \frac{d^DP}{(2\pi)^D\omega} \left\{ \frac{1}{\tanh \left[ \frac{\omega - \mu}{2} \right]} + \frac{1}{\tanh \left[ \frac{\omega + \mu}{2} \right]} \right\}, \tag{28} \]

Isolating the (diverging) zero temperature contribution we find:

\[ I_m = S_g(\beta = \infty) + \frac{\pi}{6} V_{D-1} \times \]

\[ \int \frac{d^DP}{(2\pi)^D\omega} \left\{ \frac{1}{e^{(\omega - \mu)\beta} - 1} + \frac{1}{e^{(\omega + \mu)\beta} - 1} \right\}. \tag{29} \]

For the system in hand the zero temperature entanglement entropy reads:

\[ S_g(\beta = \infty) = \frac{\pi}{3} V_{D-1} \int \frac{d^DP}{(2\pi)^D\omega} \]

\[ \rightarrow \frac{1}{6} \frac{V_{D-1}}{(4\pi)^{D/2} m^{D-1} \Gamma \left( \frac{D - 1}{2} \right) m^2 / \Lambda^2}. \tag{30} \]

To reveal the physical content of our results we shall focus on the well-studied $D = 3$ case which hosts the Bose-Einstein condensation. As is well-known [12–14] the quantitative realization of the phenomenon is different at the two opposite limits, the non-relativistic $\rho \ll m^3$ and the ultra-relativistic one $\rho \gg m^3$, as these are defined by the total charge density of the system.

Beginning from the non-relativistic case, in which the charge density is very low and the anti-particle contribution can be omitted [14], we rewrite Eq. (29) in the form:

\[ I_m^{NR}(\beta) = S_g(\beta = \infty) \]

\[ + \frac{\pi V_2}{6 m} \int \frac{d^DP}{(2\pi)^3 (e^{\frac{(\omega - \mu)^3}{m^{3/2}} - \mu_{NR})^3 - 1)} \tag{31} \]

where we noted as $\mu_{NR}(\beta) = \mu - m \leq 0$ the non-relativistic chemical potential. The integral appearing in Eq. (31) is the total density of particles occupying excited states:

\[ \rho_c = \int \frac{d^DP}{(2\pi)^3 (e^{\frac{(\omega - \mu)^3}{m^{3/2}} - \mu_{NR})^3 - 1} \]

\[ \left( \frac{m}{2\pi \beta} \right)^{3/2} \sum_{n=1}^{\infty} \frac{z_{NR}^n}{n^{3/2}}, \quad z_{NR} = e^{\beta \mu_{NR}} \leq 1. \tag{32} \]

For temperatures below a certain critical value $T_C$, $\mu(T_C) = m$ and this number is a constant:

\[ \rho_c = \left( \frac{2\pi m}{\beta} \right)^{3/2} \zeta \left( \frac{3}{2} \right), \quad 0 \leq T \leq T_C. \tag{33} \]

At exactly the critical temperature the number (33) becomes the conserved total particle density of the system:

\[ \rho_c = \rho = \left( \frac{2\pi m}{\beta} \right)^{3/2} \zeta \left( \frac{3}{2} \right). \tag{34} \]

As an immediate consequence we get for the mutual information:

\[ I_m^{NR} = S_g(T = 0) + \frac{\pi V_2}{6 m} \rho \left( \frac{T}{T_C} \right)^{3/2}, \quad T < T_C. \tag{35} \]

Above the critical temperature the system passes to the gas phase in which all of the particles occupy excited states. The mutual information reads now:

\[ I_m^{NR} = S_g(T = 0) + \frac{\pi V_2}{6 m} \rho, \quad T > T_C. \tag{36} \]

Thus, the Bose-Einstein condensation and the relevant phase transition are reflected in a discontinuity of the derivative of the mutual information:

\[ \frac{\partial}{\partial T} I_m^{NR} \bigg|_{T=T_C^-} - \frac{\partial}{\partial T} I_m^{NR} \bigg|_{T=T_C^+} = \frac{\pi^2}{2} \zeta \left( \frac{3}{2} \right) V_2 \rho_0^{3/2}. \tag{37} \]

When $\rho \gg m^3$ we are approaching the ultra-relativistic limit, the critical temperature rises at relativistic high values $T_C = (3|\rho|/m)^{1/2} \gg m$ and the behavior of the system changes. Below the critical temperature, one easily finds that [12, 13]:

\[ I_m = S_g(T = 0) \]

\[ + \frac{\pi V_2}{6} \int \frac{d^DP}{(2\pi)^3 (e^{\frac{(\omega - \mu)^3}{m^{3/2}} - \mu_{NR})^3 - 1} + \frac{1}{e^{(\omega + \mu)^3} - 1}} \]
\[
\approx S_b(T = 0) + \frac{\pi V_2 |\rho|}{12 m} \left( T_{bC}\right)^2, \quad T < T_C. \tag{38}
\]

The integral that appears in Eqs. (37) and (38) is not the charge density of the system and, consequently, is not a conserved quantity even for temperatures above the critical one. However, it is not hard to confirm [12] that at high temperatures \( T > T_C \) it behaves as following:

\[
\int \frac{d^3 p}{(2\pi)^3} \omega \left( \frac{1}{e^{\beta (\omega - p \beta)} - 1} + \frac{1}{e^{\beta (\omega + p \beta)} - 1} \right) = \frac{T^2}{6} - \frac{T}{2\pi} (m^2 - \mu^2) / 2 - \frac{m^2}{4\pi^2} \ln \left( C \frac{m}{T} \right) + \frac{1}{4\pi^2} (m^2 - \mu^2) + O(\frac{m^2}{T^2}) \tag{39}
\]

where \( C = e^{\gamma_E - 1}/4\pi \) and \( \gamma_E \) is the Euler-Macheroni constant.

As in the non-relativistic case, the derivative of the mutual information with respect to the temperature possesses a discontinuity that reflects the underlying phase transition:

\[
\frac{\partial}{\partial T} I_m \bigg|_{T = T_C^{-}} - \frac{\partial}{\partial T} I_m \bigg|_{T = T_C^{+}} = - \frac{\pi \sqrt{3}}{9} V_2 \left( |\rho| / m \right)^{1/2}. \tag{40}
\]

The last result completes our study for the influence of the Bose-Einstein condensation on the entropy of entanglement in an ideal Bose system at finite temperature and non-zero chemical potential.

**IV. CONCLUSION**

In the current work we have performed two types of calculations and we have arrived at results with a clear physical content. First, we calculated the geometric entropy in an ideal Bose system at finite temperature and we confirmed the expected result: It combines the genuine quantum correlations with the classical thermal fluctuations, and it becomes an extensive quantity for an infinite system. Due to the simplicity of the system under consideration, we were able to explicitly subtract from the geometric entropy the thermal one. In this way we defined the so-called mutual information that exhibits a property which, as it is believed, characterizes quantum correlations: it grows like the surface that bounds the space region in which a system lives. The second calculation we performed refers to a Bose system at finite temperature and non-zero chemical potential. We examined the impact of the Bose-Einstein condensation on the quantum correlations of the system as these are represented by the mutual information. We found that, at the critical temperature, the temperature derivative of the mutual information exhibits a finite discontinuity, and we explicitly calculated it.

**APPENDIX A**

In this Appendix we shall prove those of the formulas appearing in the text for which summations over \( m \) or \( \nu \) must be performed. To begin with, let us discuss Eq. (14) of the text. The relevant integral diverges and calls for the introduction of a cutoff:

\[
\frac{1}{2it} \sum_m \int_0^\infty dp e^{-\frac{p^2}{2it} - \frac{|\nu m|}{\pi T} J_{|\nu m|} (\frac{p^2}{2t})}
\]

\[
to \frac{1}{2t} \sum_m \int_0^R dp e^{-\frac{p^2}{2t} - \frac{|\nu m|}{\pi T} J_{|\nu m|} (\frac{p^2}{2t})}. \tag{A1}
\]

To handle the last integral we make an intermediate step by introducing the following expression:

\[
F_n (\alpha) = \frac{1}{2} \sum_{m=-\infty}^{\infty} \int_0^\infty dq e^{-\alpha q} I_{|\nu m|} (q)
\]

\[
= \frac{1}{2\sqrt{\alpha^2 - 1}} \coth \left( \frac{\alpha + \sqrt{\alpha^2 - 1}}{2} \right) \tag{A2}
\]

which is also a regularized version (for \( \alpha \to 1^+ \)) of the integral entering Eq. (14). Taking the limit \( \alpha = 1 + \epsilon, \epsilon \to 0^+ \) one easily finds that:

\[
F_n (\alpha) = \frac{n}{2\epsilon} + \frac{1}{12} \left( \frac{1}{n} - n \right) + O(\epsilon). \tag{A3}
\]

We immediately see that the diverged part of the integral appears for \( n = 1 \). In this case the integration in (A.1) is trivial and we are led to the conclusion:

\[
\frac{1}{\epsilon} \approx \frac{\pi}{4\pi T} = \frac{V_2}{4\pi T}. \tag{A4}
\]

Combining this identification with the finite part appearing in (A3) we get the confirmation of Eq. (14).

Our next concern is eq. (16). Using the identities:

\[
e^{-\frac{|\nu m|^2}{4\pi T}} = (4\pi T)^{D+1} \int \frac{d^D+1 p}{(2\pi)^{D+1}} e^{-T p^2 + ip_{\nu} \beta}
\]

\[
\sum_{\nu=-\infty}^{\infty} e^{ip_{\nu} \beta} = \frac{2\pi}{\beta} \sum_{k=-\infty}^{\infty} \delta(p_0 - \omega_k), \quad \omega_k = \frac{2\pi k}{\beta} \tag{A5}
\]
we recast the integral appearing in Eq. (16) into the form:

\[
\frac{1}{2} V_{D-1} V_2 \int_0^\infty \frac{dT}{T^{\frac{D+1}{2}}} e^{-T m^2} \sum_\nu e^{-(\nu\beta)^2 / T} = \frac{V_D}{2} \int \frac{d^D p}{(2\pi)^D} \sum_k \int_0^\infty \frac{dT}{T^{\frac{D+1}{2}}} e^{-T \left[(\beta\omega)^2 + (2\pi k)^2\right]}. \tag{A6}
\]

To obtain the last result we wrote \( V_2 = \int_0^\infty \frac{d\tau}{\beta \omega} \int dx_1 \to \beta L \), we rescaled \( T \to T/\beta^2 \) and we used the abbreviation \( \omega^2 = p^2 + m^2 \).

Performing the integral over \( T \) and neglecting an irrelevant (infinite) constant we get:

\[
\int_0^\infty \frac{dT}{T^{\frac{D+1}{2}}} e^{-T \left[(\beta\omega)^2 + (2\pi k)^2\right]} = -\ln \left((\beta\omega)^2 + (2\pi k)^2\right). \tag{A7}
\]

The summation over \( k \) is standard [13]:

\[
\frac{1}{2} \sum_\nu \ln \left((\beta\omega)^2 + (2\pi k)^2\right) = \frac{1}{2} \beta \omega \ln(1 - e^{-\beta \omega}). \tag{A8}
\]

The last result proves Eq. (16) of the text.

Following the same line of reasoning we can prove Eq. (18). Using, once again, the identities (A5) we rewrite the relevant integral in the form:

\[
\frac{1}{12} \frac{V_{D-1}}{(4\pi)^{\frac{D+1}{2}}} \int_0^\infty \frac{dT}{T^{\frac{D+1}{2}}} e^{-T m^2} \sum_\nu e^{-(\nu\beta)^2 / T} = \pi \frac{V_{D-1}}{3^3} \int \frac{d^D p}{(2\pi)^D} \sum_k \frac{1}{\omega^2 + \omega_k^2}. \tag{A9}
\]

The summation is easily performed:

\[
\sum_k \frac{1}{\omega^2 + \omega_k^2} = \frac{\beta}{2\omega} \tanh(\beta\omega). \tag{A10}
\]

Combining Eqs. (A9) and (A10) we immediately obtain Eq. (18) of the text.

Our next concern is Eq. (15). The relevant integral diverges and the introduction of a cutoff is necessary. To this end let us discuss the integral:

\[
\sum_\nu \int_0^\infty d\rho e^{-\rho^2 / 2T} \int_1^\infty d\nu \frac{\nu^3}{2\nu} \int_0^\infty d\nu' \frac{1}{\nu' T^{\frac{D+1}{2}}} e^{-T m^2} \sum_\nu e^{-(\nu\beta)^2 / T}.
\]

which can be considered (at the limit \( R \to \infty \)) as a regularized version of the integral appearing in Eq. (15). Note that the divergence in (A11) is independent of \( n \) and, contrary to (A3), there is no finite part for \( n \neq 1 \). This completes the proof of Eq. (15).

To prove eq. (17) it is enough to follow the road we followed to arrive at (A8). Beginning from the relation:

\[
\frac{V_{D-1}}{2} \frac{V_2}{(4\pi)^{\frac{D+1}{2}}} \int_0^\infty \frac{dT}{T^{\frac{D+1}{2}}} e^{-T m^2} \sum_\nu e^{-(\nu\beta)^2 / T} = \frac{V_D}{2n} \int \frac{d^D p}{(2\pi)^D} \sum_k \int_0^\infty \frac{dT}{T^{\frac{D+1}{2}}} e^{-T \left[(\beta\omega)^2 + (2\pi k)^2\right]} \tag{A12}
\]

we only have to perform a differentiation with respect to \( n \) to arrive at the result indicated in Eq. (17):

\[
\frac{\partial}{\partial n} \left[ \frac{1}{n} \int \frac{d^D p}{(2\pi)^D} \left(-\frac{1}{2} \beta \omega n - \ln \left(1 - e^{-\beta \omega n}\right) \right) \right]_{n=1} = \int \frac{d^D p}{(2\pi)^D} \left[\ln \left(1 - e^{-\beta \omega}\right) - \frac{\beta \omega}{e^{\beta \omega} - 1}\right]. \tag{A13}
\]

The last relation we have to prove is Eq. (28) of the text. We begin by using the Poisson summation formula to find that:

\[
\sum_\nu e^{-(\nu\beta)^2 / T} = \frac{\sqrt{4\pi T}}{\beta} \sum_k e^{-T(\omega_k + \nu\mu)^2}. \tag{A14}
\]

With the help of this result we get the mutual information:

\[
I_m = \frac{2\pi}{3^3} \frac{V_{D-1}}{V_D} \int \frac{d^D p}{(2\pi)^D} \sum_k \frac{1}{(\omega_k + \nu\mu)^2 + \omega^2}. \tag{A15}
\]

The sum in the last expression can be easily performed if we rewrite it in the form:

\[
\sum_k \frac{1}{(\omega_k + \nu\mu)^2 + \omega^2} = \frac{1}{2} \frac{\omega - \mu}{\omega} \sum_k \frac{1}{\omega_k^2 + (\omega - \mu)^2} + \frac{1}{2} \frac{\omega + \mu}{\omega} \sum_k \frac{1}{\omega_k^2 + (\omega + \mu)^2}. \tag{A16}
\]
Using for each term the formula (A10) we get the result indicated in Eq. (28).

It would be useful to compare our result indicated in Eq. (19) with the corresponding result derived in the framework of a two dimensional conformal scalar field theory [5] with central charge $c = 1/2$. This can be done by indentifying the ultraviolet cutoff $\Lambda$ with the inverse lattice spacing $1/\alpha$ and the mass $m$ with the inverse finite size of excluded interval $1/l$ (that is, the infrared cutoff).

Given that our result is valid at the limit $L^2 m^2 \to \infty$, the comparison is meaningful only for $\beta/\alpha \to \infty$ or $l/\beta \to \infty$.

Applying Eq. (19) for $D = 1$ we get the result:

$$S_g = \begin{cases} \frac{1}{6} \ln \left( \frac{\Lambda}{m} \right) & \text{if } \beta \to \infty \\ \frac{2}{6} \ln \left( \frac{L}{m} \right) - \frac{2}{3} \frac{1}{\beta} & \text{if } \beta \to 0 \end{cases}.$$  (A17)